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Transition from order to chaos in SU(2) Yang–Mills–Higgs system

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Abstract. Time-dependent spherically symmetric $SU(2)$ Yang–Mills–Higgs system is shown to be chaotic near the 't Hooft–Polyakov monopole solution by calculating the maximal Lyapunov exponents. A phase transition like behaviour from order to chaos is observed as a parameter depending on the self interaction constant of scalar fields increases.

Keywords. Chaos; Yang–Mills–Higgs system; monopoles; Lyapunov exponents.

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Classical Yang–Mills theories are intrinsically non-linear and in general chaotic. Yang–Mills classical mechanics (YMCM) where only spatially homogeneous fields are considered, is highly chaotic and has been proved to be a K-system (Matinyan *et al* 1981a; Savvidy 1984). Recently Matinyan *et al* (1986, 1988) studied the more general space-time dependent Yang–Mills system and found that it is also chaotic. Chaotic behaviour of spatially homogeneous Yang–Mills system with Higgs scalar fields (YMHCM) was studied by Matinyan *et al* (1981b). They observed a phase transition like behaviour from order to chaos in YMHCM as the vacuum expectation value of Higgs field is changed. Recently Matinyan *et al* (1989) studied the time-dependent spherically symmetric $SU(2)$ Yang–Mills–Higgs system (SSYMH) and showed that there can be chaos. Using the technique of Painlevè analysis we (Joy and Sabir 1989) have recently shown that time-dependent spherically symmetric $SU(2)$ Yang–Mills and Yang–Mills–Higgs systems are non-integrable. Whether there is an order to chaos transition in SSYMH similar to YMHCM is still an open question. Here we present the results of a numerical study on the chaotic behaviour of SSYMH system. We consider specifically the 't Hooft–Polyakov monopole solution and find a phase-transition like behaviour from order to chaos as we tune the parameter which depends on the self interaction constant of scalar fields. We studied the chaos in the system by calculating the maximal Lyapunov exponents (LE).

We consider the Georgi–Glashow model with gauge group $SU(2)$ broken down to $U(1)$ by Higgs triplets. Lagrangian of the model is,

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - V(\phi) \tag{1}$$

where,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon_{\rm abc}A^b_\mu A^c_\nu$$

$$D_\mu\phi_a = \partial_\mu\phi_a + g\varepsilon_{\rm abc}A^b_\mu\phi_c$$

and

$$V(\phi)=\frac{\lambda}{4}\left(\phi^2-\frac{m^2}{\lambda}\right)^2.$$

The vacuum expectation values of the scalar field and Higgs boson mass are $\langle\phi^2\rangle=F^2=m^2/\lambda$ and $M_H=\sqrt{2\lambda}F$ respectively. Mass of the gauge boson is $M_W=gF$. We use the time-dependent 't Hooft–Polyakov ansatz (Mecklenberg and O'Brien 1978),

$$A_0^a=0,\quad A_i^a=-\varepsilon_{ain}r_n\frac{[1-K(r,t)]}{r^2} \tag{2}$$

$$\phi_a=\frac{1}{g}r_a\frac{H(r,t)}{r^2}$$

where $r_n=x_n$ and r is the radial variable. With $\beta=\lambda/g^2=M_H^2/2M_W^2$, and introducting the variables $\xi=M_W r$ and $\tau=M_W t$, the equations of motion become,

$$(\partial_\xi^2-\partial_\tau^2)K=K(K^2+H^2-1)/\xi^2$$
$$(\partial_\xi^2-\partial_\tau^2)H=H(2K^2+\beta(H^2-\xi^2))/\xi^2. \tag{3}$$

Total energy of the system E is given by

$$C(\beta)=\frac{g^2E}{4\pi M_W}=\int_0^\infty\left\{K_\tau^2+\frac{H_\tau^2}{2}+K_\xi^2+\frac{1}{2}\left(H_\xi-\frac{H}{\xi}\right)^2\right.$$
$$\left.+\frac{1}{2\xi^2}(K^2-1)^2+\frac{K^2H^2}{\xi^2}+\frac{\beta}{4\xi^2}(H^2-\xi^2)^2\right\}\mathrm{d}\xi. \tag{4}$$

Time-independent version of the ansatz (2) gives the 't Hooft–Polyakov monopole solution with winding number 1. 't Hooft–Polyakov monopole is nonsingular and has finite energy. In the limit $\beta\to 0$ known as the Prasad–Sommerfeld (PS) limit, we have exact static solutions (Prasad and Sommerfeld 1975).

For our study we discretize the system (3) to obtain a set of coupled anharmonic oscillators. Discretization is done by taking the second derivative in space as follows,

$$\partial_\xi^2K(i,t)=\frac{K(i+1,t)-2K(i,t)+K(i-1,t)}{h^2},$$

$$\partial_\xi^2H(i,t)=\frac{H(i+1,t)-2H(i,t)+H(i-1,t)}{h^2}, \tag{5}$$

where $i=1,\ldots,N-1$ and h is the discretization step. N is the number of oscillators. For convenience we use t instead of τ from here onwards. For studying chaos we calculate the maximal LE. Lyapunov exponent (LE) is the average rate of exponential divergence of nearby trajectories. If the maximal LE is greater than zero the system is said to be chaotic; there is sensitive dependence on initial conditions. Lyapunov exponent is defined as

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{y(t)}{y(0)}, \tag{6}$$

where $y(t)$ is the distance between two nearby solutions at time t (Contopoulos *et al* 1978). If one chooses the initial variation $y(0)$ at random, one obtains maximal LE. For calculating LE we have to solve the system (3) along with the variational system which is obtained by discretizing the following equations

$$(\partial_\xi^2 - \partial_\tau^2)\delta K = \frac{(3K^2 + H^2 - 1)\delta K + 2HK\delta H}{\xi^2}$$

$$(\partial_\xi^2 - \partial_\tau^2)\delta H = \frac{(2K^2 + 3\beta H^2 - \beta\xi^2)\delta H + 4KH\delta K}{\xi^2}. \tag{7}$$

In the system, there exist two parameters, the energy and the value of β. Since we are interested in the evolution of 't Hooft–Polyakov monopole solutions we take $K(i,0)$ and $H(i,0)$ in such a way that the energy functional is a minimum for a fixed β. Static monopole solutions occur at the minimum of energy functional $C(\beta)$. Static solutions $K(i,0)$ and $H(i,0)$ are found using a finite difference method for solving boundary value problems. We use the asymptotic form of the solutions for fixing the boundary values. We use fixed boundary conditions and numerically solve the system (3) with static solutions as initial conditions along with the discretized system obtained from (7). For our calculations we take $N = 100$ and the discretization step $h = 0{\cdot}1$. We calculate up to $t = 1000$ which is sufficient for obtaining asymptotic values of LE. Calculations are done in CYBER 180/830 computer with high accuracy. Lyapunov exponents for $\beta = 0, 0{\cdot}1, 0{\cdot}5, 1, 2, 5, 10, 50, 75, 100, 200, 500, 1000$ and 5000 are calculated. For all β values less than 75, LE is zero within the numerical accuracy. For $\beta = 75$ LE becomes positive and reaches an asymptotic value $1{\cdot}54 \times 10^{-3}$. For higher β values we get higher and higher positive LEs. However LE is not seen increasing indefinitely with β. In figure 1, $\log(\lambda)$ versus $\log(\beta)$ is plotted. Transition occurs near $\beta = 75$. At the transition region the increase is rapid but as β increases further the rate of increase in LE falls. For $\beta = 5000$ LE is $0{\cdot}1$. We have repeated the calculations with $N = 16, 32$ and 64 also. Results are qualitatively the same as that of $N = 100$. Details of the calculations will be published elsewhere.

Our calculations show that there is a phase-transition-like behaviour from order to chaos in SSYMH system. This result is in agreement with that obtained for YMHCM, where Higgs field manifests only as the vacuum expectation value F and there are no terms dependent on the self interaction constant. Here we consider the time evolution of both gauge and scalar fields and there exist two parameters $C(\beta)$ and β. β depends on the self interaction constant λ. Since we are interested in monopole solutions we took the minimum value of energy functional $C(\beta)$ for a specific β value. It is known that as β increases the effect of Higgs field decreases and when $\beta \to \infty$ system becomes purely Yang–Mills, which is highly chaotic. The effect of Higgs scalar fields is to reduce the stochasticity of the YM system. From our study one can see that 't Hooft–Polyakov monopole solutions show irregular behaviour in time, and they are exponentially unstable.

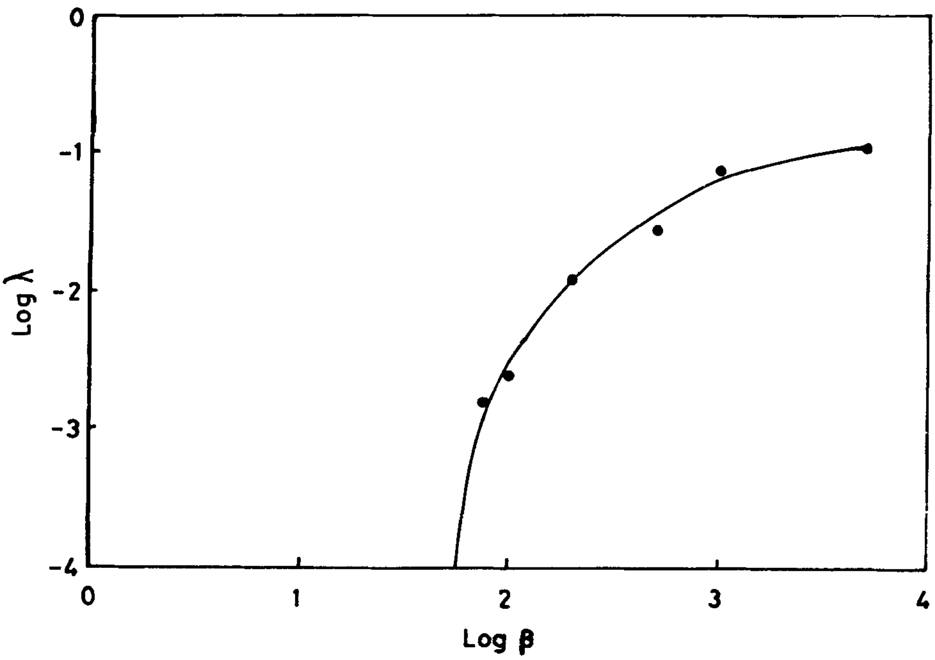

Figure 1. Log of maximal LE (λ) vs $\log(\beta)$.

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