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Path integral analysis of harmonic oscillators with time-dependent mass

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Abstract. Two cases of forced harmonic oscillators with time dependent mass for which exact propagators can be evaluated are presented. From the exact propagators, normalized solutions of the corresponding Schrödinger equations are arrived at. Time-dependent invariants are also found.

Keywords. Exact propagators; time dependent mass; Schrödinger equation; time-dependent invariants.

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1. Introduction

The path integral formulation of Feynman provides an approach to solve quantum mechanical problems, alternative to the well-known methods of Heisenberg and Schrödinger. The applicability of this formulation has, however, been limited because explicit expressions for propagators are available only for a few cases. The propagator for general quadratic Lagrangian has been evaluated by many authors (Khandekar and Lawande 1986). Physically, this corresponds to the motion of a generalized harmonic oscillator with time-dependent mass and with variable frequency under the action of a perturbative force. Though the procedure for obtaining the propagator is valid for arbitrary mass $M(t)$, one can note that explicit expression for the propagator cannot be obtained for all time varying mass-functions, since the procedure involves the solution of a non-linear differential equation. A solution of the problem of a variable mass, or variable frequency oscillator would be welcomed in several branches of physics. Looking through the literature one can find that only the case of an oscillator with an exponentially varying mass which can be used for describing the effect of damping has been treated via the path integral method so far. Khandekar and Lawande (1979) have considered this case with variable frequency also. The same problem with constant frequency has been treated by path integral method by many authors (Caldirola 1941; Kanai 1948; Cheng 1984; Landovitz *et al* 1979; Dodnov *et al* 1979).

In this paper we present two more exactly solvable cases of forced harmonic oscillators with time-dependent mass and constant frequency:

1. A strongly pulsating mass obeying $M(t) = M_0 \cos^2 \alpha t$
2. A mass growing with time according to $M(t) = M_0(1 + \alpha t)^2$.

We have evaluated exact closed form expressions for the propagators for these two cases. Following Khandekar and Lawande (1975), we have shown that the propagators (in the absence of applied force) admit expansion in terms of normalized solutions of the corresponding time-dependent Schrödinger equation. These solutions are also the eigenfunctions of an invariant operator associated with the problem.

The first case has been considered already by Colegrave and Abdalla (1982) to describe the electric and magnetic field intensities in a Fabry–Perot cavity. They applied a time-dependent canonial transformation for solving the corresponding Schrödinger equation. We show that the quasiperiodic solutions arrived at by them are easily obtainable from the exact propagator.

2. Evaluation of exact propagators

A general one-dimensional quadratic action is characterized by the Lagrangian

$$L = \tfrac{1}{2}[M(t)\dot{x}^2 - b(t)x^2] + c(t)x \tag{1}$$

where $M(t)$, $b(t)$ and $c(t)$ are well-bahaved function of time. Physically the Lagrangian corresponds to a forced harmonic oscillator with a time-dependent mass $M(t)$ and with a variable frequency $\omega(t) = (b/M)^{1/2}$ under the action of a force $c(t)$. The exact propagator for the above Lagrangian has been derived by many authors. A standard derivation of the propagator based on the conventional polygonal path approach (Khandekar and Lawande 1986) will give

$$\begin{aligned} K(x'',t'';x',t') = {} & \left[\frac{\eta'\eta''(\dot{\mu}'\dot{\mu}'')^{1/2}}{2\pi i\hbar \sin\varphi(t'',t')}\right]^{1/2} \exp\left(\frac{i}{2\hbar}\left[M(t)x^2\frac{\dot{\rho}\eta-\rho\dot{\eta}}{\eta\rho}\right]_{t'}^{t''}\right) \\ & \times \exp\left\{\frac{i}{2\hbar\sin\varphi(t'',t')}\left[(M''\dot{\mu}''x''^2 + M'\dot{\mu}'x'^2)\cos\phi(t'',t')\right.\right. \\ & - 2\eta'\eta''(\dot{\mu}'\dot{\mu}'')^{1/2}x'x'' + \frac{2x''\eta''}{\rho''}\int_{t'}^{t''} G(t)\sin\varphi(t,t')\mathrm{d}t \\ & + \frac{2x'\eta'}{\rho'}\int_{t'}^{t''} G(t)\sin\varphi(t'',t)\mathrm{d}t \\ & \left.\left. - 2\int_{t'}^{t''}\int_{t'}^{t} G(t)G(s)\sin\varphi(t'',t)\sin\varphi(s,t')\mathrm{d}s\mathrm{d}t\right]\right\} \end{aligned}$$

where

$$G(t) = \frac{c(t)\rho(t)}{\eta(t)},\ \eta = \sqrt{M(t)}. \tag{2}$$

In the above expression we have set $r' = r(t')$ and $r'' = r(t'')$ for any function $r(t)$ of time t. ρ and μ appear in connection with the solution of the time-dependent oscillator.

$$\ddot{v} + \Omega^2(t)v = 0 \tag{3}$$

with

$$\Omega^2(t) = \frac{1}{2}\left[\frac{\dot{M}^2}{2M^2} - \frac{\ddot{M}}{M}\right] + \frac{b}{M} \tag{4}$$

$\rho(t)$ and $\mu(t)$ are the amplitude and phase of the oscillator and they satisfy,

$$\ddot{\rho} + \Omega^2(t)\rho - \rho^{-3} = 0 \tag{5}$$

and

$$\varphi(t,s) = \mu(t) - \mu(s) = \int_s^t \rho^{-2}(t)\mathrm{d}t \text{ or } \dot{\mu}\rho^2 = 1. \tag{6}$$

The above expression for the propagator cannot be evaluated explicitly for all functions $M(t)$. This is because ρ is to be obtained as a solution of the non-linear differential equation (5) which is known as the Pinney's equation (Pinney 1950). Also we note that when $\Omega^2(t)$ becomes a constant, (3) reduces to that of a simple harmonic oscillator and in this case ρ will have a simple, time-independent, real solution $\rho = 1/\sqrt{\Omega}$. The two cases which we have given above become exactly solvable ones because in these cases $\Omega^2(t)$ becomes constant. The exact propagators can be easily evaluated as follows.

2.1 *Forced harmonic oscillator of frequency ω with a strongly pulsating mass*

We consider a forced harmonic oscillator with constant frequency ω and variable mass given by $M(t) = M_0\cos^2\alpha t$.

The Lagrangian for this problem, eq. (1), is,

$$L = \frac{1}{2}(M_0\cos^2\alpha t)[\dot{x}^2 - \omega^2 x^2] + f(t)(M_0\cos^2\alpha t)\cdot x \tag{7}$$

where $f(t)$ is the force applied per unit mass. From (4) we obtain

$$\Omega^2(t) = \omega^2 + \alpha^2 = \text{constant}. \tag{8}$$

As pointed out above, ρ, φ and $\dot{\mu}$ will have simple solutions,

$$\rho = \frac{1}{\sqrt{\Omega}},\ \varphi(t,s) = \mu(t) - \mu(s) = \int_s^t \rho^{-2}(t)\mathrm{d}t = \Omega(t-s);\ \dot{\mu} = \frac{1}{\rho^2} = \Omega. \tag{9}$$

With these values, the exact propagator for our dynamical system can be readily written using (2) as

$$\begin{aligned} K(x'',t'';x',t') = {} & \left[\frac{M_0\Omega(\cos\alpha t')(\cos\alpha t'')}{2\pi i\hbar\sin\Omega(t''-t')}\right]^{1/2} \exp\left(\frac{iM_0\alpha}{4\hbar}\left[x^2\sin 2\alpha t\right]_{t'}^{t''}\right) \\ & \times \exp\left\{\frac{i}{2\hbar\sin\Omega(t''-t')}\left[M_0\Omega\{x''^2\cos^2\alpha t'' + x'^2\cos^2\alpha t'\}\cos\Omega(t''-t')\right.\right. \\ & - 2M_0\Omega(\cos\alpha t')(\cos\alpha t'')x'x'' \\ & + 2M_0 x''\cos\alpha t''\int_{t'}^{t''} f(t)\cos\alpha t\sin\Omega(t-t')\mathrm{d}t \\ & + 2M_0 x'\cos\alpha t'\int_{t'}^{t''} f(t)\cos\alpha t\sin\Omega(t''-t)\mathrm{d}t \\ & \left.\left. - 2\frac{M_0}{\Omega}\int_{t'}^{t''}\int_{t'}^{t} f(t)\cos\alpha t\cdot f(s)\cos\alpha s\sin\Omega(t''-t')\sin\Omega(s-t')\mathrm{d}s\,\mathrm{d}t\right]\right\}. \end{aligned} \tag{10}$$

The evaluation of the integral will require the form of $f(t)$. If we assume for simplicity that $f(t) = f$ = constant the integrals appearing in the exponent can be obtained after lengthy but straightforward integrations.

The first integral $I_1 = \int_{t'}^{t''} f \cdot \cos\alpha t \cdot \sin\Omega(t - t')\mathrm{d}t$

$$= \frac{-f}{2}\left[\frac{\cos\{(\Omega+\alpha)t - \Omega t'\}}{(\Omega+\alpha)} + \frac{\cos\{(\Omega-\alpha)t - \Omega t'\}}{(\Omega-\alpha)}\right]_{t'}^{t''}. \tag{11}$$

The second integral $I_2 = \int_{t'}^{t''} f\cos\alpha t \cdot \sin\Omega(t'' - t)\mathrm{d}t$

$$= \frac{f}{2}\left[\frac{\cos\{\Omega t'' - (\Omega-\alpha)t\}}{(\Omega-\alpha)} + \frac{\cos\{\Omega t'' - (\Omega+\alpha)t\}}{(\Omega+\alpha)}\right]_{t'}^{t''}. \tag{12}$$

The double integral in the propagator

$$I_3 = f^2\Bigg[\frac{\Omega t}{4(\alpha^2 - \Omega^2)}\left(\sin\Omega(t'' - t') + \frac{\cos\{\Omega(t'' + t') - 2\Omega t\}}{2\Omega t}\right) +$$
$$\frac{1}{2}\frac{\Omega\cos\alpha t'}{(\Omega^2 - \alpha^2)}\left(\frac{\cos\{\Omega t'' - (\Omega-\alpha)t\}}{(\Omega-\alpha)} + \frac{\cos\{\Omega t'' - (\Omega+\alpha)t\}}{(\Omega+\alpha)}\right) +$$
$$\frac{1}{8}\frac{1}{(\Omega+\alpha)}\left(\frac{\cos[2\alpha t + \Omega(t'' - t')]}{2\alpha} - \frac{\cos[\Omega(t' + t'') - 2(\alpha+\Omega)t]}{2(\Omega+\alpha)}\right) +$$
$$\frac{1}{8(\Omega-\alpha)}\left(\frac{\cos[2(\alpha-\Omega)t + \Omega(t' + t'')}{2(\alpha-\Omega)} - \frac{\cos[\Omega(t'' - t') - 2\alpha t]}{2\alpha}\right)\Bigg]_{t'}^{t''}. \tag{13}$$

2.2 *Forced harmonic oscillator of frequency ω with a variable mass obeying $M(t) = M_0(1 + \alpha t)^2$*

The Lagrangian for a forced harmonic oscillator of constant frequency ω and with a variable mass given by the mass-law $M(t) = M_0(1 + \alpha t)^2$ can be written as

$$L = \tfrac{1}{2}M_0(1 + \alpha t)^2[\dot{x}^2 - \omega^2 x^2] + M_0(1 + \alpha t)^2 f(t)\cdot x. \tag{14}$$

Where $f(t)$ is the force applied per unit mass. From (4) we can obtain in this case

$$\Omega^2(t) = \omega^2 = \text{constant}. \tag{15}$$

Equation (5) and (6) will have the simple solutions.

$$\rho = \frac{1}{\sqrt{\Omega}} = \frac{1}{\sqrt{\omega}},\ \varphi(t,s) = \mu(t) - \mu(s) = \int_s^t \rho^{-2}\mathrm{d}t = \omega(t - s)$$
$$\dot{\mu} = \omega. \tag{16}$$

The exact propagator for this dynamical system will become

$$K(x'',t'';x',t') = \left[\frac{M_0(1 + \alpha t')(1 + \alpha t'')\omega}{2\pi i\hbar\sin\omega(t'' - t')}\right]^{1/2}\exp\left(\frac{-iM_0\alpha}{2\hbar}\left[x^2(1 + \alpha t)\right]_{t'}^{t''}\right)$$

$$
\begin{aligned}
&\exp\left\{\frac{i}{2\hbar\sin\omega(t''-t')}\left[M_0\omega\{(1+\alpha t'')^2x''^2\right.\right.\\
&+(1+\alpha t')^2x'^2\}\cos\omega(t''-t')-2M_0\omega(1+\alpha t')(1+\alpha t'')x'x''\\
&+2M_0x''(1+\alpha t'')\int_{t'}^{t''}f(t)(1+\alpha t)\sin\omega(t-t')\mathrm{d}t\\
&+2M_0x'(1+\alpha t')\int_{t'}^{t''}f(t)(1+\alpha t)\sin\omega(t''-t)\,\mathrm{d}t\\
&-2\left(\frac{M_0}{\omega}\right)\int_{t'}^{t''}\int_{t'}^{t}f(t)(1+\alpha t)f(s)\,(1+\alpha s)\\
&\qquad\left.\left.\times\sin\omega(t''-t)\sin\omega(s-t')\,\mathrm{d}s\,\mathrm{d}t\right]\right\}. \qquad (17)
\end{aligned}
$$

If we again assume that the applied force $f(t)=f=$ a constant, the evaluation of the integrals in the exponent can be done by a lengthy but straightforward calculation, as in the previous case.

From both cases we can obtain the harmonic oscillator propagator if we put $\alpha=0$ and $f=0$.

3. Schrödinger equation, invariants and the propagator

Lewis and Riesenfield (1969) have shown that for a quantal system characterized by a time-dependent Hamiltonian $H(t)$ and a hermitian invariant operator $I(t)$, the general solution of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)=H(t)\psi(x,t) \qquad (18)$$

is given by

$$\psi(x,t)=\sum_n C_n\exp[i\alpha_n(t)]\phi_n(x,t) \qquad (19)$$

where $\phi_n(x,t)$ are normalized eigenfunctions of the invariant operator I:

$$I\phi_n(x,t)=\lambda_n\phi_n(x,t) \qquad (20)$$

λ_n are the eigenvalues which are time-independent. It follows quite easily from the definition of the propagator and (19) that

$$K(x'',t'';x',t')=\sum_n\psi_n{}^*(x',t')\psi_n(x'',t'') \qquad (21)$$

where

$$\psi_n(x,t)=\exp[i\alpha_n(t)]\phi_n(x,t) \qquad (22)$$

is the normalized solution of the time-dependent Schrödinger equation (18). Khandekar and Lawande (1975) applied Mehler's formula (Erdeyl 1953, Vol. 2 p. 194) for expanding the propagator for a harmonic oscillator with constant mass but with variable frequency in the form given in (21). Following their method we can expand the propagator given in (2). From here onwards we assume that $c(t)=0$. Consequently, (2) can be re-written as

$$K(x'',t'';x',t')=\left(\frac{\eta'\eta''}{\pi\hbar}\right)^{1/2}(\dot{\mu}'\dot{\mu}'')^{1/4}\frac{\exp(-i\varphi/2)}{[1-\exp(-2i\varphi)]^{1/2}}$$
$$\times\exp\left[\frac{i}{2\hbar}\left(M(t)x^2\frac{\dot{\rho}\eta-\rho\dot{\eta}}{\eta\rho}\right)_{t'}^{t''}\right]$$
$$\times\exp\left[\frac{i}{2\hbar}(M''\dot{\mu}''x''^2+M'\dot{\mu}'x'^2)\right]$$
$$\times\exp\left[\frac{-1}{\hbar[1-\exp(-2i\varphi)]}(M''\dot{\mu}''x''^2+M'\dot{\mu}'x'^2$$
$$-2\eta'\eta''(\dot{\mu}'\dot{\mu}'')^{1/2}x'x''\exp(-i\varphi)\Big)\right]. \tag{23}$$

Mehler's formula is

$$\frac{\exp[-(x^2+y^2-2xyz)/(1-z^2)]}{\sqrt{1-z^2}}$$
$$=\exp[-(x^2+y^2)]\sum_{n=0}^{\infty}\frac{z^n}{2^n n!}H_n(x)H_n(y) \tag{24}$$

where $H_n(x)$ is the nth Hermite polynomial.

Defining

$$z=\exp(-i\varphi)[\varphi=\mu''-\mu']$$
$$x=\sqrt{M''\dot{\mu}''/\hbar}\,x'',\ y=\sqrt{M'\dot{\mu}'/\hbar}\,x' \tag{25}$$

and using (24) the propagator in (23) can be expanded in the form given in (21). From this we obtain the normalized solution of the time-dependent Schrödinger equation as

$$\psi_n(x,t)=\exp[i\alpha_n(t)]\phi_n(x,t) \tag{26a}$$

with

$$\alpha_n(t)=-(n+\tfrac{1}{2})\mu(t) \tag{26b}$$

and

$$\phi_n(x,t)=\left[\frac{1}{2^n n!}\left(\frac{M(t)\dot{\mu}}{\pi\hbar}\right)^{1/2}\right]^{1/2}\exp\left[\frac{iM(t)}{2\hbar}\left(\frac{\dot{\rho}\eta-\dot{\eta}\rho}{\eta\rho}+i\dot{\mu}\right)x^2\right]$$
$$\times H_n\left(\sqrt{\frac{M(t)\dot{\mu}}{\hbar}}x\right). \tag{26c}$$

Referring to Colegrave and Abdalla (1983) the hermitian invariant operator for an oscillator with time-dependent mass can be obtained as

$$I=\frac{M(t)x^2}{\rho^2}+\left[(\dot{\rho}\eta-\rho\dot{\eta})x-\frac{\rho p}{\eta}\right]^2 \tag{27}$$

where $p = -i\hbar\dfrac{\partial}{\partial x}$ and ρ is given by (5). Using this it can be verified that

$$I\phi_n(x,t) = \hbar(n+\tfrac{1}{2})\phi_n(x,t) \tag{28}$$

and

$$\hbar\frac{\mathrm{d}\alpha_n(t)}{\mathrm{d}t} = \left\langle \varphi_n \left| i\hbar\frac{\partial}{\partial t} - H \right| \varphi_n \right\rangle. \tag{29}$$

The normalized solutions of the Schrödinger equation for the two cases which we have considered (with $f(t)=0$) can be easily obtained from (26). For harmonic oscillator with strongly pulsating mass

$$\psi_n(x,t) = \left[\frac{1}{2^n n!}\left(\frac{M(t)\Omega}{\pi\hbar}\right)^{1/2}\right]^{1/2} \exp[-i(n+\tfrac{1}{2})\Omega t]$$
$$\times \exp\left[\frac{-M(t)x^2}{2\hbar}(\Omega - i\alpha\tan\alpha t)\right] H_n\left(\sqrt{\frac{M(t)\Omega}{\hbar}}\,x\right). \tag{30}$$

For the harmonic oscillator obeying the mass law $M(t) = M_0(1+\alpha t)^2$

$$\psi_n(x,t) = \left[\frac{1}{2^n n!}\left(\frac{M(t)\omega}{\pi\hbar}\right)^{1/2}\right]^{1/2} \exp[-i(n+\tfrac{1}{2})\omega t]$$
$$\times \exp\left[\frac{-M(t)x^2}{2\hbar}(\omega + i\alpha/(1+\alpha t))\right] H_n\left(\sqrt{\frac{M(t)\omega}{\hbar}}\,x\right). \tag{31}$$

We note that (30) agrees with the result of Colegrave and Abdalla (1982). The time-dependent invariants can be obtained from (27) by simple substitution of ρ and η. For harmonic oscillator with strongly pulsating mass

$$I = M_0\sqrt{(\omega^2+\alpha^2)}(\cos^2\alpha t)x^2$$
$$+ \frac{1}{\sqrt{(\omega^2+\alpha^2)}}\left[\sqrt{M_0}\,\alpha(\sin\alpha t)x + \frac{i\hbar}{\sqrt{M_0}\cos\alpha t}\frac{\partial}{\partial x}\right]^2. \tag{32}$$

For harmonic oscillator with mass law $M(t) = M_0(1+\alpha t)^2$

$$I = M_0\omega(1+\alpha t)^2x^2 + \frac{1}{\omega}\left[-\sqrt{M_0}\,\alpha x + \frac{i\hbar}{\sqrt{M_0}(1+\alpha t)}\frac{\partial}{\partial x}\right]^2. \tag{33}$$

4. Discussion

The study of quantal harmonic oscillators with time-dependent mass has assumed significance because of the connection between these problems and many others belonging to different areas of physics like plasma physics, quantum optics, quantum chemistry etc. Looking through the literature one can notice in this context that the path integral method has been used to solve the problem of an exponentially varying mass only, which is used for describing damping. We have presented two more cases of forced oscillators with time-dependent mass for which exact propagators can be evaluated. The first example of strongly pulsating mass has an objectionable feature from a physical point of view in that its mass becomes zero periodically. Colegrave and Abdalla (1982) have remarked that a mass which could become zero could arise in an ideal situation for a Fabri–Perot cavity. Still a gentler variation of mass which avoids periodic vanishing is preferred. But when the mass law is modified to avoid vanishing, an exact solution will not be obtained. We note that the quasi-periodic states obtained by Colegrave and Abdalla (1982) can be easily obtained by the path integral method.

The second case of growing mass has no problem of mass vanishing. Even though not much used, this model can also be applied to describe the effect of damping, like the exponentially varying mass.

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