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The radial force on a charged particle in superimposed magnetic fields on Schwarzschild spacetime

A R PRASANNA and SAI IYER

Physical Research Laboratory, Navrangpura, Ahmedabad 380009, India

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Abstract. Following the approach of optical reference geometry we derive the expression for the total force in the radial direction acting on a charged particle in magnetic fields superimposed on the static Schwarzschild background and show the possible existence of bound orbits for particles in the field of ultra compact objects at distances $r \leqslant 3m$ wherein the Lorentz force counterbalances both the gravitational and centrifugal forces.

Keywords. Centrifugal force; magnetic fields; general relativity; neutron stars.

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1. Introduction

The study of the trajectories of particles in a given spacetime depicts the dynamics of the associated physical system that is in equilibrium under different force fields. It is indeed well known that in most astrophysical scenarios one has electromagnetic processes taking place under the influence of strong gravitational fields and thus has equilibrium of configurations under gravitational, inertial and electromagnetic forces. As strong gravitational fields are described by general relativity in terms of spacetime structure, there have been several studies of charged particle motion in terms of orbital dynamics in the associated spacetime. However, in general relativity, as one does not distinguish the terms in the 4-force expression in terms of individual forces, it is difficult to appreciate the role of different parameters while discussing the dynamics of particles. In order to realize whether there exist circular orbits for charged particles in a certain configuration of interacting electromagnetic and gravitational fields, one will have to first consider the effective potential to see the possible limits on energy and angular momentum for having a circular orbit. On the other hand, if it is possible to ascertain the 'radial force' acting on a particle, one can directly get the sequence of all possible 'circular orbits' by varying the free parameters and looking for the zeros of the 'radial force equation'. This procedure would be of particular use while considering 'stable orbits' for particles close to compact and ultra-compact (Iyer *et al* 1983) objects. Motivated by this consideration, one looks for the description of the total force acting on a particle in the 3-space in the sense of Newtonian physics. This has become possible through the analysis of particle orbits in what is referred to as the optical reference geometry (Abramowicz *et al* 1988). As depicted by Abramowicz *et al*, one can consider a 3 + 1 splitting of the 4-space, if it is stationary, by using the time-like Killing vector and an appropriate conformal 3-space such that the geodesic equation of the 4-space when projected onto the 3-space shows a natural splitting of the total force into gravitational and inertial forces. The first interesting and important application of this analysis was that of Abramowicz and Prasanna (1990) who illustrated that in the static Schwarzschild spacetime the force equation splits in such a way that the *centrifugal part* vanishes at $r = 3m$, the location of the photon circular orbit. The reversal of the centrifugal force would naturally mean that there are no bound orbits in the region $r < 3m$ as both gravitational and centrifugal forces act inwards for $r < 3m$. This important result appears in a most natural way when one sees the expression for the radial force as given by ORG, while in the description of geodesics in 4-space it is not at all apparent.

Consequent to the reversal of centrifugal force across $r = 3m$, freely falling particles in Schwarzschild space-time will have only plunge orbits once they cross $r = 3m$. However, it is possible that the presence of other fields besides the gravitational field could give rise to circular orbits even for $r < 3m$. Abramowicz and Bicak (1990) have considered the role of an electric field in this context by analysing the circular orbits in the Reissner–Nordstrøm geometry and shown that ultra-relativistic charged particles below the photon orbit should be of the same sign as that of the source while those just outside the photon orbit should be of opposite sign.

In order to consider the role of magnetic fields in this context, Prasanna (1991) analysed the centrifugal force reversal effect in the Ernst space-time, which represents a mass embedded in a uniform magnetic field whose geometry corresponds to that of a Schwarzschild space-time embedded in Melvin's magnetic universe. As this space-time has two circular photon orbits, Prasanna found that the centrifugal force reversal occurs twice, once close to $r = 3m$ and the other at a distance very far away depending essentially on the parameter Bm, the product of the magnetic field strength and mass in geometrised units. Also, this example bears out the result that the centrifugal force acts away from the maximum of the photon effective potential and acts towards the minimum of the photon effective potential.

However, it is more natural to assume the existence of compact bodies with magnetic fields anchored to them (like neutron stars or ultra-compact bodies). In the region outside such bodies, charged particles would be subjected, apart from gravitational and inertial forces, to the Lorentz force too. The existence of the Lorentz force could indeed influence the total force such that for a certain range of physical parameters, even in the case of ultra-compact bodies, charged particles could have stable orbits for $r < 3m$ if the Lorentz force is strong enough to counterbalance the gravitational and centrifugal forces. The ideal way to look for such a possibility is through the analysis of total force in terms of ORG elements including the Lorentz force.

2. Radial force equation

With this background, we would now like to consider the effects in the presence of an external magnetic field (test field superimposed on the Schwarzschild background) wherein again the reversal would be only at $r = 3m$. Our interest at present is to consider the possible circular orbits for charged particles in the presence of a superimposed magnetic field that is either dipolar or uniform at infinity.

The force acting on a particle in a stationary space-time as expressed in the optical reference geometry (Abramowicz *et al* 1988) is given by

$$m_0 f_0 = -P^i \partial_i \mathscr{E}, \tag{1}$$

$$m_0 \Phi (f_i - 2\alpha_i f_0) = p^j \tilde{\nabla}_j p_i + \tfrac{1}{2} m_0^2 \partial_i \Phi + 2\mathscr{E} p^j \omega_{ij}, \tag{2}$$

wherein i running from 1 to 3 represents the spatial variables, p_i the spatial 3-momentum and $\tilde{\nabla}_j$ the covariant derivative in the 3-space obtained through the conformal splitting of the 4-space as depicted by

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \Phi[\tilde{g}_{ij} dx^i dx^j - (dt + \alpha_i dx^i)^2]. \tag{3}$$

In terms of Newtonian forces the three terms on the r.h.s of (2) represent the centrifugal force, the gravitational force and the Coriolis force in the absolute 3-space.

Restricted to a static space-time ($\alpha_i = 0$), the force equation reduces to (Abramowicz and Prasanna 1990)

$$m_0 \Phi f_i = p^j \tilde{\nabla}_j p_i + \tfrac{1}{2} m_0^2 \partial_i \Phi. \tag{4}$$

The force acting on a charged particle in a magnetic field superimposed on the given space-time is

$$f_\alpha = e F_{\alpha\beta} P^\beta, \tag{5}$$

which, in the absence of an electric field, reduces to the 3-force

$$f_i = e F_{ik} P^k. \tag{6}$$

The spatial part P^k of the 4-momentum P^α is related to the 3-momentum p^k through the relation

$$p^k = \Phi P^k. \tag{7}$$

Using (6) and (7) in (4) one can get the force balance equation

$$m_0 \Phi e F_{ik} P^k = \Phi P^j (\partial_j p_i - \tfrac{1}{2} p^k \partial_i \tilde{g}_{jk}) + \tfrac{1}{2} m_0^2 \partial_i \Phi, \tag{8}$$

which may be written explicitly as

$$e F_{ik} U^k = U^j \partial_j (\tilde{g}_{ik} \Phi U^k) - \tfrac{1}{2} \Phi U^j U^k \partial_i \tilde{g}_{jk} + \tfrac{1}{2} \Phi^{-1} \partial_i \Phi, \tag{9}$$

U^k being the spatial part of the 4-velocity U^α.

We consider the Schwarzschild background where

$$\Phi = (1 - 2m/r);\ \tilde{g}_{rr} = (1 - 2m/r)^{-2},$$
$$\tilde{g}_{\theta\theta} = r^2 (1 - 2m/r)^{-1};\ \tilde{g}_{\phi\phi} = r^2 \sin^2\theta (1 - 2m/r)^{-1}, \tag{10}$$

and look at the force acting on the particle in two different cases of superimposed magnetic fields:

(1) magnetic field dipolar at infinity (Prasanna and Varma 1977)

$$F_{\phi r} = \frac{3\mu \sin^2\theta}{4m^2} \left[\left(1 - \frac{2m}{r}\right)^{-1} + \frac{r}{m} \ln\left(1 - \frac{2m}{r}\right) + 1 \right], \tag{11}$$

$$F_{\phi\theta}=\frac{3\mu\sin\theta\cos\theta}{4m^3}\left[r^2\ln\left(1-\frac{2m}{r}\right)+2mr\left(1+\frac{m}{r}\right)\right], \tag{12}$$

and
(2) magnetic field uniform at infinity (Wald 1974)

$$F_{\phi r}=-B_0 r, \tag{13}$$

$$F_{\phi\theta}=-B_0 r^2\sin\theta\cos\theta. \tag{14}$$

Restricting the analysis to the case of a particle confined to the equatorial plane ($\theta=\frac{1}{2}\pi$, $U^r=0$, $U^\theta=0$), eq. (9) reduces to

$$eF_{r\phi}U^\phi=-\tfrac{1}{2}(1-2m/r)(U^\phi)^2\partial_r\tilde{g}_{\phi\phi}+\frac{m}{r^2}(1-2m/r)^{-1}, \tag{15}$$

with U^ϕ expressed in terms of the canonical angular momentum l and the vector potential A_i as

$$U_\phi+eA_\phi=l. \tag{16}$$

Using the appropriate expression for A_ϕ one gets the total force acting on the particle in the radial direction to be

$$F_r=-\frac{1}{\rho^2}+(U^\phi)^2(\rho-3)-\frac{3\lambda U^\phi}{4}[(\rho-2)\ln(1-2/\rho)+2(1-1/\rho)], \tag{17}$$

with

$$U^\phi=\frac{L}{\rho^2}+\frac{3\lambda}{8}\left[\ln(1-2/\rho)+\frac{2}{\rho}(1+1/\rho)\right],$$

$$\rho=r/m,\quad L=l/m,\quad \lambda=e\mu/(m_0c^2m^2) \tag{18}$$

for the dipolar field and

$$F_r=-\frac{1}{\rho^2}+(U^\phi)^2(\rho-3)+\lambda U^\phi(\rho-2), \tag{19}$$

with

$$U^\phi=\frac{L}{\rho^2}-\frac{\lambda}{2},\quad \lambda=eB_0m \tag{20}$$

for the uniform field. The expression for F_r has clearly the three terms corresponding to the gravitational force, the centrifugal force and the Lorentz force, and $F_r=0$ corresponds to the case of a circular orbit for the particle.

3. Discussion

As it is a conservative system, the force represents the derivative of the potential and thus the zeros of F correspond to the extrema of the effective potential V_{eff} which is given by

$$V^2=(1-2/\rho)[1+\rho^2(U^\phi)^2]. \tag{21}$$

Table 1. Zeros of F_r for the dipole field.

λ	L	Zeros			λ	L	Zeros		
	17·6954	2·0563	3·1157	6·1084		17·6954	2·0840	7·0465	15·3679
	21·2344	2·0505	2·8783	5·5162		21·2344	2·0814	6·1709	12·8306
	24·7735	2·0454	2·7098	5·1148		24·7735	2·0790	5·5315	11·2095
	31·5458	2·0371	2·4970	4·6191		31·5458	2·0744	4·6961	9·2742
27·5	34·4136	2·0341	2·4338	4·4723	100·0	34·4136	2·0726	4·4399	8·7090
	41·1247	2·0281	2·3235	4·2132		41·1247	2·0685	3·9795	7·7159
	70·7816	2·0124	2·1122	3·6772		70·7816	2·0532	2·9989	5·6522
	81·2964	2·0094	2·0812	3·5844		81·2964	2·0487	2·8271	5·2905
	99·5468	2·0058	2·0478	3·4716		99·5468	2·0419	2·6191	4·8481
	17·6954	2·0720	4·3298	8·8587		−17·6954	2·1728		
	21·2344	2·0678	3·8874	7·7241		−21·2344	2·1925		
	24·7735	2·0638	3·5687	6·9672		−24·7735	2·2136		
	31·5458	2·0568	3·1579	6·0398		−31·5458	2·2573		
50·0	34·4136	2·0541	3·0334	5·7656	27·5	−34·4136	2·2767		
	41·1247	2·0484	2·8119	5·2817		−41·1247	2·3229		
	70·7816	2·0299	2·3560	4·2754		−70·7816	2·5027		
	81·2964	2·0253	2·2804	4·1001		−81·2964	2·5508		
	99·5468	2·0191	2·1923	3·8865		−99·5468	2·6175		

Table 2. Zeros of F_r for the uniform field.

λ	L	Zeros		λ	L	Zeros	
	10	3·0041	13·5411		10	2·5763	4·3783
	20	2·9788	19·7196		20	2·7089	6·2877
0·1	30	2·9807	24·3126	1	30	2·7805	7·7234
	50	2·9859	31·5158		50	2·8528	9·9875
	100	2·9920	44·6690		100	2·9191	14·1363

Tables 1 and 2 give the locations of the zeros of F_r for the case of a dipolar and a uniform magnetic field, respectively. When compared with the locations of the extrema of V_{eff} given in Prasanna and Varma (1977), the matching with the locations of the corresponding zeros for various values of L and λ, given in table (1), are indeed perfect. For $L > 0$ the three zeros correspond to the two maxima and one minimum close to the compact object, and for $L < 0$ the single zero corresponds to the maximum. As depicted in figures 1–3, the force is attractive very close to $\rho = 2$, then becomes repulsive and remains so till the minimum of V_{eff}, and then again is attractive till it reverses sign at the second maximum of V_{eff}. The zone between the first and third zeros of F_r corresponds to the potential well wherein a particle can have gyrating orbits confined to the potential well.

In the absence of a magnetic field ($\lambda = 0$), both (17) and (19) reduce to the same expression which for $\rho \leqslant 3$ is always attractive, and thus cannot give any bound orbit for the particle, reflecting the non-availability of bound orbits behind the photon orbit in the Schwarzschild space-time. In the presence of a magnetic field the situation changes for a charged particle, as the Lorentz force can counterbalance both the gravitational and centrifugal forces for $\rho \leqslant 3$, thus allowing both gyrating and circular orbits depending upon the energy, angular momentum and the magnetic field strength.

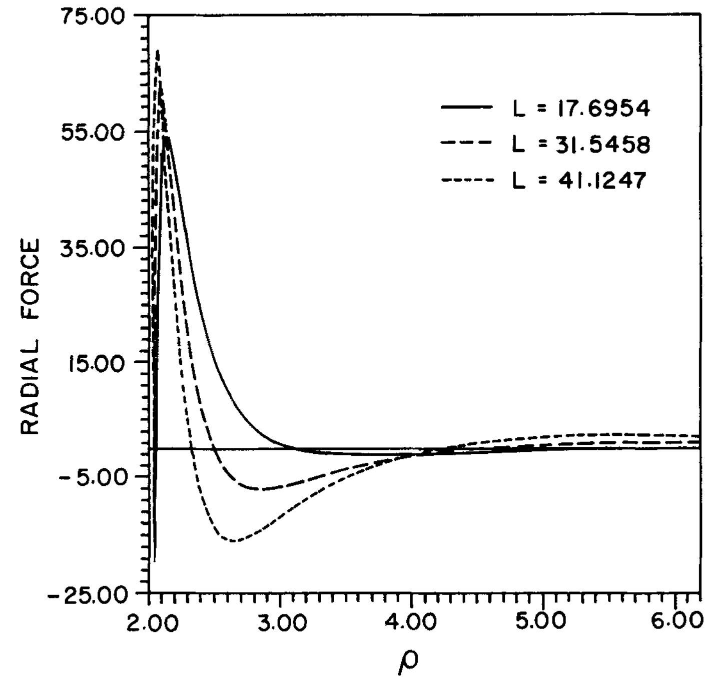
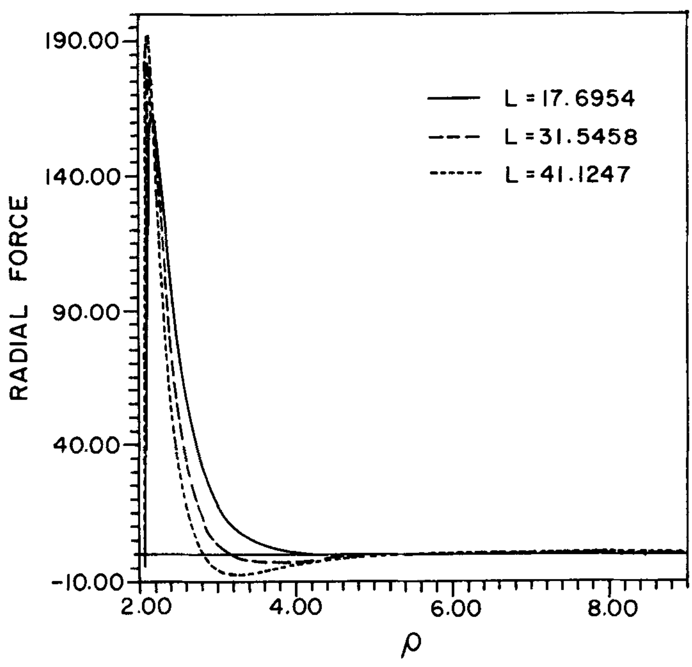


Figure 1.



Figures 1 and 2. F_r as a function of ρ, depicting the three zeros for $\lambda = 27{\cdot}5$ (1) and $\lambda = 50$ (2) and various values of L for the dipole field.

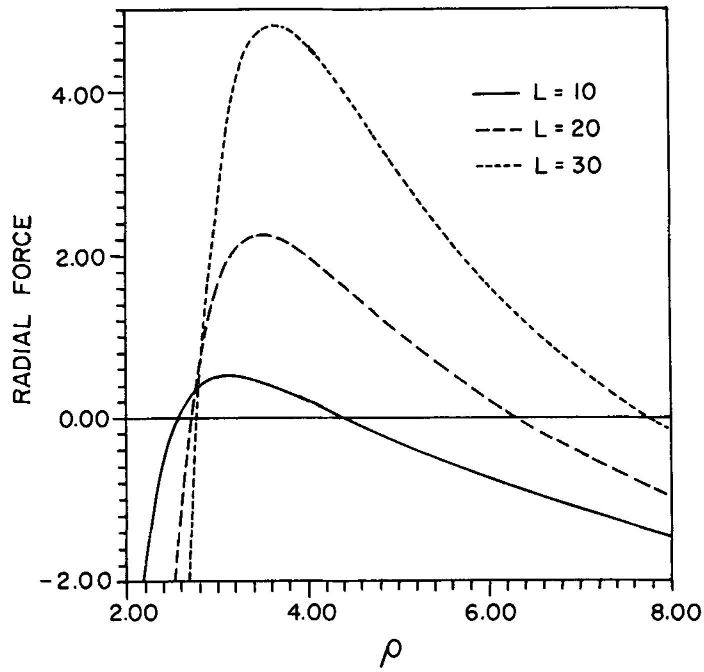


Figure 3. F_r as a function of ρ depicting the two zeros for $\lambda = 1$ and different L values for the uniform field.

For large ρ the centrifugal force is always repulsive, and for the total force one finds that

$$F_r \approx -\frac{1}{\rho^2} + (\rho - 3)(U^\phi)^2 - \frac{\lambda U^\phi}{\rho^3}(\rho + 1),$$
$$U^\phi \approx \frac{L}{\rho^2} - \frac{\lambda}{\rho^3}\left(1 + \frac{3}{2\rho}\right) \tag{22}$$

for the dipolar field, whereas

$$F_r \approx -\frac{1}{\rho^2} + \rho U^\phi(U^\phi + \lambda) \approx -\frac{1}{\rho^2} + \frac{L^2}{\rho^3} - \frac{\lambda^2 \rho}{4} \tag{23}$$

for the uniform field. Thus, in both cases, at distances farther than $\rho = 3$ bound orbits may exist only for sufficiently large values of L.

On the other hand, let us take the case of a charged particle falling in radially at infinity (i.e., $L = 0$). Then we find that the force F_r is given by

$$F_r = -\frac{1}{\rho^2} + \frac{9\lambda^2}{64}\left[\ln\left(1 - \frac{2}{\rho}\right) + \frac{2}{\rho}\left(1 + \frac{1}{\rho}\right)\right]$$
$$\times \left[(1 - \rho)\ln\left(1 - \frac{2}{\rho}\right) - 2\left(1 + \frac{3}{\rho^2}\right)\right], \tag{24}$$

which in principle has two zeros in the case of the dipole field, one between $r = 2m$

Table 3. Zeros of F_r for the dipole field when $L = 0$.

λ	Zeros	
10	2·0997	6·2955
10^3	2·0981	126
10^4	2·0981	585
10^6	2·0981	12600

and $3m$ and the other sufficiently far away depending upon the value of λ. The locations of the zeros for different values of λ are given in table 3. In the case of the uniform field, when $L = 0$ there is no root of $F = 0$ beyond $\rho = 1$ and thus one cannot have any circular orbit for the radially falling particle. Thus when one looks at the dynamics of charged particles close to a magnetized compact object, it is only in the case of an ultra-compact object that one can have circular orbits, irrespective of the fact whether $L = 0$ or not. *This has been rendered possible only due to the centrifugal force reversal at $r = 3m$.*

4. Conclusion

Considering the fields close to the compact object, one knows that the gravitational potential being large, matter would be heated and hence be in an ionized rather than neutral form. The motion of ionized matter produces currents which in turn could produce magnetic fields. Thus it is more significant to consider the motion of charged particles, which, as seen above, can have circular or gyrating orbits for $\rho \leqslant 3$. On the other hand, as the centrifugal force reverses sign at $\rho = 3$, neutral particles cannot have bound orbits at distances $r \leqslant 3m$.

In the case of a normal neutron star ($R > 3m$), the force acting on a charged particle would be given by (22). As λ would be very large ($\lambda \approx 10^{-4} B$), bound orbits can exist only if the angular momentum parameter L is sufficiently large such that U^ϕ is positive. On the other hand, for ultra compact objects with a radius R such that $2m < R < 3m$, the radial force is given by (17) or (19) depending upon the nature of the magnetic field, which can yield bound orbits. *Thus there would be an intrinsic difference in the dynamics of charged particles in the vicinity of an ultra compact star as compared to that in the vicinity of a normal neutron star.*

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