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Relation between Green's functions for different potentials

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Abstract. A general relation between the energy-dependent Green's functions for different potentials is derived in a simple and direct manner. This interesting connection enables the eigenstates of one physical system to be deduced from those of a related system. The derivation is based on the Schrödinger equation and provides an independent justification for the technique of path-dependent time transformation used in path integration.

Keywords. Propagator; Green's function; path integral; hydrogen atom; solvable potentials.

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1. Introduction

The present work deals with an interesting general relation between the energy-dependent Green's functions of different physical problems. The connection takes a remarkably compact form when expressed in terms of the radial part of the Green's function for spherically symmetric potentials. Such a relation was first obtained by Steiner (1984a) in the course of his work on radial path integrals. He made use of it to give a path-integral treatment of the hydrogen atom, by relating the radial Coulomb Green's function for a given l to that of an oscillator having an unphysical angular momentum value of $2l + \frac{1}{2}$ (Steiner 1984b). His analysis involved a nonlinear spacetime transformation of the radial path integral, with a path-dependent change of the time variable.

Our principal aim here is to demonstrate that the aforesaid general connection between the Green's functions for different potentials is derivable in an elementary and straight-forward manner, directly from the differential equation satisfied by the Green's function. There is no need to invoke any path-dependent time transformation. An interesting aspect of our method of approach is that it not only leads to the connection between two 3-d problems, as noted by Steiner, but also serves to connect certain 1-d problems with 3-d ones. As specific illustrations, we show that the Green's functions for the standard Coulomb and the 1-d Morse potentials are both expressible in terms of the 3-d harmonic oscillator Green's function. Consequently, the eigenvalues and eigenfunctions of these problems can be got from those of the oscillator. We also discuss the symmetric Rosen-Morse potential in one dimension. Its Green's function is shown to be related, not to that of an oscillator, but to that of another solvable 3-d system, the rigid rotator. In this case also the correct eigenvalues and eigenfunctions are deduced from those of the rotator.

2. The Coulomb-oscillator connection

The propagator $K(\mathbf{r}t, \mathbf{r}_0 t_0)$ of a particle of unit mass moving from an initial position $\mathbf{r}_0$ at time t_0 to a final position $\mathbf{r}$ at time t in a spherically symmetric potential $V(r)$ satisfies the differential equation ($\hbar = 1$)

$$\left[i\frac{\partial}{\partial t} + \frac{1}{2}\nabla^2 - V(r)\right]K = i\delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0). \tag{1}$$

For our purposes it is adequate to deal with the energy Green's function defined by

$$G(\mathbf{r}\mathbf{r}_0, E) = i\int_0^\infty \mathrm{d}T \exp(iET)K(\mathbf{r}\mathbf{r}_0, T) \tag{2}$$

where $T = t - t_0$ and E is to be understood as $E + i\delta$ whenever necessary, together with the limit $\delta \to 0+$. G satisfies the equation

$$[E + \tfrac{1}{2}\nabla^2 - V(r)]G = -\delta(\mathbf{r} - \mathbf{r}_0). \tag{3}$$

Making the partial wave expansion

$$G = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{rr_0} g_l(r, r_0, E)\, Y^*_{l,m}(\theta, \phi)\, Y_{l,m}(\theta_0, \phi_0) \tag{4}$$

we separate the angular dependence. On substituting (4) in (3) we get the following equation for the radial Green's function g_l:

$$\left[E + \frac{1}{2}\frac{\partial^2}{\partial r^2} - V(r) - \frac{l(l+1)}{2r^2}\right]g_l = -\delta(r - r_0). \tag{5}$$

Clearly, the dynamics of the problem is contained in this function and therefore we focus attention on g_l in the rest of our analysis.

We now consider changing the variables in (5) in such a way that the resulting equation is of the same form but with new parameters. One simple way of achieving this is to define

$$\rho = r^\alpha, \quad g_l = \frac{1}{\alpha}(\rho\rho_0)^{(1-\alpha)/2\alpha}\tilde{g}(\rho) \tag{6}$$

where $\alpha > 0$ is a free parameter. The equation for $\tilde{g}$ has the form

$$\left[\frac{1}{2}\frac{\partial^2}{\partial \rho^2} - W(\rho) - \frac{\lambda(\lambda+1)}{2\rho^2}\right]\tilde{g} = -\delta(\rho - \rho_0) \tag{7}$$

where

$$\lambda = -\frac{1}{2} + \frac{1}{\alpha}\left(l + \frac{1}{2}\right)$$

$$W(\rho) = \frac{1}{\alpha^2}\rho^{2(1-\alpha)/\alpha}[V(\rho^{1/\alpha}) - E]. \tag{8}$$

The structure of (7) enables us to interpret $\tilde{g}$ as the radial Green's function of a particle moving in a potential $W(\rho)$ with zero energy and angular momentum λ. Accordingly we write

$$\tilde{g} = g_\lambda^W(\rho, \rho_0; 0). \tag{9}$$

This identification leads us, via (6), to the following formal relation between the radial Green's functions of two different potentials:

$$g_l^V(r, r_0; E) = \frac{1}{\alpha}(\rho\rho_0)^{(1-\alpha)/2\alpha} g_\lambda^W(\rho, \rho_0; 0). \tag{10}$$

This result of ours is identical to the one derived by Steiner (1984a) in the context of path-integral evaluation of Green's functions. In practice the above relation is useful whenever we can find, for a given $V(r)$, a value α for which $W(\rho)$ represents a problem with known g_λ^W. Happily, this turns out to be the case for the Coulomb problem.

For the attractive Coulomb potential $V(r) = -e^2/r$, the choice $\alpha = 1/2$ in (6) and (8) gives

$$\rho = \sqrt{r}, \quad W(\rho) = -4e^2 - 4E\rho^2, \quad \lambda = 2l + \tfrac{1}{2}. \tag{11}$$

The function $\tilde{g}$ obeys the equation (eq. (7))

$$\left[\frac{1}{2}\frac{\partial^2}{\partial\rho^2} + 4e^2 + 4E\rho^2 - \frac{\lambda(\lambda+1)}{2\rho^2}\right]\tilde{g} = -\delta(\rho - \rho_0). \tag{12}$$

This equation shows that $\tilde{g}$ can be identified as the radial Green's function of an isotropic oscillator of frequency $\sqrt{-8E}$, energy $4e^2$ and angular momentum $\lambda = 2l + 1/2$:

$$\tilde{g} = g_\lambda^{\mathrm{OSC}}(\rho, \rho_0; \varepsilon = 4e^2, \omega = \sqrt{-8E}). \tag{13}$$

Coupling this with (10) we get

$$g_l^{\mathrm{Cou}}(r, r_0; E) = 2(rr_0)^{\frac{1}{4}} g_{2l+\frac{1}{2}}^{\mathrm{OSC}}(\sqrt{r}, \sqrt{r_0}, 4e^2, \sqrt{-8E}). \tag{14}$$

This relation gives the connection between the Coulomb radial Green's function for a given l and that of an oscillator with angular momentum continued to a value $2l + 1/2$.

The result (14) may be exploited to solve the Coulomb eigenvalue problem completely, given the eigenvalues and eigenfunctions of the oscillator. In doing this the following point is to be noted. While a negative value of E, the energy parameter of the Coulomb problem, leads to the usual confining oscillator potential with $\omega^2 > 0$, positive E corresponds to an oscillator with imaginary frequency. Thus, in order to get both the bound and the scattering states of the Coulomb problem, we must consider the oscillator eigenfunctions for positive as well as negative ω^2.

Consider first the case $E < 0$. We then have an oscillator with real frequency, whose radial Green's function at energy ε and angular momentum λ can be conveniently represented by its eigenfunction expansion:

$$g_\lambda^{\mathrm{OSC}}(\rho, \rho_0; \varepsilon, \omega) = \sum_{n_r=0}^{\infty} \frac{u_{n_r,\lambda}(\rho, \omega) u_{n_r,\lambda}^*(\rho_0, \omega)}{\varepsilon_{n_r} - \varepsilon - i\delta}. \tag{15}$$

Using this and the relation (14) we obtain the following explicit expression g^{Cou} for $E<0$:

$$g_l^{\mathrm{Cou}}(r,r_0;E)=2(rr_0)^{\frac{1}{4}}\sum_{n_r=0}^{\infty}\frac{u_{n_r,2l+\frac{1}{2}}(\sqrt{r})u^*_{n_r,2l+\frac{1}{2}}(\sqrt{r_0})}{(2n_r+2l+2)\sqrt{-8E-4e^2-i\delta}}. \tag{16}$$

In (15) and (16), $u_{n_r,\lambda}$ is the normalized reduced radial wave function of an oscillator of frequency $\omega=\sqrt{-8E}$ corresponding to radial quantum number n_r and angular momentum λ. Explicitly

$$u_{n_r,\lambda}(\rho,\omega)=\left[\frac{2\sqrt{\omega}\,n_r!}{\Gamma^3(n_r+\lambda+\frac{3}{2})}\right]^{\frac{1}{2}}(\sqrt{\omega}\rho)^{\lambda+1}e^{-\frac{1}{2}\omega\rho^2}L^{\lambda+\frac{1}{2}}_{n_r+\lambda+\frac{1}{2}}(\omega\rho^2). \tag{17}$$

The associated Laguerre polynomial is defined by

$$L^p_{n+p}(x)=\frac{\Gamma^2(n+p+1)}{n!\,\Gamma(p+1)}F(-n,p+1;x) \tag{18}$$

where F is the confluent hypergeometric function.

It follows from (16) that g^{Cou} considered as a function of E has simple poles at the values

$$E=E_n=-\frac{e^4}{2(n_r+l+1)^2} \tag{19}$$

which give the Coulomb bound state energies. These are the only singularities in the left half of the complex E-plane. From the residues of g^{Cou} at these poles, the bound state wave functions can be determined at once (to within an overall phase). They are

$$\begin{aligned}v_{n,l}(r)&=\frac{e}{\sqrt{2}(n_r+l+1)}r^{\frac{1}{4}}u_{n_r,2l+\frac{1}{2}}(\sqrt{r},\omega=\sqrt{-8E_n})\\&=\left[\frac{(n-l-1)!}{an^2(n+l)^3}\right]^{\frac{1}{2}}\left(\frac{2r}{na}\right)^{l+1}\exp(-r/na)\,L^{2l+1}_{n+l}\left(\frac{2r}{na}\right)\end{aligned} \tag{20}$$

where $n=n_r+l+1$ and $a=e^{-2}$ is the Bohr radius. $v_{n,l}$ given by (20) is seen to be the correctly normalized, reduced radial wave function.

We now consider the Coulomb Green's function for positive values of E. For $E>0$, we see from (12) that $\tilde{g}$ will be the radial Green's function of an oscillator whose frequency is imaginary (inverted oscillator):

$$\omega=\sqrt{-8E}=i\sqrt{8E}=2ik. \tag{21}$$

Here we have replaced E in terms of k, the asymptotic wave number. For an inverted oscillator the energy spectrum is the entire real continuum, the eigenfunctions being only Dirac delta normalizable (see Appendix). Using the orthogonality and completeness relations (A8 and A10) of the eigenfunctions u_λ, we can write the analogue of (15) for the inverted oscillator in the form

$$g_\lambda^{\mathrm{OSC}}(\rho,\rho_0;\varepsilon,2ik)=\int_{-\infty}^{\infty}\frac{\mathrm{d}\varepsilon'}{\varepsilon'-\varepsilon-i\delta}u_\lambda(\rho,\varepsilon',2ik)u^*_\lambda(\rho_0,\varepsilon',2ik). \tag{22}$$

Using this we get the following expression for the Coulomb radial Green's function for $E > 0$ (with $\sigma = \varepsilon'/4k$)

$$g_l^{\mathrm{Cou}}(r, r_0; E) = 2(rr_0)^{\frac{1}{4}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\sigma}{\sigma - (e^2/k) - i\delta} u_{2l+\frac{1}{2}}(\sqrt{r}, \sigma, 2ik) u^*_{2l+\frac{1}{2}}(\sqrt{r_0}, \sigma, 2ik). \tag{23}$$

The structure of the rhs of (23) shows that g^{Cou} as a function of E is nonanalytic everywhere along the positive real E axis. This implies in turn that every $E > 0$ is an eigenvalue of the Coulomb problem. To determine the Coulomb wave function $v_l(r, E)$ belonging to the continuum, we use the prescription

$$\mathrm{Im}\, g_l^{\mathrm{Cou}}(r, r; E) = \pi v_l(r, E) v_l^*(r, E) \tag{24}$$

and the identity

$$\frac{1}{\sigma - (e^2/k) - i\delta} = P \frac{1}{\sigma - (e^2/k)} + i\pi\delta(\sigma - (e^2/k)). \tag{25}$$

The result is

$$\begin{aligned} v_l(r, E) &= \sqrt{2} r^{\frac{1}{4}} u_{2l+\frac{1}{2}}(\sqrt{r}, \sigma = (e^2/k), 2ik) \\ &= \frac{\exp(\pi e^2/2k)}{\sqrt{2\pi k}\, \Gamma(2l+2)} \left| \Gamma\left(\frac{ie^2}{k} + l + 1 \right) \right| (2kr)^{l+1} \exp(-ikr) \\ &\quad \times F\left(\frac{ie^2}{k} + l + 1, 2l + 2, 2ikr \right). \end{aligned} \tag{26}$$

This is the correct Coulomb-reduced radial wave function which is normalized according to

$$\int_0^{\infty} v_l^*(r, E) v_l(r, E') \mathrm{d}r = \delta(E - E'). \tag{27}$$

3. The Morse-oscillator connection

The problem of the 1-d Morse potential

$$V(x) = V_0(\exp(-2ax) - 2\exp(-ax)), \quad V_0 > 0, \quad -\infty < x < \infty \tag{28}$$

is now solved by relating its Green's function to the radial Green's function of a 3-d oscillator at fixed energy. Consider the equation for the Morse Green's function g^{MOR}

$$\left[\frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + E - V_0(\exp(-2ax) - 2\exp(-ax)) \right] g^{\mathrm{MOR}}(x, x_0; E) = -\delta(x - x_0). \tag{29}$$

By changing the variables to

$$\rho = \exp(-ax/2), \quad g^{\mathrm{MOR}} = \frac{2}{a} (\rho\rho_0)^{-\frac{1}{2}} \tilde{g}(\rho) \tag{30}$$

(29) is transformed to

$$\left[\frac{1}{2}\frac{\partial^2}{\partial\rho^2}+\frac{8V_0}{a^2}-\frac{4V_0}{a^2}\rho^2+\frac{1}{2\rho^2}\left(\frac{8E}{a^2}+\frac{1}{4}\right)\right]\tilde{g}=-\delta(\rho-\rho_0). \tag{31}$$

This shows that $\tilde{g}$ is the radial Green's function at energy $8V_0/a^2$ and angular momentum $\lambda=-1/2+\sqrt{-8E}/a$ of an harmonic oscillator of frequency $\omega=\sqrt{8V_0}/a$, i.e.,

$$\tilde{g}=g_\lambda^{\mathrm{OSC}}\left(\rho,\rho_0;\varepsilon=\frac{8V_0}{a^2},\omega=\frac{\sqrt{8V_0}}{a}\right). \tag{32}$$

By an analysis similar to that of the previous section, the correct eigenvalues and wavefunctions of the Morse problem can be deduced from (32) and (30). In particular we have the result that the number of bound states is finite. The details are not presented here.

4. Rosen-Morse-rigid rotator connection

In this section we solve the 1-d problem of the symmetric Rosen-Morse potential by relating its Green's function to that of the 3-d rigid rotator. For brevity we limit the discussion to the bound states only.

The Green's function for the symmetric Rosen-Morse potential

$$V(x)=-V_0\,\mathrm{sech}^2\alpha x,\quad V_0>0,\quad -\infty<x<\infty \tag{33}$$

satisfies the equation

$$\left[\frac{1}{2}\frac{\partial^2}{\partial x^2}+E+V_0\,\mathrm{sech}^2\alpha x\right]g^{\mathrm{RM}}(x,x_0;E)=-\delta(x-x_0). \tag{34}$$

With the substitutions (taking $E<0$)

$$\rho=\tanh\alpha x,\quad \lambda(\lambda+1)=\frac{2V_0}{\alpha^2},\quad \mu^2=-\frac{2E}{\alpha^2},\quad \mu>0 \tag{35}$$

the equation is transformed to

$$\left[(1-\rho^2)\frac{\partial^2}{\partial\rho^2}-2\rho\frac{\partial}{\partial\rho}+\lambda(\lambda+1)-\frac{\mu^2}{1-\rho^2}\right]\tilde{g}(\rho)=-\delta(\rho-\rho_0) \tag{36}$$

where

$$g^{\mathrm{RM}}(x,x_0;E)=\frac{2}{\alpha}\tilde{g}. \tag{37}$$

Equation (36) shows that $\tilde{g}$ is formally the Green's function for a rigid rotator with an angular momentum λ and 'magnetic quantum number' μ. Hence we identify

$$g^{\mathrm{RM}}(x,x_0;E)=\frac{2}{\alpha}g_\lambda^{\mathrm{ROT}}(\rho,\rho_0;\mu). \tag{38}$$

To obtain an expression for g^{ROT}, consider the homogeneous equation

$$\left[(1-\rho^2)\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}-2\rho\frac{\mathrm{d}}{\mathrm{d}\rho}+\lambda(\lambda+1)-\frac{\mu^2}{1-\rho^2}\right]\Phi(\rho)=0, \quad -1<\rho<1. \tag{39}$$

This equation has solutions which remain finite as $\rho\to\pm 1$ only if λ takes one of the values

$$\lambda=\lambda_n=n+\mu, \quad n=0,1,2,\ldots \tag{40}$$

The corresponding eigenfunctions are the associated Legendre functions:

$$\Phi_n(\rho)=P_{n+\mu}^{-\mu}(\rho). \tag{41}$$

These functions are even (odd) functions of ρ for even (odd) n. For a fixed value of μ, the $P_{n+\mu}^{-\mu}$ form an orthogonal and complete set in $-1<\rho<1$ with respect to well-behaved funotions which vanish at $\rho=\pm 1$. We can therefore write for g^{ROT} an eigenfunction expansion:

$$g_\lambda^{\mathrm{ROT}}(\rho,\rho_0;\mu)=\sum_{n=0}^{\infty}\frac{C_{n,\mu}^2P_{n+\mu}^{-\mu}(\rho)P_{n+\mu}^{-\mu}(\rho_0)}{(n+\mu)(n+\mu+1)-\lambda(\lambda+1)-i\delta}. \tag{42}$$

In (42) $C_{n,\mu}$ is a normalization constant. Using the relation

$$\int_{-1}^{+1}\mathrm{d}\rho\,[P_{n+\mu}^{-\mu}(\rho)]^2=\frac{2}{2n+2\mu+1}\frac{\Gamma(n+1)}{\Gamma(n+2\mu+1)} \tag{43}$$

obtained by analytic continuation of the corresponding result for the P_l^m, we get

$$C_{n,\mu}=\left[\frac{(2n+2\mu+1)\Gamma(n+2\mu+1)}{2\Gamma(n+1)}\right]^{\frac{1}{2}}. \tag{44}$$

Having determined g^{ROT}, we can write down the Green's function for the Rosen-Morse potential using (38). We have

$$g^{\mathrm{RM}}(x,x_0;E)=\frac{2}{\alpha}\sum_{n=0}^{\infty}\frac{C_{n,\mu}^2P_{n+\mu}^{-\mu}(\rho)P_{n+\mu}^{-\mu}(\rho_0)}{(n+\mu)(n+\mu+1)-\dfrac{2V_0}{\alpha^2}-i\delta}. \tag{45}$$

It is apparent from (45) that g^{ROT} has simple poles at μ values satisfying the relation

$$(n+\mu)(n+\mu+1)-\frac{2V_0}{\alpha^2}=0. \tag{46}$$

The positive root of this quadratic equation gives the energy levels E_n through the relation

$$\mu_n=(-2E_n/\alpha^2)^{\frac{1}{2}}=-(n+\tfrac{1}{2})+\tfrac{1}{2}[1+(8V_0/\alpha^2)]^{1/2}. \tag{47}$$

Since the rhs of (47) must be positive, we see that the number of bound states is finite and equal to the integer part of $-\frac{1}{2}+\frac{1}{2}(1+(8V_0/\alpha^2)]^{1/2}$.

To find the wave function of the n-th eigenstate, we consider the limit

$$\lim_{E \to E_n} [(E_n - E) g^{\mathrm{RM}}(x, x_0; E)]. \tag{48}$$

Using (45) and expanding the denominator in the neighbourhood of $E = E_n$, we find the limit (48) to be

$$\left[\frac{2\alpha\mu C_{n,\mu}^2 P_{n+\mu}^{-\mu}(\tanh \alpha x) P_{n+\mu}^{-\mu}(\tanh \alpha x_0)}{2n + 2\mu + 1} \right]_{E = E_n} \tag{49}$$

From this we obtain the wavefunction

$$\psi_n(x) = \left[\frac{\alpha\mu_n \Gamma(n + 2\mu_n + 1)}{\Gamma(n+1)} \right]^{\frac{1}{2}} P_{n+\mu_n}^{-\mu_n}(\tanh \alpha x). \tag{50}$$

Using the formula

$$\int_0^1 [P_{n+\mu}^{-\mu}(\rho)]^2 \frac{\mathrm{d}\rho}{1 - \rho^2} = \frac{\Gamma(n+1)}{2\mu \Gamma(n + 2\mu + 1)} \tag{51}$$

it may be verified that ψ_n given by (50) is correctly normalized.

The determination of the Rosen-Morse bound state normalization constant has been the subject of a detailed investigation by Nieto (1978). It is a noteworthy feature of the present analysis that this constant comes out in a direct manner.

5. Discussion

We have shown above how a general relation connecting the quantum mechanical Green's functions for two different potentials can be derived in a simple manner. Our analysis is based on the Schrödinger equation, and the main result is expressed in (10). We point out that in our approach there is no need to make any transformation of the time variable. This is in sharp contrast to the method of Steiner who derived (10) in a different context by making use of a path-dependent time tranformation. Since it has a starting point different from that of Steiner; our work may be considered as providing independent justification to his technique of transforming space and time variables within the path integral. As for the application of (10) to the Coulomb potential, our treatment goes farther than that of Steiner, who dealt with only the bound states. We obtain the complete set of Coulomb eigenfunctions, including the continuum states.

The relation (10) is exact, and provides a closed form expression for the Green's function for one potential, if the other related one is known in closed form. It is however not necessary to know the closed form expressions to obtain physically important quantities, as shown above.

The specific potentials that we have considered though non-trivial are nevertheless exactly solvable, and serve to illustrate the use of the relation (10). To go beyond these solvable cases is the next natural step. That the method has a wider region of applicability is shown by examples such as the following. The linear plus Coulomb potential, a nonsolvable potential of considerable interest in charmonium physics, can be related through (10) to a quartic anharmonic oscillator potential. As there exists a large body of accurate, though nonexact, results for the latter system, one is naturally led to inquire how best this information can be exploited, in conjunction with (10), in the study of the former. Investigation of this and other related issues is in progress.

Appendix

The inverted oscillator

We consider here the radial eigenvalue problem for the potential

$$V(\rho) = -\tfrac{1}{2}\beta^2\rho^2, \quad \beta > 0 \tag{A1}$$

which corresponds to an 'oscillator' whose frequency is imaginary: $\omega = i\beta$. The radial Schrödinger equation for a given angular momentum λ is

$$\left[\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \varepsilon + \frac{1}{2}\beta^2\rho^2 - \frac{\lambda(\lambda+1)}{2\rho^2}\right]u = 0. \tag{A2}$$

In the above u is the reduced radial function and ε is the energy. By a standard procedure we arrive at the following explicit form for the solution

$$u = N(\beta\rho^2)^{\frac{1}{2}(\lambda+1)}\exp(-\tfrac{1}{2}i\beta\rho^2)F\left(\frac{i\varepsilon}{2\beta} + \frac{\lambda}{2} + \frac{3}{4}, \lambda + \frac{3}{2}, i\beta\rho^2\right) \tag{A3}$$

where F is the confluent hypergeometric function, and N is a constant. The above solution is finite everywhere in $0 < \rho < \infty$ for arbitrary ε and vanishes at $\rho = 0$ provided $\lambda + 1 > 0$. It is however not square-integrable, as is clear from its asymptotic form for $\rho \to \infty$:

$$\begin{aligned} &u \to NC(\beta\rho^2)^{-\frac{1}{4}}\sin\left[\frac{1}{2}\beta\rho^2 + \frac{\varepsilon}{2\beta}\ln\beta\rho^2 + \phi\right], \\ &C = 2\Gamma(\lambda + \tfrac{3}{2})\exp(-\pi\varepsilon/4\beta)\left|\Gamma\left(\frac{i\varepsilon}{2\beta} + \frac{\lambda}{2} + \frac{3}{4}\right)\right|^{-1}, \\ &\phi = \frac{\pi}{4}\left(\frac{1}{2} - \lambda\right) - \arg\Gamma\left(\frac{i\varepsilon}{2\beta} + \frac{\lambda}{2} + \frac{3}{4}\right). \end{aligned} \tag{A4}$$

We thus see that the spectrum of this oscillator is the entire continuum $-\infty < \varepsilon < \infty$, there being no bound states.

The set of eigenfunctions u given by (A3) is a complete orthogonal set. To fix the overall normalization factor N in (A3) we proceed as follows. Let u_1 and u_2 be two solutions corresponding to energies ε_1 and ε_2. Using (A2) for u_1 and u_2, with λ fixed, we get

$$u_1^* u_2'' - u_1^{*\prime\prime} u_2 + 2(\varepsilon_2 - \varepsilon_1)u_1^* u_2 = 0. \tag{A5}$$

Integrating this and using the fact that $u(0) = 0$, we have

$$\int_0^\infty u_1^* u_2 \, d\rho = \frac{1}{2(\varepsilon_2 - \varepsilon_1)} [u_1^{*\prime} u_2 - u_1^* u_2']_{\rho \to \infty}. \tag{A6}$$

To evaluate the rhs of (A6), we substitute the asymptotic expression (A4) for u and keep only terms which survive in the limit $\rho \to \infty$. A short calculation yields the result

$$\int_0^\infty u_1^* u_2 \, d\rho = \tfrac{1}{2}(N_1 C_1)^* N_2 C_2 \sqrt{\beta} \lim_{\rho \to \infty} \left\{ \frac{\sin[(\varepsilon_2 - \varepsilon_1) \ln \beta\rho^2/2\beta]}{\varepsilon_2 - \varepsilon_1} \right\}$$

$$= \tfrac{1}{2}(N_1 C_1)^* N_2 C_2 \sqrt{\beta}\, \pi \delta(\varepsilon_2 - \varepsilon_1). \tag{A7}$$

This not only proves the orthogonality of the eigenfunctions, but also shows that the choice

$$N = \sqrt{\frac{2}{\pi}} C^{-1} \beta^{-\frac{1}{4}}$$

in (A3) gives an eigenfunction which is δ-normalized. We adopt this normalization and denote the resulting function $u_\lambda(\rho, \varepsilon)$. These functions satisfy the relation

$$\int_0^\infty u_\lambda^*(\rho, \varepsilon) u_\lambda(\rho, \varepsilon') \, d\rho = \delta(\varepsilon - \varepsilon'). \tag{A8}$$

Explicitly the constant N has the value

$$N = \frac{\beta^{-\frac{1}{4}} \exp(\pi\varepsilon/4\beta)}{\sqrt{2\pi}\, \Gamma(\lambda + \frac{3}{2})} \left| \Gamma\left(\frac{i\varepsilon}{2\beta} + \frac{\lambda}{2} + \frac{3}{4} \right) \right|. \tag{A9}$$

The completeness relation for the functions u_λ reads

$$\int_{-\infty}^{+\infty} u_\lambda(\rho, \varepsilon) u_\lambda^*(\rho', \varepsilon) \, d\varepsilon = \delta(\rho - \rho'). \tag{A10}$$

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