Pramana - J. Phys., Vol. 38, No. 1, January 1992, pp. 11-20. © Printed in India.

BRST invariance and the conical pendulum

S G KAMATH

Department of Mathematics, Indian Institute of Technology, Madras 600036, India

MS received 23 April 1991; revised 13 September 1991

Abstract. The Hamiltonian formulation of the BRST method for quantizing constrained systems developed recently by Nemeschansky *et al* is applied to the well-known problem of the conical pendulum in classical mechanics. The similarity of the system to a gauge theory wherein the two constraints serve as generators of local Abelian gauge transformations is also pointed out. The definition of the physical states of the system as a gauge theory and also as a BRST invariant theory is then discussed in some detail.

Keywords. BRST invariance; conical pendulum.

PACS No. 11·15

1. Introduction

In a recent study by Nemeschansky *et al* (1988) (hereafter referred to as I) a Hamiltonian formulation of the BRST method (Becchi *et al* 1974, 1975, 1976) for quantizing constrained systems was developed in a manner so as to render the subject accessible, in particular, to the pedestrian. The quantization of a simple classical system, namely, that of a free particle of mass m constrained to move on a circle of radius a was taken up to: (a) establish the equivalence between a theory with constraints and a gauge theory, and (b) rewrite the gauge theory as a quantum system with BRST symmetry.

An immediate generalization of the aforesaid example is to that of a free particle constrained to move on a sphere. It is easily seen, however that this replacement of the constraint $((x^2 + y^2)^{1/2} - a)$ by $((x^2 + y^2 + z^2)^{1/2} - a)$ adds little to the content of § 2 in I apart, of course, from appropriate changes in the definitions. Thus, in the case of motion on a sphere one would define

$$\mathbf{p}_\theta = \mathbf{r} \times \mathbf{p}$$

$$p_r = \frac{1}{2r}(\mathbf{r}\cdot\mathbf{p}) + (\mathbf{p}\cdot\mathbf{r})\frac{1}{2r} \tag{1}$$

with $\mathbf{r}$ and $\mathbf{p}$ as 3-component vectors, instead of 2-component vectors as in eqs (2.6) and (2.7) of I. In this context, the extra constraint (θ = constant) implied by the problem of motion of a free particle on a cone (instead of a sphere) turns out to be particularly interesting. Thus, at the level of a local gauge theory, there would be two kinds of gauge transformations, with the generators given by the constraint equations (r = constant) and (θ = constant). Besides, within the framework of the BRST formalism, one would have instead of a doublet of generators $(Q, \bar{Q})$ for the case of the sphere — a pair of doublets which we denote here by $(Q_i, \bar{Q}_i, i = 1, 2)$.

The motion of a pendulum—be it the simple, spherical or the conical—is a concrete example wherein the three types of motions discussed above are realized physically. Because the extra potential energy term relevant to the pendulum motion in each case commutes with the respective constraint equations, the quantization procedure adopted for the free particle motion will go through without alteration when extended to include the case of the pendulum motion. In other words, one could, for example, include the potential energy term $mg(r-x)$ for the simple pendulum in the Lagrangian (2.1) of I and simply repeat the arguments given in that section. Likewise a separate discussion for the spherical pendulum—with the potential energy given by $mg(r-z)$—is also unnecessary because of the trivial generalization referred to earlier in respect of free particle motion on a circle and a sphere. We therefore confine our attention here to the problem of the conical pendulum only.

The rest of this paper is organized as follows. In § 2 we use the momenta given in (1) to derive a Hamiltonian for the conical pendulum which commutes with the constraints $r=r_0$ and $\theta=\theta_0, r_0$ and θ_0 being constants. Defining a physical state as one which is annihilated by the constraint operators $(r-r_0)$ and $(\theta-\theta_0)$ the invariance of the Hamiltonian under local Abelian gauge transformations is then worked out. In § 3 the gauge theory discussed in § 2 is rewritten following I as a quantum system which possesses BRST symmetry. Because the BRST invariant Lagrangian that we shall write breaks gauge invariance, the definition of a physical state adopted in § 2 will now be altered to a new one using the BRST generators $Q_i, \bar{Q}_i, (i=1,2)$. Finally, § 4 concludes the paper with a short discussion.

2. The conical pendulum as a gauge theory

The Lagrangian for the conical pendulum is taken as

$$2L=m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-2mg(r-z)-2(\mu_1\xi_1+\mu_2\xi_2) \tag{2}$$

where the point of suspension is taken as the origin with the z-axis pointing vertically downwards, and μ_1 and μ_2 are the Lagrange multipliers, with $\xi_1=(r-r_0)$ and $\xi_2=(\theta-\theta_0)$. With the usual definitions of the canonical momenta, one can now obtain the Hamiltonian as

$$2mH=(p_x^2+p_y^2+p_z^2)+2m^2g(r-z)+2m(\mu_1\xi_1+\mu_2\xi_2). \tag{3}$$

If we quantize the system by using the commutators ($\hbar=1$)

$$[x,p_x]=i, [y,p_y]=i, [z,p_z]=i$$

one finds that as in the case of the rotor, the constraints* $\xi_1=0=\xi_2$ no longer commute with the Hamiltonian.

*The equations $\xi_1=0$ and $\xi_2=0$ have the status of a first class constraint in the terminology introduced by Dirac (1964). Such constraints, which Dirac denoted by the equation $\phi_i(q,p)=0, i=1,\dots m$, when there are m constraints, physically represent generating functions of infinitesmal contact transformations that lead to changes in the q's and p's but do not affect the physical state. In the present context, the contact transformation is given by (7) below.

To construct a Hamiltonian which commutes with the constraints, we use the following commutators

$$[r, p_r] = i, \quad [r, p_\theta^{(i)}] = 0, \quad [\theta, p_r] = 0,$$

$$[\theta, p_\theta^{(1)}] = -i\frac{y}{\lambda}, \quad [\theta, p_\theta^{(2)}] = i\frac{x}{\lambda}$$

$$[\phi, p_\theta^{(3)}] = i, \quad [\theta, p_\theta^{(3)}] = 0, \quad [\theta, P_\theta] = i, \quad [r, P_\theta] = 0 \tag{4}$$

where $p_\theta^{(i)}$ is the ith component of the vector $\mathbf{p}_\theta$ given in (1), $P_\theta = (1/\lambda)(x p_\theta^{(2)} - y p_\theta^{(1)})$ and $\lambda = (x^2 + y^2)^{1/2}$. Using (1), eq. (3) can be rewritten, after normal ordering, as

$$2mH = p_r^2 + \frac{1}{r^2} p_\theta^2 + 2m^2 g(r - z) + 2m(\mu_1 \xi_1 + \mu_2 \xi_2). \tag{5}$$

With the help of the commutators given in (4), it is clear that there are three terms in (5) which do not commute with the constraints. They are the first term, and the sum $(1/r^2)(p_\theta^{(1)^2} + p_\theta^{(2)^2})$ contained in the second. Following the prescription of Dirac (1966), we drop these terms from the Hamiltonian (5), and arrive at

$$H = \frac{1}{2mr^2} p_\theta^{(3)^2} + mg(r - z) + \mu_1 \xi_1 + \mu_2 \xi_2. \tag{6}$$

The constraints ξ_1 and ξ_2 now commute with the H in (6) and therefore the three operators can be simultaneously diagonalized. The physical states of the conical pendulum will then be states for which $\xi_1 |\psi\rangle = 0$ and $\xi_2 |\psi\rangle = 0$. In the equations below we shall, for notational convenience, write p_θ for $p_\theta^{(3)}$ which appears in (6).

Following I we shall now show, that the Lagrangian constructed from (6) is invariant under local gauge transformations whose generators are ξ_1 and ξ_2. Since ξ_1 and ξ_2 commute, the group associated with the transformations will be a two-parameter Abelian group.

It is easy to see using (4) that under a gauge transformation represented by the unitary operator

$$U = \exp\{-i(f_1(t)\xi_1 + f_2(t)\xi_2)\}, \tag{7}$$

with $f_i(t)$, $i = 1, 2$ being arbitrary c-number functions of time t, that,

$$U p_r U^{-1} = p_r + i f_1(t).$$
$$U P_\theta U^{-1} = P_\theta + i f_2(t). \tag{8}$$

A first order Lagrangian constructed from (6) will be given by

$$L = p_r \dot{r} + P_\theta \dot{\theta} + p_\phi \dot{\phi} - \frac{1}{2mr^2} p_\theta^2 - mg(r - r\cos\theta) - (\mu_1 \xi_1 + \mu_2 \xi_2). \tag{9}$$

Clearly, under the increments defined by

$$\delta p_r = f_1(t), \quad \delta\mu_1 = -\dot{f}_1(t)$$
$$\delta P_\theta = f_2(t), \quad \delta\mu_2 = -\dot{f}_2(t)$$
$$\delta p_\theta = 0 = \delta r = \delta\phi = \delta p_{\mu_1} = \delta p_{\mu_2} = \delta\theta = \delta p_\phi \tag{10}$$

the change in the Lagrangian in (9) will be

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}(f_1\xi_1 + f_2\xi_2). \tag{11}$$

Since δL is a total derivative the action is invariant. Similarly, under (10), one finds easily that for the Hamiltonian given by (6),

$$\delta H = -(\dot{f}_1(t)\xi_1 + \dot{f}_2(t)\xi_2). \tag{12}$$

Unlike (11), eq. (12) is not a total time derivative; however on the sector of physical states, as defined earlier, $\delta H = 0$. Before concluding this section, we note that the momenta canonically conjugate to the Lagrange multipliers μ_i are gauge invariant, as seen from (10). In the restriction of the Hamiltonian (6) to the space of physical (or gauge invariant) states the p_{μ_i} have had no role to play, as the time derivative of μ_i does not appear in the Lagrangian. There is therefore an arbitrariness associated with the eigenvalues of p_{μ_i}, as far as the physical states of the pendulum are concerned. Following I therefore, we shall define physical states $|\psi\rangle$ as those for which $p_{\mu_1}|\psi\rangle = 0 = p_{\mu_2}|\psi\rangle$ in addition to $\xi_1|\psi\rangle = 0 = \xi_2|\psi\rangle$.

In the next section we shall rewrite the gauge theory developed here as a quantum system which possesses a BRST symmetry. It may be apt to point out here that, as in the case of rotor, there is no compelling reason for this generalization. Still, following I we shall write a gauge non-invariant (under the gauge transformations whose increments are given as in (10)), but BRST invariant Lagrangian and obtain subsequently the BRST charge*, $Q_i, \bar{Q}_i (i = 1, 2)$ alluded to in the introduction.

3. BRST transformations

We begin with a brief but necessarily incomplete (for reasons of space) introduction to the subject of BRST transformations. These transformations were first proposed (Becchi *et al* 1974, 1975, 1976) in connection with the renormalization of gauge theories. It is enough for our purposes here to restrict our attention below to renormalization in QED, though for the record, the utility of BRST is strongly evident for non-Abelian gauge theories (Becchi *et al* 1976).

The renormalization of any quantum field theory, as is well-known, involves the elimination of divergent diagrams associated with the initial Lagrangian density $\mathscr{L}$ for the field theory; it is done by redefining the initial Lagrangian i.e. by adding to ($\mathscr{L}$) a finite number of counterterms. Naturally, these counterterms should preserve the symmetries, for example, local gauge invariance, present in $\mathscr{L}$. In proving that renormalization does not violate local gauge invariance it is convenient to use the Ward identities—which are essentially relations between Green's functions†. While

*Physically, the BRST charges $Q_i, \bar{Q}_i$ will have the same status as the ξ_1, ξ_2 dealt with in this section; namely they are the generators of infinitesimal contact transformations on the coordinates and their canonical momenta (see (13) and (20) below) but without any changes on the physically realizable state.

†In QED the Ward identities ensure that the gauge invariance of the Lagrangian density ($\mathscr{L}$) is preserved under renormalization by virtue of the equality of the renormalization constants Z_1 and Z_2 to all orders.

the method of derivation of these Ward identities is largely a matter of choice, the BRST technique for QED (as well as for non-Abelian gauge theories) consists (Ramond 1983) in enlarging the number of terms in the Lagrangian density; because the extra term in QED is associated with Grassmann fields which do not interact with the fields originally present in $\mathscr{L}$, this procedure merely redefines the generating functional for the Green's functions, but with a new normalization constant. One can now show (Ramond 1983) that this new effective Lagrangian for QED is invariant under BRST transformations—a characteristic feature of which is that it is a gauge transformation but with a mixing between operators having different statistics. This is exactly what we shall meet with while constructing a BRST invariant version of eq. (9) further below.

From the above remarks it is clear that BRST was basically an aid (albeit involving fictional Grassmann variables) to the renormalization of a gauge theory. The paper of Nemeschansky *et al* (1988) is important because the equivalence between local gauge theories and physical systems with first class constraints that was pointed out by Dirac (1964), has been extended there to establish the connection between constrained systems, local gauge invariance and BRST symmetry. In order words, I show that BRST symmetry is not a mere artefact of the renormalization procedure, but a symmetry that is intrinsic to constrained systems.

We shall now show below that the Lagrangian for the conical pendulum given in (9), can be reformulated so as to admit of BRST invariance. Under a BRST transformation as mentioned earlier gauge transformation which shifts operators by c-number functions (e.g. (10)) is replaced by a transformation which mixes operators with different statistics. Additionally under BRST, $\delta^2 = 0$. Consider, for example, the following infinitesimal increments involving new anti-commuting variables $c_i, \bar{c}_i$, $(i = 1, 2)$ and a new commuting variable $b_i (i = 1, 2)$ besides the operators contained in (10):

$$\begin{aligned}
&\delta\mu_1 = -\dot{c}_1, \quad \delta\mu_2 = -\dot{c}_2, \quad \delta p_r = c_1, \quad \delta P_\theta = c_2 \\
&\delta p_{\mu_1} = \delta p_\phi = \delta\theta = \delta r = \delta p_{\mu_2} = \delta\phi = 0 = \delta p_\theta \\
&\delta c_1 = \delta c_2 = 0, \quad \delta\bar{c}_1 = b_1, \quad \delta\bar{c}_2 = b_2, \quad \delta b_1 = 0 = \delta b_2.
\end{aligned} \tag{13}$$

Under (13), the change in the following gauge non-invariant Lagrangian obtained by the addition of a gauge-fixing term to (9)

$$\begin{aligned}
L = {} & q_i\dot{\xi}_i + p_\phi\dot{\phi} - \frac{1}{2mr^2} p_\theta^2 - mgr(1 - \cos\theta) \\
& - (\mu_1\xi_1 + \mu_2\xi_2) - \delta\left(\sum_{i=1}^{2} \bar{c}_i\left(\dot{\mu}_i - q_i + \frac{1}{2}b_i\right)\right)
\end{aligned} \tag{14}$$

with $q_1 = p_r, q_2 = P_\theta$ can be easily found*.

We get

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}(c_1\xi_1 + c_2\xi_2) \tag{15}$$

*In (14) and below, there will be a sum on the repeated index i over $i = 1, 2$ unless otherwise indicated.

thus implying that the action is invariant. By choosing the gauge fixing term as in (14) we are, as in the case of rotor, identifying μ_1 and μ_2 with the time components of the electromagnetic potential in QED and p_r and P_θ with $\nabla\cdot\mathbf{A}$. The definition of δ in (13) can be used to rewrite (14) as

$$L = q_i\dot{\xi}_i + p_\phi\dot{\phi} - \frac{1}{2mr^2}p_\theta^2 - mgr(1-\cos\theta)$$
$$-\mu_i\xi_i - b_i(\dot{\mu}_i - q_i + \tfrac{1}{2}b_i) + \dot{\bar{c}}_i\dot{c}_i - \bar{c}_ic_i. \tag{16}$$

Using the Euler equation for $b_i = q_i - \dot{\mu}_i$, we see that

$$-\sum_{i=1}^{2} b_i\left(\dot{\mu}_i - q_i + \frac{1}{2}b_i\right) = \sum_{i=1}^{2}\frac{1}{2}(\dot{\mu}_i - q_i)^2. \tag{17}$$

With the identification referred above we see that the extra BRST invariant term plays the role of adding a gauge fixing term of the form $(\partial_0 A_0 - \nabla\cdot\mathbf{A})^2$ to the QED Lagrangian.

From (16), the momenta canonically conjugate to c_i, $\bar{c}_i$ and μ_i can be easily found to be $\pi_{c_i} = \dot{\bar{c}}_i, \pi_{\bar{c}_i} = \dot{c}_i$, and $p_{\mu_i} = -b_i$ respectively.

We therefore expect the following non-zero commutators:

$$[\mu_i, p_{\mu_j}] = i\delta_{ij}, \quad \{\dot{\bar{c}}_i, c_j\}_+ = i\delta_{ij}, \{\dot{c}_i, \bar{c}_j\}_+ = i\delta_{ij} \tag{18}$$

besides

$$\{c_i, c_j\} = 0 = \{\bar{c}_i, \bar{c}_j\} = \{c_i, \bar{c}_j\}.$$

Note however, that the last anticommutator above yields,

$$\{\dot{c}_i, \bar{c}_j\} = -\{c_i, \dot{\bar{c}}_j\}$$

which does not agree with the signs in (18). The correct sign, as in I, is fixed using the fermionic part of the Hamiltonian (see eq. (24) below) to get

$$[H, c_i] = -i\dot{c}_i = -\{\dot{\bar{c}}_j, c_i\}\dot{c}_j$$

so that

$$\{\dot{\bar{c}}_i, c_i\} = i\delta_{ij} = -\{\dot{c}_i, \bar{c}_j\} \tag{18a}$$

instead of the (incorrect) anticommutators in (18).

The BRST operator Q_i which effects the increments in the operators given in (13) can now be worked out using (18a) as (no sum over i below)

$$Q_i = -i\,c_i\xi_i - i\,\dot{c}_ip_{\mu_i}. \tag{19}$$

Thus there are two generators Q_1, Q_2 whose action on the physical states should be given by $Q_1|\psi\rangle = 0 = Q_2|\psi\rangle$, so as to reflect the conditions $\xi_i|\psi\rangle = 0 = p_{\mu_i}|\psi\rangle, i = 1, 2$ mentioned earlier. While deferring this issue to the latter part of this section, we can readily check that instead of (13), if the set of increments were written as,

$$\delta\mu_i = \dot{\bar{c}}_i, \quad \delta q_i = -\bar{c}_i, \quad \delta c_i = b_i, \quad \delta\bar{c}_i = 0 \tag{20}$$

with the increments in all the other variables set to zero then, the following Lagrangian

$$L = q_i\dot{\xi}_i + p_\phi\dot{\phi} - \frac{1}{2mr^2}p_\theta^2 - mgr(1-\cos\theta) - \mu_i\xi_i$$
$$-\delta\left(\sum_{i=1}^{2}(\dot{\mu}_i - q_i + \tfrac{1}{2}b_i)c_i\right) \tag{21}$$

is changed under (20) by

$$\delta L = -\frac{\mathrm{d}}{\mathrm{d}t}(\bar{c}_1\xi_1 + \bar{c}_2\xi_2)$$

so that the action is again invariant. The reader will notice that the gauge fixing term in (21) differs from that in (14). However, using (20) in (21), one can recover the Lagrangian in (16) easily.

Equation (20) is associated with the anti-BRST operators (no sum over i below)

$$\bar{Q}_i = i\bar{c}_i\xi_i + i\dot{\bar{c}}_i p_{\mu_i}. \tag{22}$$

The $\bar{Q}_i$ in (22) are the adjoints of the Q_i in (19), and eq. (18a) yields the following anti-commutators:

$$\{Q_i, \bar{Q}_j\} = 0 = \{Q_i, Q_j\} = \{\bar{Q}_i, \bar{Q}_j\}. \tag{23}$$

Thus, unlike the case of the rotor, one now has a pair of doublets $(Q_i, \bar{Q}_i, i = 1, 2)$ associated with the BRST transformations given in (13) and (20). The Hamiltonian resulting from the Lagrangian given in (16) can be worked out as (with i summed as before):

$$H = q_i\dot{\xi}_i + p_\phi\dot{\phi} + \pi_{c_i}c_i + \dot{\bar{c}}_i\pi_{\bar{c}_i} + p_{\mu_i}\dot{\mu}_i - L$$
$$= \frac{1}{2mr^2}p_\theta^2 + mgr(1-\cos\theta) + \mu_i\xi_i + p_{\mu_i}q_i + \tfrac{1}{2}p_{\mu_i}^2 + \dot{\bar{c}}_i\dot{c}_i + \bar{c}_i c_i. \tag{24}$$

Using the BRST generators in (19) and (22) one can verify with the help of the commutators in (18a) that the Hamiltonian is both BRST and anti-BRST invariant, i.e.,

$$[Q_i, H] = 0 = [\bar{Q}_i, H]. \tag{25}$$

We now discuss in some detail, for the remainder of this section, the conditions that the state vectors should obey so that they can be labelled as physical states. Much of the discussion here will parallel that in I and will hence be brief.

From (16) we notice that the Euler equation of motion for the c_i will be given by

$$\ddot{c}_i + c_i = 0 = \bar{c}_i + \ddot{\bar{c}}_i. \tag{26}$$

Equation (26) has the solution given by $c_i(t) = f_i e^{it} + g_i e^{-it}$ and, $\bar{c}_i(t) = f_i^+ e^{-it} + g_i^+ e^{it}$.

Imposing the conditions $c_i^2 = \bar{c}_i^2 = 0$ we obtain since $c_i(0) = f_i + g_i, \bar{c}_i(0) = f_i^+ + g_i^+$, the relations

$$f_i^2 + g_i^2 + \{f_i, g_i\}_+ = 0,$$
$$f_i^{+2} + g_i^{+2} + \{f_i^+, g_i^+\}_+ = 0,$$

so that $f_i^2 = 0 = f_i^{+2} = g_i^2 = g_i^{+2}$. Also $\{f_i, g_i\} = 0 = \{f_i^+, g_i^+\}$. Similarly, from the anticommutators $\{c_i, \bar{c}_j\} = 0 = \{\dot{\bar{c}}_i, \dot{c}_j\}$ and $\{\dot{\bar{c}}_i, c_j\} = i\delta_{ij}$, we obtain, as in I,

$$\{f_i, g_j^+\} = 0 = \{f_i^+, g_j\}$$
$$\{f_i, f_j^+\} = -\tfrac{1}{2}\delta_{ij} = -\{g_i, g_j^+\}. \tag{27}$$

So, by defining the vacuum as the state for which $f_i|0\rangle = g_i|0\rangle = 0$, we find that the states $f_i^+|0\rangle$ and $g_i^+|0\rangle$ have a norm equal to 1/2 and $-1/2$ respectively if the norm of $|0\rangle$ is chosen to be -1 as in I. In terms of the operators f_i, f_i^+, g_i and g_i^+ one can express the $Q_i, \bar{Q}_i$ as

$$Q_i = -i(f_i\Lambda_{+i} + g_i\Lambda_{-i})$$
$$\bar{Q}_i = i(f_i^+\Lambda_{-i} + g_i^+\Lambda_{+i}) \tag{28}$$

with

$$\Lambda_{\pm i} = \xi_i \pm i p_{\mu_i}.$$

From (28), it is clear that the condition $\xi_i|\psi\rangle = 0 = p_{\mu_i}|\psi\rangle$ used to describe the physical states $|\psi\rangle$ in the gauge theory in §2 can, for instance, be rewritten as $Q_i|\psi\rangle = 0$. But, because the f_i and g_i commute with $\Lambda_{\pm i}$, one also has $Q_i|\phi\rangle = 0$ for all states $|\phi\rangle$ for which

$$f_i|\phi\rangle = 0 = g_i|\phi\rangle. \tag{29}$$

In other words, as observed by Nemeschansky *et al*, although the set of states annihilated by the operator Q_i describes the set of states $|\psi\rangle$ for which $\xi_i|\psi\rangle = 0 = p_{\mu_i}|\psi\rangle$, it also contains additional states for which (29) holds. It is here that the anti-BRST symmetry of the Hamiltonian expressed by the second equality in (25) comes in handy. Because, by requiring as in I that $\bar{Q}_i$ also annihilate the physical states (in addition to Q_i), one easily excludes those extra states for which (29) is satisfied.

Thus, by insisting that for states given by (29) $\bar{Q}_i|\phi\rangle = 0$, we obtain,

$$f_i^+|\phi\rangle = 0 = g_i^+|\phi\rangle. \tag{30}$$

The first equality in (30) yields, using (27) and (29),

$$f_i f_i^+|\phi\rangle = 0 = (-\tfrac{1}{2} - f_i^+ f_i)|\phi\rangle = -\tfrac{1}{2}|\phi\rangle. \tag{31}$$

Similarly,

$$g_i g_i^+|\phi\rangle = 0 = \tfrac{1}{2}|\phi\rangle. \tag{31a}$$

Both (31) and (31a) show that there cannot be any free eigenstates of the fermionic part of the Hamiltonian for which (29) and (30) hold simultaneously. However with Q_i and $\bar{Q}_i$ defined by (28), for all states for which the bosonic operators ξ_i and p_{μ_i} satisfy $\xi_i|\psi\rangle = 0 = p_{\mu_i}|\psi\rangle$, the conditions $Q_i|\psi\rangle = 0 = \bar{Q}_i|\psi\rangle$, for $i = 1, 2$, continue to hold undisturbed.

Thus, as in the case of a particle moving on a circle, the extra anti-BRST symmetry is necessary in order to impose enough constraints so as to recover only the physical states for the conical pendulum.

4. Conclusions

In this paper, we have second quantized the well-known classical problem (see, for example, Symon 1978) of the conical pendulum with the two constraints $r - r_0 = 0$ and $\theta - \theta_0 = 0$. The Lagrangian for this system was first shown to be invariant up to a total time derivative under local gauge transformations, with the constraints serving as generators of the transformations. The gauge invariance of the Hamiltonian, however, required the identification of the physical states of the conical pendulum as those which are annihilated by the operators $\xi_1 = (r - r_0)$ and $\xi_2 = (\theta - \theta_0)$. Then, following I, a BRST invariant (but gauge non-invariant) Lagrangian for the system was written down. A pair of generators (Q_1, Q_2) associated with the BRST transformations were then obtained. It was then pointed out that within the BRST formalism, the identification of the physical states for the conical pendulum is made with the aid of the generators $(\bar{Q}_1, \bar{Q}_2)$ of an extra anti-BRST symmetry possessed by the BRST invariant Lagrangian.

In conclusion, we ought to mention that:

(a) The first quantization of the simple pendulum was studied long ago by Condon (1928) and more recently by Pradhan and Khare (1973) and Aldrovandi and Ferreira (1980). The last mentioned authors, in particular, have obtained the energy spectrum of the Schrödinger Hamiltonian as well as its eigenfunctions in terms of a certain class of Mathieu functions.

(b) Abdel-Rahman (1983) has studied the classical, semi-classical and quantum aspects of the conical pendulum having a negative total energy in its rotating frame. A peculiarity of the system is that it is a classic illustration of the phenomenon of spontaneous symmetry breaking.

(c) Finally, as a sequel to this work, we propose to examine in a subsequent paper, the Ward identities for the Green's functions associated with the BRST invariant Lagrangian given in (16).

Acknowledgements

The author thanks R Anishetty and H S Sharatchandra of the Institute of Mathematical Sciences, Madras for useful discussions.

References

Abdel-Rahman A-M M 1983 *Am. J. Phys.* **51** 721
Aldrovandi R and Ferreira P 1980 *Am. J. Phys.* **48** 660
Becchi C, Rouet A and Stora R 1974 *Phys. Lett.* **B52** 344
Becchi C, Rouet A and Stora R 1975 *Commun. Math. Phys.* **42** 127
Becchi C, Rouet A and Stora R 1976 *Ann. Phys.* **98** 287
Condon E U 1928 *Phys. Rev.* **31** 891

Dirac P A M 1964 *Lectures on quantum mechanics* (New York: Yeshiva University)
Dirac P A M 1966 *Lectures on quantum field theory* (New York: Yeshiva University)
Nemeschansky D, Preitschopf C and Weinstein M 1988 *Ann. Phys.* **183** 226 (referred to as I in the text)
Pradhan T and Khare A V 1973 *Am. J. Phys.* **41** 59
Ramond P 1983 *Field theory: A modern primer* (Reading: Benjamin/Cummings)
Symon K R 1978 *Mechanics* (Reading: Addison–Wesley)