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Period matching in modulated maps

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Abstract. We study discrete nonlinear maps in which the control parameter is itself "modulated" by another discrete nonlinear map. We show that for a certain class of such maps, which includes for example the logistic map, the periodicity of the modulated signal is either one, independent of the periodicity of the modulating signal, or its periodicity is an integral multiple of the periodicity of the modulating signal or it is chaotic.

Keywords. Two-dimensional discrete maps; Sarkovski ordering; Lyapunov exponents; multiple basins.

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1. Introduction

Interest in chaotic systems has been fuelled not only by the finding that simple dynamical systems can display complex dynamical behaviour (May 1976) but equally by the discovery of universality in such systems (Feigenbaum 1978). Thus, for example, all discrete one dimensional nonlinear maps with a quadratic maximum show similar period doubling routes to chaos with similar bifurcation diagrams and identical Feigenbaum constants. Such universality has been observed not only in nonlinear mathematical maps but also in a number of nonlinear physical systems (Schuster 1984).

The logistic map

$$X_{t+1} = 4\lambda X_t(1 - X_t), \quad 0 \leqslant \lambda, X \leqslant 1 \tag{1}$$

first used by Verhulst in 1845 to model insect populations with non-overlapping generations has been an important exemplar of simple quadratic maps used in the study of systems displaying the period doubling route to chaos. If the control parameter λ is made linearly dependent on time, Kapral and Mandel (1985) have shown that the onset of bifurcations is delayed and the system shows bistability and hysteresis. On the other hand, by making the control parameter λ proportional to the difference between a goal output and the actual output of the system, Huberman and Lumer (1989) demonstrated that such a system is capable of efficient adaptive control and seems to exhibit a universal relation between the maximum amplitude of perturbation from which the system can recover and the speed with which it does so. In a series of papers, Harikrishnan and Nandakumaran (1987a, b, 1988) have studied what they call the modulated logistic map in which the evolution of the control parameter λ is itself determined by a logistic map so that the resulting two dimensional system is described by

$$\lambda_{t+1} = 4\mu\lambda_t(1-\lambda_t)$$
$$X_{t+1} = 4\lambda_t X_t(1-X_t) \tag{2}$$

with $0 \leqslant \mu, \lambda, X \leqslant 1$. This yields a 3-dimensional bifurcation diagram whose projection on the (μ, λ) plane is just the bifurcation diagram of the usual logistic map. The authors show that the bifurcation structure of X as a function of μ is different from that of the usual logistic map in that X goes to a period-4 state at a smaller value of μ than does λ. The higher bifurcations of X and λ however occur at identical values of μ. This has the remarkable consequence that the different periodic windows that occur after the first onset of chaos for the $X-\mu$ map appear in exactly the same order as in the logistic map. In other words, if λ has a stable n-cycle in a particular interval of μ, then X also has a stable n-cycle in the same interval, i.e. both λ and X have the same Sarkovski ordering.

In §2 we establish a general theorem which explains the conditions under which the X system is enslaved by the λ system (where by enslavement we mean that the periodicity of the X system is identical to that of the λ system) and show that the novel behaviour of the modulated logistic map of Harikrishnan and Nandakumaran can be understood completely in terms of this theorem and its corollaries. We conclude with a brief discussion in §3.

2. Enslavement

To understand the behaviour of the modulated logistic map we start with the following theorem.

Theorem 1. *Given the two-dimensional discrete map*

$$\lambda_{t+1} = F(\mu, \lambda_t)$$
$$X_{t+1} = G(\lambda_t, X_t) \tag{3}$$

of the unit square $I \subset R^2$ (defined by $0 \leqslant \mu, \lambda, X \leqslant 1$) into itself, where G is a monotonic function of λ for fixed X; if λ and X are periodic – the periodicity of X is always an integral multiple of the periodicity of λ for any given μ.

Proof. Assume that under iterations of the map (3), λ is in a stable state of periodicity n, i.e. n is the smallest positive integer for which

$$F(\mu, \lambda_i^*) = \lambda_{i+1}^*$$
$$F^n(\mu, \lambda_i^*) \equiv \underbrace{F(\mu, F(\mu, \ldots, F(\mu, \lambda_i^*) \ldots)) = \lambda_i^*}_{n \text{ times}}$$

and

$$\lambda_{i+n}^* = \lambda_i^*, \quad i = 1, 2, \ldots, n, \tag{4}$$

and that X is in a stable state of periodicity k i.e. k is the smallest positive integer for which

$$G(\lambda_j^*, X_j^*) = X_{j+1}^*$$

$$G^k(\lambda_j^*, X_j^*) \equiv G(\lambda_{j+k-1}^*, G(\lambda_{j+k-2}^*, \ldots, G(\lambda_j^*, X_j^*)\ldots)) = X_j^*$$

and

$$X_{j+k}^* = X_j^*, \quad j = 1, 2, \ldots, k. \tag{5}$$

Since by assumption, X is in a stable state of periodicity k, k is the smallest integer for which

$$X_{j+1}^* = G(\lambda_j^*, X_j^*) = G(\lambda_{j+k}^*, X_{j+k}^*)$$
$$= G(\lambda_{j+k}^*, X_j^*), \quad j = 1, 2, \ldots, k. \tag{6}$$

Since $G(\lambda, X)$ is given to be monotonic in λ for fixed X,

$$G(\lambda_j^*, X_j^*) = G(\lambda_{j+k}^*, X_j^*) \tag{7}$$

can be true only if

$$\lambda_j^* = \lambda_{j+k}^*, \quad j = 1, 2, \ldots, k. \tag{8}$$

Also by assumption λ is in a stable state of periodicity n. Thus n is the smallest integer for which

$$\lambda_j^* = \lambda_{j+n}^* = \lambda_{j+2n}^* = \ldots, \quad j = 1, 2, \ldots, n. \tag{9}$$

Therefore condition (8) can only be satisfied if k is an integral multiple of n.

QED.

Let us now consider the conditions under which the periodicity of X can match or be a multiple of the periodicity of λ. We know (Lauwerier 1986; Lichtenberg and Lieberman 1983) that the stability of the map (3) is governed by the eigenvalues of the matrix

$$M = \prod_{i=1}^{n} J(\lambda_j^*, X_i^*) \tag{10}$$

where $J(\lambda, X)$ is the Jacobian of the transformation. In the present instance, as the function $F(\mu, \lambda)$ is independent of X, the matrix M is lower triangular and its eigenvalues are given by

$$\alpha_{\lambda,n} = \prod_{i=1}^{n} \frac{\partial F(\mu, \lambda)}{\partial \lambda}\Big|_{\lambda_i^*} \tag{11}$$

and

$$\alpha_{X,n} = \prod_{i=1}^{n} \frac{\partial G(\lambda_i^*, X)}{\partial X}\Big|_{X_i^*}. \tag{12}$$

In order that (λ_i^*, X_i^*) be a stable fixed point, we must have $|\alpha_{\lambda,n}| < 1$ and $|\alpha_{X,n}| < 1$ (Lauwerier 1986). The Lyapunov exponents of the λ and X systems of the map (3) can be defined in terms of these eigenvalues, in analogy to the one dimensional case

(Schuster 1984; Lauwerier 1986), by the expressions:

$$\mathrm{LE}_\lambda = \lim_{n\to\infty} \frac{1}{n}\log|\alpha_{\lambda,n}| \tag{13}$$

and

$$\mathrm{LE}_X = \lim_{n\to\infty} \frac{1}{n}\log|\alpha_{X,n}| \tag{14}$$

respectively. Theorem 1 can be shown to have the following consequence.

COROLLARY 1

For a given value of μ for which λ is in a stable state of periodicity n, the X system will (a) also be in a stable state of periodicity n, provided

$$|\alpha_{X,n}| = \left|\prod_{i=1}^{n} \frac{\partial}{\partial X} G(\lambda_i^*, X)|_{X_i^*}\right| < 1, \quad (n = \text{periodicity of } \lambda), \tag{15}$$

(b) be in a stable state of periodicity k higher than n, where $k = mn$, $m = 1, 2, 3, \ldots,$ if condition (15) *is violated but the Lyapunov exponent LE_X is negative and*

$$|\alpha_{X,k}| = \left|\frac{\partial}{\partial X} G^k(\lambda_r^*, X)|_{X_r^*}\right| = \left|\prod_{j=1}^{k} \frac{\partial}{\partial X} G(\lambda_j^*, X)|_{X_j^*}\right| < 1$$

with

$$\lambda_{i+n}^* = \lambda_i^* \quad \textit{and} \quad X_{i+k}^* = X_i^* \tag{16}$$

(c) be in a chaotic state if LE_X is positive, in which case neither eq. (15) *nor* (16) *is satisfied.*

Proof. Consider first the case when G has one stable attractor. For a given μ, the periodicity n of λ depends upon the number of stable fixed points of $F^n(\mu, \lambda)$ – the nth iterate of the map $F(\mu, \lambda)$. If for this value of μ there are n stable values λ_i^* of $\lambda (i = 1, 2, \ldots, n)$, and the derivative of the nth iterate $G(\lambda_{i+n-1}^*, G(\lambda_{i+n-2}^*, \ldots, G(\lambda_i^*, X)\ldots))$ of $G(\lambda, X)$ evaluated at any of the fixed points X_i^* lies in the open interval $]-1, 1[$, (which is equivalent to condition (15) being valid) then to each of the n equations obtained by the cyclic permutations of $G^n(\lambda_i^*, X_i^*) = X_i^*$, there will correspond a unique stable solution X_i^*. This implies that there exists a unique sequence of n ordered pairs (λ_i^*, X_i^*) which constitute the stable fixed points of the two dimensional map for the given value of μ. The n values X_i^* constitute a stable cycle of period n to which the X system will converge asymptotically. Thus the periodicity of X will be the same as that of λ.

However if condition (15) is violated but the Lyapunov exponent LE_X remains negative, the n fixed points X_i^* will be unstable and the X system will be in a state of higher periodicity $k (k = mn, m = 2, 3, \ldots)$, where k is the smallest positive integer for which eq. (16) is satisfied. In this case the k ordered pairs $(\lambda_i^*, X_{i+jn}^*)$, $i = 1, 2, 3, \ldots, n$; $j = 0, 1, 2, \ldots, m-1$, constitute the k stable fixed points of the two dimensional map for a given value of μ. Finally if LE_X is positive, X will be in a chaotic state. Therefore when λ is in a periodic state, either the periodicity of X is an integral multiple of the periodicity of λ or X is in a chaotic state.

When the map $G(\lambda, X)$ has more than one stable attractor, the proof given above holds separately for each stable attractor and the corollary is valid in each of the basins associated with the stable attractors. This does not imply that X will have the same periodicity in all the basins but only that in each basin either the periodicity of X will be an integral multiple of the periodicity of λ or X will be in a chaotic state.

QED.

COROLLARY 2

In map(3), when λ is in an aperiodic state, the X system is also in an aperiodic state.

Proof. Under iterations of map(3) for a value of μ for which λ is in an aperiodic state assume that X is in a stable state of periodicity k. Then following the arguments used in the proof of theorem 1, we can show that $\lambda_j = \lambda_{j+k}$ for all $j = 1, 2, \ldots, k$. This contradicts the assumption that λ is in an aperiodic state, hence X must also be aperiodic when λ is aperiodic. QED.

The theorem and its corollaries proved so far cannot be applied immediately to the analysis of the modulated logistic map of Harikrishnan and Nandakumaran (1987a, 1988) because their function $G(\lambda, X) = 4\lambda X(1 - X)$ is not monotonic in λ for $X = 0$ or 1. We therefore consider the following corollary.

COROLLARY 3

In map(3), when $G(\lambda, X)$ is a monotonic function of λ for fixed X in the open interval $]0, 1[$ with $G(\lambda, 0) = G(\lambda, 1) = 0$ for all λ (i.e. $X = 0$ is a fixed point of $G(\lambda, X)$ for all λ), then either (a) $(\lambda_i^*, X = 0)$ is a stable fixed point of the map in which case the X system will always be in a period 1 state irrespective of the periodicity of λ (periodic or aperiodic); or (b) $(\lambda_i^*, X = 0)$ is an unstable fixed point of the map in which case the previous theorem and corollaries will hold.

Proof. The proof proceeds along the same lines as that of theorem 1 with the difference that in this case eq. (7) can be satisfied either for $\lambda_{j+k}^* = \lambda_j^*$ with $X_j^* \neq 0$ or for $X_j^* = 0$, as $X = 0$ is a fixed point of $G(\lambda, X)$ for all values of λ. Note that there is no need to consider the point $X = 1$ in this analysis, as it is not a fixed point of $G(\lambda, X)$ for any λ. The X system will converge to zero, i.e. $(\lambda_i^*, X = 0)$ will be a stable fixed point of the map only when LE_X for $X = 0$ is negative i.e.,

$$\mathrm{LE}_X|_{X=0} = \lim_{n\to\infty} \frac{1}{n} \log|\alpha_{X,n}|_{X=0}| < 0.$$

So for those values μ for which $\mathrm{LE}_X|_{X=0}$ is negative, the X system will always be in a period 1 state with $X^* = 0$, irrespective of the state of the λ system (periodic or aperiodic). On the other hand, when $\mathrm{LE}_X|_{X=0} > 0$, the X system will never converge to zero. So X_j^* cannot be equal to zero. This implies that (7) will be satisfied only for $\lambda_{j+k}^* = \lambda_j^*$. Thus the periodicity of the X system will be in accordance with theorem 1 and corollaries 1 and 2.

QED

The behaviour of the modulated logistic map, discussed by Harikrishnan and Nandakumaran can be understood in terms of our theorem and its corollaries in the special circumstance that the functional form of $G(\lambda, X)$ is identical to that of $F(\mu, \lambda)$, both being logistic maps. In this case $\alpha_{\lambda,n} = \Pi_{i=1}^{n} 4\mu(1-2\lambda)|_{\lambda=\lambda_i^*}$ and $\alpha_{X,k} = \Pi_{j=1}^{k} 4\lambda_j^* \times (1-2X)|_{X=X_j^*}$ where n and k are the smallest positive integers for which (4) and (5) are satisfied, i.e. n and k are the periodicities of the λ and X systems respectively. To apply corollary 3 to this system, we first evaluated $\mathrm{LE}_X|_{X=0}$ numerically for this map and its plot as a function of μ is shown in figure 1. Notice that LE_X for the fixed point $(\lambda^*, 0)$ is negative for $0 \leqslant \mu \leqslant 1/3$, which implies that the X system will be in a period 1 state and will always converge to $X^* = 0$ for all values of μ lying in this range. For these values of μ, $|\alpha_{\lambda,1}| < 1$, so λ is also in a period 1 state. On the other hand for $1/3 < \mu < 1$, $\mathrm{LE}_X|_{X=0}$ is always positive, implying that the X system will never converge to zero for μ in this range. Thus for $\mu \leqslant 1/3$ in the modulated logistic map both λ and X will be in stable period 1 states while for $\mu > 1/3$, the behaviour of the map can be understood in terms of corollaries 1 and 2.

One of the interesting observations reported by Harikrishnan and Nandakumaran was that in the modulated logistic map the periodicity of X was identical to that of λ except in the range of values $0{\cdot}848 \lesssim \mu \lesssim 0{\cdot}860$ wherein the periodicity of λ was 2 while that of X was 4. We have already shown that the periodicities of λ and X systems will be one for $0 \leqslant \mu \leqslant 1/3$. We have also verified numerically that in the case of the modulated logistic map, condition (15) is valid for $0 \leqslant \mu \lesssim 0{\cdot}848$ and therefore, in accordance with corollary 1, the X system remains enslaved to the λ system even as the latter makes a transition from a stable period 1 state to a stable period 2 state at $\mu = 0{\cdot}75$. We show this graphically in figure 2 which is a plot of $\alpha_{\lambda,n}$ and $\alpha_{X,k}$ as functions of μ for $0{\cdot}72 \leqslant \mu \leqslant 0{\cdot}885$. For $\mu < 0{\cdot}75$, $|\alpha_{\lambda,1}|$ and $|\alpha_{X,1}|$ are less than unity,

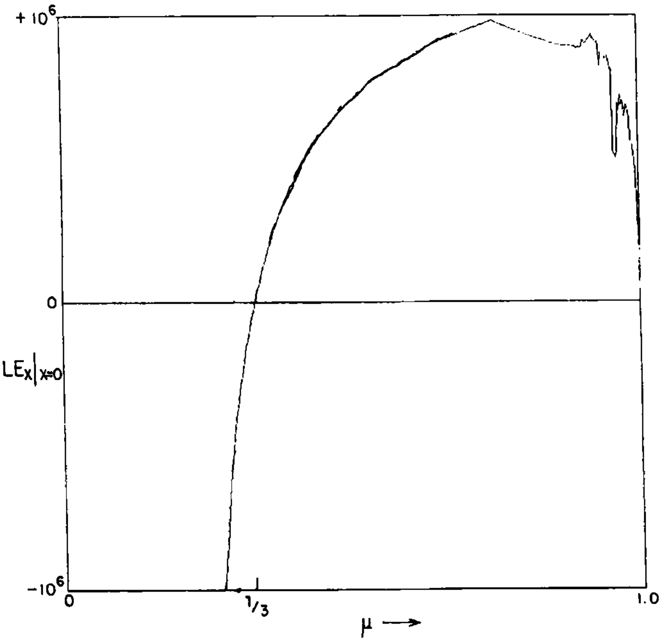

Figure 1. Plot of the sub Lyapunov exponent $\mathrm{LE}_X|_{X=0}$ of the modulated logistic map of Harikrishnan and Nandakumaran as a function of μ for $1/4 < \mu < 1$ where $\mathrm{LE}_X|_{X=0}$ is calculated for 8000 iterations after ignoring the first 1000 iterations.

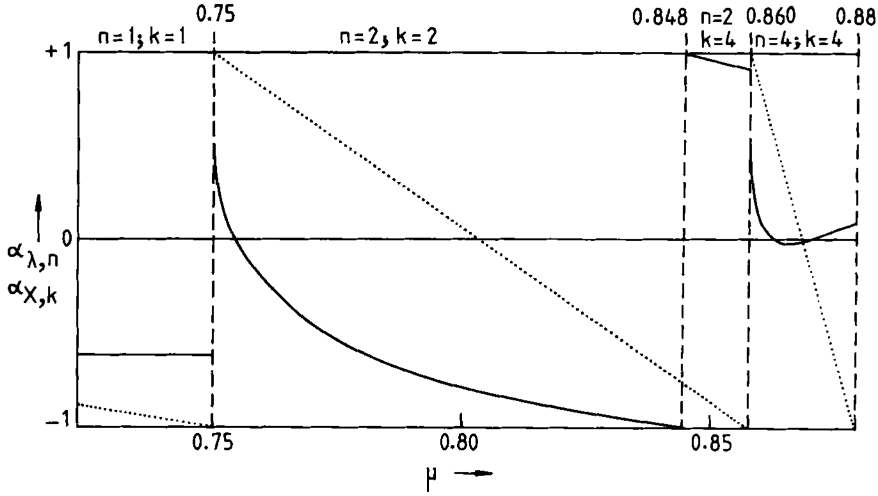

Figure 2. Plots of $\alpha_{\lambda,n}$ (dotted lines) and $\alpha_{X,k}$ (solid lines) of the modulated logistic map of Harikrishnan and Nandakumaran as functions of μ for $0{\cdot}72 \leqslant \mu \leqslant 0{\cdot}885$, with n and k being the smallest positive integers for which (4) and (5) are satisfied respectively.

so that λ and X are both in stable period 1 states. In the range $0{\cdot}75 < \mu < 0{\cdot}848$, $\alpha_{\lambda,1}$ becomes less than -1 but $|\alpha_{\lambda,2}|$ remains less than 1 so that λ makes a transition from a stable period 1 state to a stable period 2 state at $\mu = 0{\cdot}75$, and as a consequence of the theorem we have proved, X is also forced to make a transition from a stable period 1 state to a stable period 2 state. For $0{\cdot}848 \lesssim \mu \lesssim 0{\cdot}860$ condition (15) does not remain valid for $k = 2$ but (16) is satisfied for $k = 4$ so that the X system goes into a stable period 4 state while λ remains in period 2 state. This can also be seen in figure 2 where for these values of μ, $\alpha_{X,2} < -1$ while $-1 < \alpha_{X,4} < +1$ whereas $|\alpha_{\lambda,2}| < 1$ throughout. At $\mu \simeq 0{\cdot}86$, $\alpha_{\lambda,2} < -1$ but $-1 < \alpha_{\lambda,4} < +1$ so that the λ system bifurcates to a stable period 4 state and the periodicities of λ and X once again become equal. Since condition (15) remains valid for $\mu > 0{\cdot}86$ whenever λ is in a stable periodic regime, the periodicity of X remains enslaved to the periodicity of λ even in the windows in the chaotic region of λ, with X becoming aperiodic whenever λ is aperiodic (corollary 2). Consequently X displays the same Sarkovski ordering as λ, (which is that of the standard logistic map). This then serves to explain all the important findings of Harikrishnan and Nandakumaran in relation to the modulated logistic map.

In the general situation where the functional form $F(\mu, \lambda)$ is different from $G(\lambda, X)$, we have verified numerically that the behaviour of the periodicities of the λ and X systems is in accordance with our theorems and corollaries when the functions F and G are chosen to be any combination of the following:

$$f_1(a,b) = 4ab(1-b) \qquad \text{(Schuster 1984)}$$

$$f_2(a,b) = (3\sqrt{3}/2)ab(1-b^2) \qquad \text{(Becker and Dörfler 1989)}$$

$$f_3(a,b) = 4^{4/3}/3ab(1-b^3) \qquad \text{(Becker and Dörfler 1989)}$$

$$f_4(a,b) = a\sin(\pi b) \qquad \text{(Cvitanovic 1984).} \tag{17}$$

Note that all the functions have been normalized so that the variables a and b always lie in the interval $[0,1]$. Also all the functions listed are monotonic in the variable a for b fixed in the open interval $]0,1[$.

COROLLARY 4

The necessary and sufficient condition for the bifurcation diagrams of the λ and X systems in maps of the kind discussed above to have identical window structures is that condition (15) *be valid for all μ beyond the first onset of chaos in the λ system. In such a situation, if the λ system displays Sarkovski ordering then so shall the X system. This immediately enlarges the class of two-dimensional discrete maps which display Sarkovski ordering.*

3. Conclusions

For discrete, modulated two-dimensional maps in which the modulated map $G(\lambda, X)$ is a monotonic function of the driving variable λ for fixed X, we have shown that either the periodicity of X is an integral multiple of the periodicity of λ, or X is aperiodic. When the modulated map $G(\lambda, X)$ is monotonic in λ for all fixed X except the end points $X=0$ and $X=1$, as is true in the case of the modulated logistic map, the periodicity of X is either as given above or it is unity irrespective of the periodicity of λ provided $X=0$ is a stable fixed point of the system.

We have also shown that as long as condition (15) remains valid, the periodicity of X remains enslaved to that of λ. This provides us an easy technique for generating two-dimensional maps which display Sarkovski ordering.

Finally, it would be wrong to conclude that the enslavement of X by the driving signal λ is a trivial consequence of the fact that whereas λ affects X, X does not affect λ. The dynamics of X plays a significant role in determining its periodicity even when it is driven by λ. This can be easily demonstrated if we take the modulating map as in eq. (2) but choose the modulated map to be $G(\lambda, X) = \lambda X(1-X)$. The fixed point $X=0$ of the modulated map will now be stable for all allowed λ in the interval $[0,1]$, and the periodicity of X will always be unity independent of the periodicity of the driving signal λ.

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