Pramana – J. Phys., Vol. 38, No. 1, January 1992, pp. 31–35. © Printed in India.

Energy and momentum in Vaidya spacetime

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MS received 26 April 1991; revised 30 September 1991

Abstract. The components of the energy-momentum pseudotensors of Einstein, Tolman, Landau and Lifshitz, and Møller are evaluated for the Vaidya radiating spacetime. These pseudotensors are found to be traceless for this spacetime. The pseudotensors of Einstein and Tolman give exactly same result for all their components. Unlike in the case of the Kerr-Newman field, the pseudotensor of Møller gives the same energy as given by that of the Einstein, Tolman or Landau and Lifshitz.

Keywords. Vaidya spacetime; energy-momentum; pseudotensors.

PACS No. 04·20; 04·40

1. Introduction

The energy and momentum in curved spacetime has been a subject of extensive research since the early days of general relativity. Following the first energy-momentum pseudotensor of Einstein, a large number of energy-momentum complexes as instruments to calculate energy and momentum in curved spacetimes have been proposed by many authors which are cited in a paper by Virbhadra (1990c). There are mutually opposing views that authors share regarding the physical importance of the energy-momentum pseudotensors as well as a possibility of successful localization of the energy and momentum in general relativistic systems (see references, Misner *et al* 1973; Cooperstock and Sarracino 1978; Palmer 1980; Bondi 1990). Using the energy-momentum pseudotensors of Einstein, Tolman, Landau and Lifshitz and Møller (ETLLM), Virbhadra (1990a, c) calculated energy in the Kerr-Newman (K–N) spacetime. LL pseudotensor, being symmetric, allows one to evaluate angular momentum in asymptotically flat spacetimes. Virbhadra (1990b) evaluated angular momentum in the K–N spacetime and found convincing results.

The pseudotensors of Einstein, Tolman and LL (ETLL) give meaningful results only if these are evaluated in quasi-Galilean coordinates. Such calculations are usually very lengthy. Enjoying the freedom of divergence relation, Møller (1958) constructed a new energy-momentum complex and claimed that one is no more restricted to the use of the quasi-Galilean coordinates. The energy density component of the Møller's complex transforms like a scalar density under purely spatial transformations. Only three years after the new energy-momentum complex was proposed, Møller (1961) realized that unlike in the Einstein's prescription, the total energy-momentum vector of a closed physical system does not transform like a four-vector with respect to the Lorentz transformations. Later Kovacs (1985) showed that Møller was wrong in his conclusion. However, Novotny (1987) pointed out the mistake of Kovacs and wrote that Møller was correct to say that the energy-momentum vector in his prescription is not a Lorentz four-vector.

Vaidya (1952) started with a line element for the general non-static spherically symmetric spacetime, transformed it into quasi-Galilean coordinates and then he found the energy density in this spacetime in Tolman's prescription. Lindquist *et al* (1965) evaluated the total energy and momentum in the prescription of LL for the Vaidya radiating spacetime. In this paper, we evaluate all the components of the pseudotensors of ETLLM for the same spacetime in order to investigate the possible symmetries and to compare the results for the energy and momentum in this spacetime in these prescriptions.

The paper is organized as follows: Sections 2 and 3 give respectively the Vaidya spacetime and the pseudotensors of ETLLM. Section 4 presents the calculation of the components of the pseudotensors for the Vaidya spacetime, and § 5 gives a discussion of the investigations in this paper. We use the usual geometrized units ($G = c = 1$) and follow the convention that the Latin indices take values 0 to 3 where x^0 is the time coordinate t.

2. Vaidya radiating spacetime

The Vaidya radiating spacetime (Vaidya 1951, 1953) is a non-static generalization of the Schwarzschild spacetime and it gives the gravitational field due to a spherically symmetric radiating star. The line element describing this spacetime in the Schwarzschild coordinates is

$$ds^2 = B\,dt^2 - A\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

with

$$B = \left[\frac{\dot{M}}{f(M)}\right]^2 \left[1 - \frac{2M}{r}\right] \tag{2}$$

and

$$A = \left[1 - \frac{2M}{r}\right]^{-1} \tag{3}$$

where $M = M(r,t)$, $\dot{M} = \partial M/\partial t$ and $f(M)$ is an arbitrary function of the mass parameter $M(r,t)$.

Vaidya (1952) wrote the line element (1) in quasi-Galilean coordinates as

$$ds^2 = B\,dt^2 - dx^2 - dy^2 - dz^2 - \frac{A-1}{r^2}[x\,dx + y\,dy + z\,dz]^2 \tag{4}$$

where

$$r^2 = x^2 + y^2 + z^2. \tag{5}$$

However, the same spacetime in Kerr-Schild cartesian coordinates is given by the line element (Lindquist *et al* 1965):

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{2M(u)}{r}\left[dt - \frac{1}{r}(x\,dx + y\,dy + z\,dz)\right]^2 \tag{6}$$

where

$u = t - r$, and r as defined by (5).

3. Pseudotensors

The energy-momentum complexes of Einstein (Møller 1958), Tolman (Vaidya 1952). Landau and Lifshitz (1985), and Møller (1958) which are given below are extensively discussed in literature.

$$\theta_i^k = (1/16\pi)\left[\frac{g_{in}}{\sqrt{-g}}\{-g(g^{kn}g^{lm} - g^{ln}g^{km})\}_{,m}\right]_{,l} \tag{7}$$

(g stands for the determinant of the metric),

$$t_i^k = (1/8\pi)[\sqrt{-g}\{-g^{lk}\, V^j{}_{il} + (1/2)g_i^k g^{lm}\, V^j{}_{lm}\}]_{,j} \tag{8}$$

with

$$V^i{}_{jk} = -\Gamma^i{}_{jk} + (1/2)g^i{}_j\Gamma^m{}_{mk} + (1/2)g^i{}_k\Gamma^m{}_{mj} \tag{9}$$

$$L^{mn} = L^{nm} = (1/16\pi)[-g(g^{mn}g^{jk} - g^{mk}g^{jn})]_{,jk} \tag{10}$$

$$\mathscr{T}_i{}^k = \frac{1}{8\pi}[\sqrt{-g}(g_{in,m} - g_{im,n})g^{km}g^{ln}]_{,1} \tag{11}$$

$\theta_i{}^k$, $t_i{}^k$, L^{ik}, $\mathscr{T}_i{}^k$ are the energy-momentum complexes of Einstein, Tolman, Landau and Lifshitz, and Møller, respectively.

4. Calculations

Vaidya (1952) used the line element given by (4) and obtained the energy density in Tolman's prescription for the radiating spacetime which is given as follows:

$$t_0{}^0 = \frac{1}{8\pi r^2}\frac{\partial}{\partial r}[(B/A)^{1/2}r(A-1)]. \tag{12}$$

As the Vaidya spacetime in the Kerr–Schild form is simpler, we start with the line element (6) and evaluate all the components of the pseudotensors of ETLLM. These are given as follows:

$$\theta_0{}^0 = t_0{}^0 = L^{00} = \mathscr{T}_0{}^0 = -M'/(4\pi r^2),$$

$$\theta_1{}^0 = -\theta_0{}^1 = t_1{}^0 = -t_0{}^1 = -L^{01} = -\mathscr{T}_0{}^1 = xM'\alpha,$$

$$\theta_2{}^0 = -\theta_0{}^2 = t_2{}^0 = -t_0{}^2 = -L^{02} = -\mathscr{T}_0{}^2 = yM'\alpha,$$

$$\theta_3{}^0 = -\theta_0{}^3 = t_3{}^0 = -t_0{}^3 = -L^{03} = -\mathscr{T}_0{}^3 = zM'\alpha,$$

$$\theta_1{}^1 = t_1{}^1 = -L^{11} = \beta x^2 M',$$

$$\theta_2{}^2 = t_2{}^2 = -L^{22} = \beta y^2 M',$$

$$\theta_3{}^3 = t_3{}^3 = -L^{33} = \beta z^2 M',$$

$$\theta_1{}^2 = \theta_2{}^1 = t_1{}^2 = t_2{}^1 = -L^{12} = \beta xyM',$$

$$\theta_2{}^3 = \theta_3{}^2 = t_2{}^3 = t_3{}^2 = -L^{23} = \beta yzM',$$

$$\theta_3{}^1 = \theta_1{}^3 = t_3{}^1 = t_1{}^3 = -L^{31} = \beta zxM',$$

$$\mathscr{T}_1{}^0 = x\beta\mu, \quad \mathscr{T}_2{}^0 = y\beta\mu, \quad \mathscr{T}_3{}^0 = z\beta\mu,$$

$$\mathscr{T}_1{}^1 = \gamma[M(r^2 - 3x^2) + M'rx^2],$$

$$\mathscr{T}_2{}^2 = \gamma[M(r^2 - 3y^2) + M'ry^2],$$

$$\mathscr{T}_3{}^3 = \gamma[M(r^2 - 3z^2) + M'rz^2],$$

$$\mathscr{T}_1{}^2 = \mathscr{T}_2{}^1 = xy\nu\gamma, \quad \mathscr{T}_2{}^3 = \mathscr{T}_3{}^2 = yz\nu\gamma,$$

and

$$\mathscr{T}_3{}^1 = \mathscr{T}_1{}^3 = zx\nu\gamma \tag{13}$$

where

$$\alpha = 1/(4\pi r^3), \quad \beta = \alpha/r, \quad \gamma = \beta/r,$$

$$\mu = -2M + rM', \quad \nu = \mu - M. \tag{14}$$

The prime denotes the derivative with respect to the coordinate u. It is clear from (13) with (14) that the integrated values of the components of these pseudotensors over the three space (spatial) are finite.

Traces of these pseudotensors for the spacetime under investigation are

$$\theta_i{}^i = t_i{}^i = L_i{}^i = \mathscr{T}_i{}^i = 0. \tag{15}$$

5. Discussion

Virbhadra (1990a, c) showed that the pseudotensors of ETLL give the same energy in the Kerr–Newman spacetime whereas that of Møller gives twice the value obtained in these prescriptions. However, it is found that these prescriptions yield the same energy for the Vaidya spacetime. The total energy and momentum associated with this spacetime is $p^i = (M, 0, 0, 0)$ as found in all these four prescriptions. We find that these energy-momentum complexes are traceless for the radiating spacetime. It is worth noting that $\theta_i{}^k = t_i{}^k$ for all values of the indices i and k.

For the Schwarzschild spacetime ($M =$ constant), all the components of the pseudotensors of ETLL vanish whereas that of Møller, none but only the energy and momentum density components are zero. The symmetry of the spacetime is clearly reflected in the components of the pseudotensors.

In passing, it is worth mentioning that we have now evaluated all the components of the pseudotensors of ETLLM for the K–N spacetime. We have found, as in the Vaidya spacetime, that these pseudotensors are traceless for the K–N spacetime and the pseudotensors of Einstein and Tolman give exactly same result for all their components.

Acknowledgements

Thanks are due to Professor P C Vaidya for his guidance during the work and for correcting the manuscript. Thanks are also due to Professor L K Patel for helpful discussions and to Dr D Kovacs for bringing to my knowledge the paper by Dr J Novotny.

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