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Phase-function method for Hulthén-modified separable potentials[†]

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Abstract. We have treated the Hulthén-modified separable potentials within the framework of the phase-function method and obtained a closed form expression for s-wave scattering phase shift. Specializing to a rank one separable potential we have found out the limiting conditions in which the Hulthén-modified phase shift goes over to its Coulomb counterpart. We demonstrate the usefulness of our approach by means of a model calculation.

Keywords. Nuclear reactions; scattering theory; phase function method; Hulthén-modified separable potential; phase shifts.

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1. Introduction

In the recent past one of us (UL) (Sett *et al* 1988) adapted the phase method (Calogero 1967) to deal with Coulomb plus nonlocal separable potentials and derived a closed form expression for the scattering phase shifts. In this paper we present the results of a similar investigation by using a Hulthén potential in place of the pure Coulomb interaction. The Hulthén potential has often been used as a model for screened and cut-off Coulomb interactions (Lindhard and Winther 1971; Berezin 1972, 1979; Kraeft *et al* 1983). Since the effect of screening should invariably affect the theory and interpretation of data relating to charged hadron scattering, it is expected that the analysis of this report will be of interest to a wide variety of physicists. Incidentally, one may note that pure Coulomb potentials never really occur in nature. For example, in the famous Rutherford experiment the Coulomb field of the gold nucleus was completely shielded at a few angstroms by the atomic electrons.

In § 2 we obtain a closed form expression for the phase shift induced by a Hulthén distorted rank n separable potential and present a case study. We devote § 3 to examine the Coulomb limit of the results of § 2 and present some concluding remarks by analysing the results for scattering phase shifts within the framework of our approach.

2. Hulthén-distorted separable potentials

The Hulthén potential

$$V_H(r) = V_0 \exp(-r/a)/(1 - \exp(-r/a)), \quad a > 0 \tag{1}$$

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behaves like a Coulomb potential at small values of r whereas for large values of r it decreases exponentially. The potential in (1) allows analytical solutions of the Schrödinger equation for s-wave only. Therefore, in the following we shall deal only with the s-wave scattering and omit the subscript $l=0$. We shall work in units in which $\hbar^2/2m$ is unity. The Schrödinger equation for the Hulthén plus rank N separable potential is given by

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-V_0\frac{\exp(-r/a)}{(1-\exp(-r/a))}\right)\phi^{HS}(k,r)$$
$$=\sum_{i=1}^{N}\lambda^{(i)}h^{(i)}(r)\int_0^\infty \mathrm{d}t\,g^{(i)}(t)\phi^{HS}(k,t) \tag{2}$$

where the strength parameter V_0 for the Hulthén potential is real and positive. The coupling constants $\lambda^{(i)}$'s of the separable interaction may either be positive or negative but always real. The quantities $h^{(i)}(r)$ and $g^{(i)}(r)$ stand for the form factors of the separable interaction. Here $\phi^{HS}(k,r)$ represents the wavefunction for scattering on Hulthén plus separable potentials. Henceforth we shall use superscripts H and HS for quantities related to pure Hulthén and Hulthén-distorted separable potentials respectively. For the regular boundary condition (1) may be converted into an integral equation (Newton 1982)

$$\phi^{HS}(k,r)=k\phi^{H}(k,r)+\sum_{i=1}^{N}\lambda^{(i)}\mathrm{d}^{(i)HS}(k)\int_0^r \mathrm{d}r'\,h^{(i)}(r')G^{H(R)}(r,r'), \tag{3}$$

where

$$\mathrm{d}^{(i)HS}(k)=\int_0^\infty \mathrm{d}t\,g^{(i)}(t)\phi^{HS}(k,t). \tag{4}$$

In (3) the regular Green function $G^{H(R)}(r,r')$ for motion in the Hulthén potential is given by

$$\begin{aligned}G^{H(R)}(r,r')&=|f^{H}(k)|^{-2}[\phi^{H}(k,r)\mathrm{Re}\{f^{H}(-k)f^{H}(k,r')\}\\&\quad-\phi^{H}(k,r')\mathrm{Re}\{f^{H}(-k)f^{H}(k,r)\}]\quad\text{for } r'<r\\&=0\qquad\qquad\text{for } r'>r.\end{aligned} \tag{5}$$

Here $\phi^{H}(k,r)$ and $f^{H}(k,r)$ stand for the regular and irregular Hulthén functions (Newton 1982; Hulthén 1942) given by

$$\phi^{H}(k,r)=a\exp(ikr)(1-\exp(-r/a))\,{}_2F_1(1+A,1+B;2;1-\exp(-r/a)) \tag{6}$$

and

$$f^{H}(k,r)=\exp(ikr)\,{}_2F_1(A,B;C;\exp(-r/a)) \tag{7}$$

with

$$A=-iak+ia\rho,\ B=-iak-ia\rho,\ C=1-2iak \text{ and } \rho=(k^2+V_0)^{1/2}. \tag{8}$$

The Jost function corresponding to (7) can be written as (Newton 1982)

$$f^{H}(k)=\frac{\Gamma(C)}{\Gamma(1+A)\Gamma(1+B)}. \tag{9}$$

Also for real k we have $f^H(-k,r) = f^{*H}(k,r)$ and $f^H(-k) = f^{*H}(k)$.

From (3) and (5) we get

$$\phi^{HS}(k,r) = \alpha^{HS}(k,r)[k\phi^H(k,r)\cos\delta^{HS}(k,r) + \mathrm{Re}\{f^H(-k)f^H(k,r)\}\sin\delta^{HS}(k,r)]. \tag{10}$$

In writing (10) we have introduced the phase and amplitude functions $\delta^{HS}(k,r)$ and $\alpha^{HS}(k,r)$ through the equations

$$\alpha^{HS}(k,r)\cos\delta^{HS}(k,r) = 1 + k^{-1}|f^H(k)|^{-2}\sum_{i=1}^{N}\lambda^{(i)}\mathrm{d}^{(i)HS}(k) \times \int_0^r \mathrm{d}r' h^{(i)}(r')\mathrm{Re}\{f^H(-k)f^H(k,r')\} \tag{11}$$

and

$$\alpha^{HS}(k,r)\sin\delta^{HS}(k,r) = -|f^H(k)|^{-2}\sum_{i=1}^{N}\lambda^{(i)}\mathrm{d}^{(i)HS}(k)\int_0^r \mathrm{d}r' h^{(i)}(r')\phi^H(k,r') \tag{12}$$

with $\delta^{HS}(k,0) = 0$ and $\alpha^{HS}(k,0) = 1$. The phase shift $\delta^{HS}(k) = \lim_{r\to\infty}\delta^{HS}(k,r)$. Structures of (10), (11) and (12) clearly indicate that we have not started the calculation of the phase function or phase shift with kinetic energy as zero order Hamiltonian. Instead, following the prescription in Calogero (1967), we started calculating the states, Green function etc. for a model Hamiltonian that involves the Hulthén potential. From (11) and (12) the phase function is obtained in the form

$$\tan\delta^{HS}(k,r) = -\sum_{i=1}^{N}\lambda^{(i)}\mathrm{d}^{(i)HS}(k)\int_0^r \mathrm{d}r' h^{(i)}(r')\phi^H(k,r')\Big[|f^H(k)|^2 + k^{-1} \times \sum_{i=1}^{N}\lambda^{(i)}\mathrm{d}^{(i)HS}(k)\int_0^r \mathrm{d}r' h^{(i)}(r')\mathrm{Re}\{f^H(-k)f^H(k,r')\}\Big]^{-1} \tag{13}$$

The quantities $\mathrm{d}^{(i)HS}(k)$'s in (13) still involve the wavefunction $\phi^{HS}(k,r)$. Thus (13) is not the final expression that can be used to obtain $\tan\delta^{HS}(k,r)$ and/or $\tan\delta^{HS}(k)$ in closed analytic form. However, we note that $\mathrm{d}^{(i)HS}(k)$ can be obtained in terms of the Hulthén functions and form factors of the separable potential since (3) is an integral equation with degenerate kernel. We have found

$$\mathrm{d}^{(i)HS}(k) = \frac{1}{\det_N A^{HS}(k)}\sum_{j=1}^{N} a^{(i,j)HS}(k)\, Y^{(j)HS}(k). \tag{14}$$

The elements of the Fredholm determinant $\det_N A^{HS}(k)$ are

$$A^{(i,j)HS}(k) = \delta_{ij} - \lambda^{(j)}\int_0^\infty\int_0^r \mathrm{d}r\,\mathrm{d}r' g^{(j)}(r)G^{H(R)}(r,r')h^{(i)}(r'). \tag{15}$$

The quantities $a^{(i,j)HS}(k)$ stand for the cofactors of $A^{(i,j)HS}(k)$. We also have

$$Y^{(j)HS}(k) = k\int_0^\infty \mathrm{d}r\,\phi^H(k,r)g^{(j)}(r). \tag{16}$$

Combining all equations from (13)–(16) and letting $r \to \infty$ we obtain $\tan \delta^{HS}(k)$ in the final form

$$\tan \delta^{HS}(k) = \sum_{i,j=1}^{N} \lambda^{(i)} a^{(i,j)HS}(k)\, Y^{(j)HS}(k) X^{(i)HS}(k) \Bigg[k|f^{H}(k)|^2 \det{}_N A^{HS}(k) + \sum_{i,j=1}^{N} \lambda^{(i)} a^{(i,j)HS}(k)\, Y^{(j)HS}(k) Z^{(i)HS}(k) \Bigg]^{-1} \tag{17}$$

with

$$X^{(i)HS}(k) = k \int_0^\infty \mathrm{d}r' \phi^{H}(k,r') h^{(i)}(r') \tag{18}$$

and

$$Z^{(i)HS}(k) = \int_0^\infty \mathrm{d}r' h^{(i)}(r') \operatorname{Re}\{ f^{H}(-k) f^{H}(k,r')\}. \tag{19}$$

The result in (17) can be used to construct an exact analytic expression for the scattering phase induced by the Hulthén plus any rational separable potential. However, in the following we specialize (17) to a rank one potential with Yamaguchi form factors and demonstrate certain interesting features of $\delta^{HY}(k)$ (Y for Yamaguchi). For the Yamaguchi potential $g(r) = h(r) = e^{-\alpha r}$ and $X^{HY}(k) = Y^{HY}(k)$ and we have

$$\tan \delta^{HY}(k) = -\lambda [Y^{HY}(k)]^2 [k|f^{H}(k)|^2 A^{HY}(k) + \lambda Y^{HY}(k) Z^{HY}(k)]^{-1} \tag{20}$$

with

$$Y^{HY}(k) = a^2 k \frac{\Gamma(\alpha a - ika)\Gamma(\alpha a + ika)}{\Gamma(1 + \alpha a - ia\rho)\Gamma(1 + \alpha a + ia\rho)} \tag{21}$$

$$Z^{HY}(k) = \operatorname{Re}\{ |f^{H}(k)|^2 [(\alpha - ik)(1 + A)]^{-1} {}_3F_2(1, A, -B + 1 + (\alpha - ik)a; 2 + A, 1 + (\alpha - ik)a; 1)\} \tag{22}$$

and

$$A^{HY}(k) = 1 - \lambda \bar{G}^{H(R)}(\alpha, \alpha). \tag{23}$$

In (23), $\bar{G}^{H(R)}(\alpha,\alpha)$ is given by

$$\bar{G}^{H(R)}(\alpha,\alpha) = \bar{G}^{H(+)}(\alpha,\alpha) + a^2 \frac{\Gamma(\alpha a - ika)\Gamma(\alpha a + ika)}{\Gamma(1 + \alpha a - ia\rho)\Gamma(1 + \alpha a + ia\rho)} \times [(\alpha - ik)(1 + A)]^{-1} {}_3F_2(1, A, -B + 1 + (\alpha - ik)a; 2 + A, 1 + (\alpha - ik)a; 1) \tag{24}$$

with

$$\bar{G}^{H(+)}(\alpha,\alpha) = -[2\alpha(\alpha + ik)^2]^{-1} + [a(\alpha^2 + k^2)^2]^{-1} \times [B\, {}_4F_3(1, -a(\alpha + ik), -a(\alpha + ik), A; 1 + A, 1 + a(\alpha - ik), 1 + a(\alpha - ik); 1) + A\, {}_4F_3(1, -a(\alpha + ik), -a(\alpha + ik), B; 1 + B, 1 + a(\alpha - ik), 1 + a(\alpha - ik); 1)]. \tag{25}$$

The results in (21) and (22) have been obtained by using an integral representation for the ${}_2F_1(\cdot)$ functions that occur in (16) and (19). Some manipulations with the help of the generalized Dixon theorem (Slater 1966) have also been made. The value of $\bar{G}^{H(+)}(\alpha,\alpha)$ has been taken from the de Maag (de Maag 1984).

3. Coulomb–limit

For extremely large values of the screening length a the Hulthén potential V_H goes over to the Coulomb potential V_C. It is well known that (Ford 1964, 1966) all objects derived from a screened Coulomb potential do not have an "unscreening limit". Thus it will be interesting to examine under what limiting conditions $\delta^{HY}(k)$ will reproduce the Coulomb–Yamaguchi phase shift $\delta^{CY}(k)$ given elsewhere (Sett *et al* 1988). To that end we observe that the unscreening limit should be defined in an appropriate manner. We demand that as $a \to \infty$ and $V_0 \to 0$, their product aV_0 remains a constant and equals to $2k\eta$ with η the Sommerfeld parameter. In this case we have

$$\lim_{\substack{a\to\infty \\ aV_0 = 2k\eta}} V_0 e^{-r/a}/(1 - e^{-r/a}) = 2k\eta/r = V_C(r). \tag{26}$$

In this limit

$$A \to i\eta, \quad B \to -i\eta - 2iak \quad \text{and} \quad C \to -2iak. \tag{27}$$

Using (27) we have found

$$\lim_{a\to\infty} Y^{HY}(k) = \frac{k}{(\alpha^2 + k^2)} \left(\frac{\alpha - ik}{\alpha + ik} \right)^{i\eta}, \tag{28a}$$

$$\lim_{a\to\infty} Z^{HY}(k) = {}_2F_1\left(1, i\eta; 2 + i\eta; \frac{\alpha + ik}{\alpha - ik} \right) \tag{28b}$$

and

$$\lim_{a\to\infty} G^{H(R)}(\alpha,\alpha) = \bar{G}^{C(+)}(\alpha,\alpha) + \frac{1}{(1 + i\eta)(\alpha^2 + k^2)(\alpha - ik)} \times \left(\frac{\alpha - ik}{\alpha + ik} \right)^{i\eta} {}_2F_1\left(1, i\eta; 2 + i\eta; \frac{\alpha + ik}{\alpha - ik} \right). \tag{28c}$$

The result for $\bar{G}^{C(+)}(\alpha,\alpha)$ has been given by Talukdar *et al* (1985). The expression for $\tan \delta^{CY}(k)$ (Sett *et al* 1988) follows directly from (20) and (28).

It would be desirable to extend our results to higher values of l. Such an effort will require analytical solutions for the Hulthén potential for $l > 0$. Unfortunately, no such exact solutions exist. One can, however, try to work with approximate analytical solutions (Laha *et al* 1988) recently constructed by using the concepts of supersymmetric quantum mechanics (Witten 1981). However we feel that the usefulness of our expressions for the scattering phase shift in (17) and (20) can be appreciated better in terms of appropriate numerical results obtained from either of them. To that end we proceed as follows.

The result in (20) refers to scattering on Hulthén-modified Yamaguchi potential. For the Yamaguchi potential we have chosen to work with $\lambda = -2{\cdot}405\,\text{fm}^{-3}$ and $\alpha = 1{\cdot}1\,\text{fm}^{-1}$. These parameters were used by van Haeringen (1975) to compute realistic values for scattering length and effective range for the $p-p$ scattering. We took $(2k\eta)^{-1} = 28{\cdot}80\,\text{fm}$. This is the proton Bohr radius. Besides λ and α, the expression for $\tan \delta^{HY}(k)$ is parametrized by V_0 and a. To have the correct Coulomb limit of (20) we assume that $V_0 = 2k\eta a^{-1}$. The results for our computed scattering phase shift as a function of laboratory energy, E_{lab}, are shown in figure 1. The Coulomb–

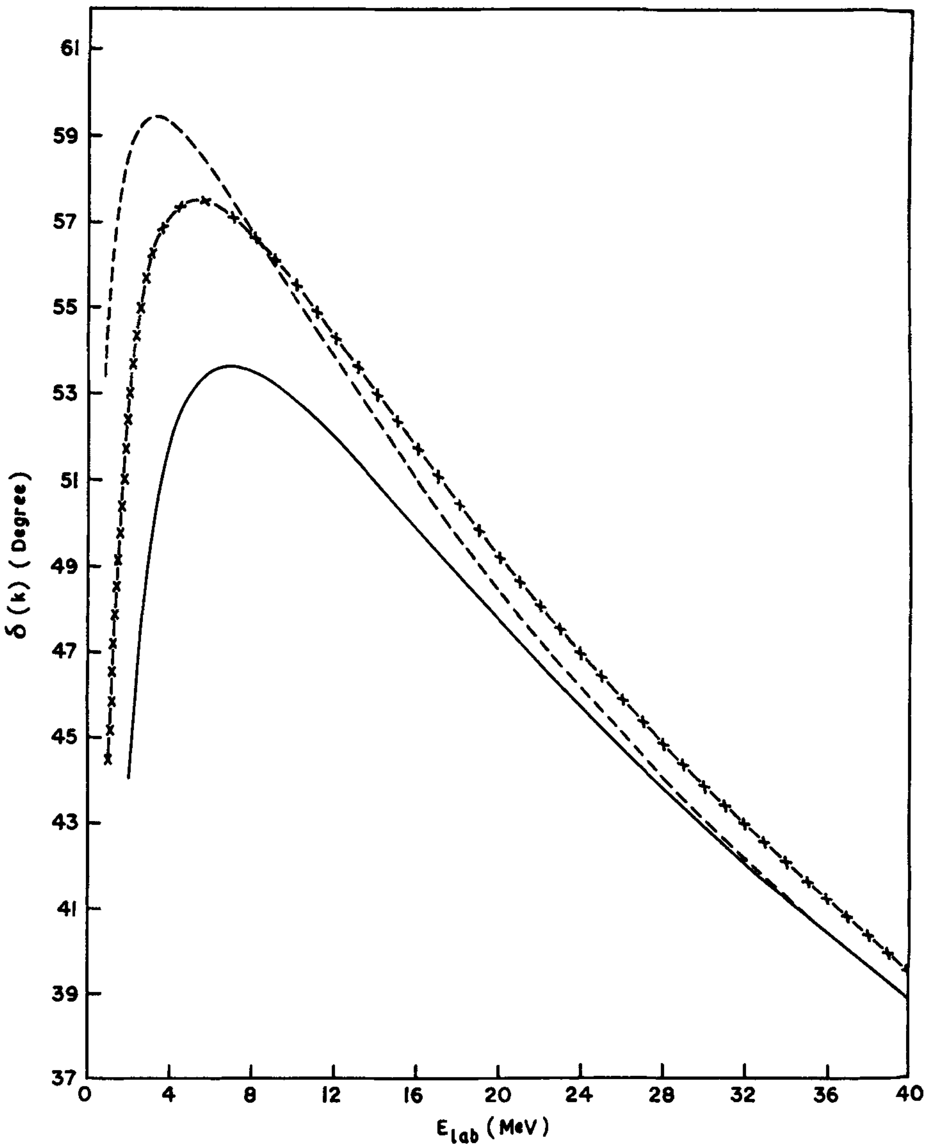

Figure 1. The s-wave p–p scattering phase shifts $\delta(k)$ as a function of E_{lab}. The solid and dashed curves give the numbers for scattering on Coulomb plus Yamaguchi, and pure Yamaguchi potentials respectively. The dashed curve with cross represents the corresponding results for Hulthén plus Yamaguchi potential for $ka = 10$.

Yamaguchi phase shifts as obtained from Sett *et al* (1988) are shown by the solid curve. The dashed curve, on the other hand, represents the variation of phase shifts induced by the pure Yamaguchi potential computed by using $V_0 = 0$ in (20). The curve for $\delta^{CY}(k)$ lies considerably below that for $\delta^{Y}(k)$. Understandably, this can be attributed to the action of the repulsive Coulomb interaction for the p–p system on the purely attractive nuclear interaction. Looking closely into these curves we see that the Coulomb effect is dominant at low energies and at high energies the Coulomb potential plays a less dominant role.

The dashed curve with cross displays the variation of $\delta^{HY}(k)$ for $ka = 10$. Obviously, we are now considering a physical situation in which the Coulomb field is somewhat screened. This screened Coulomb field is less repulsive than the pure Coulomb potential and, expectedly, our curve for $\delta^{HY}(k)$ lies in between those for $\delta^{Y}(k)$ and $\delta^{CY}(k)$. We have verified that for very large values of ka ($\sim 10^4$) the dashed curve with cross coincides with the solid curve. This observation is consistent with what has been demonstrated in (26)–(28).

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