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A nonperturbative variational approach to the vacuum structure in quantum chromodynamics

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Abstract. We study the vacuum structure in QCD in a nonperturbative manner using a variational approach with gluon condensates. We show that in Coulomb gauge as the coupling becomes moderately strong, the perturbative vacuum of QCD becomes unstable leading to gluon condensates and a gauge dependent effective mass for the gluons related to the gauge independent value of $\langle \mathrm{vac}|G^a_{\mu\nu}G^{a\mu\nu}|\mathrm{vac}\rangle$ of Shifman *et al.*

Keywords. Quantum chromodynamics; vacuum structure; phase transition; Bogoliubov transformation; variational methods.

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1. Introduction

We study the vacuum structure of quantum chromodynamics (QCD) through gluon condensates using a variational approach. For this purpose we do a nonperturbative calculation of the energy density corresponding to a state with gluon condensates, and show that when the coupling constant is above a critical value, this energy density is lower than that of the perturbative vacuum. Through a redefinition of vacuum with gluon condensates, we thus demonstrate that the perturbative vacuum in QCD is unstable above a critical value of the coupling constant. Although the above calculation is done in Coulomb gauge, we correlate the variational function for the condensates with the "known" *gauge independent* parameter of Shifman *et al* (Shifman *et al* 1979) for the same. The fact that the physical vacuum also contains quark condensates, associated with chiral symmetry breaking, has been considered earlier (Finger and Mandula 1982; Amer *et al* 1983; Adler and Davis 1984; Alkofer and Amundsen 1988). It was shown that the perturbative vacuum becomes unstable to the creation of quark-antiquark pairs when the effective coupling due to gluon exchange is greater than a critical value $g^2/4\pi = 9/8$ (Finger and Mandula 1982). We see that a similar phenomenon also occurs for the gluon condensates. The method considered here is a non-perturbative one as we use only equal time quantum algebra and is limited by the choice of ansatz function in the variational approach. For gluon condensates, the critical coupling for $g^2/4\pi$ appears to be as small as about 0·39 only as opposed to the larger value quoted above for chiral symmetry breaking.

2. Vacuum with gluon condensates

As we shall be considering only the gluon condensates of QCD vacuum, the relevant Lagrangian density is given as

$$\mathscr{L} = \tfrac{1}{2} G^{a\mu\nu}(\partial_\mu W^a{}_\nu - \partial_\nu W^a{}_\mu + g f^{abc} W^b{}_\mu W^c{}_\nu) + \tfrac{1}{4} G^a{}_{\mu\nu} G^{a\mu\nu}. \tag{1}$$

where $W^a{}_\mu$ are the SU(3) color gauge fields. Clearly, the equation of motion for $G^a{}_{\mu\nu}$ yields

$$G^a{}_{\mu\nu} = \partial_\mu W^a{}_\nu - \partial_\nu W^a{}_\mu + g f^{abc} W^b{}_\mu W^c{}_\nu. \tag{1a}$$

We then write the electric field, G^a_{0i} in terms of the transverse and longitudinal parts as

$$\begin{aligned} G^a_{0i} &\equiv \partial_0 W^a_i - \partial_i W^a_0 + g f^{abc} W^b_0 W^c_i \\ &= {}^T G^a_{0i} + \partial_i f^a, \end{aligned} \tag{2}$$

where the form of f^a is to be determined. We shall be working in the Coulomb gauge for which the subsidiary condition and the equal time algebra for the gauge fields are given as (Schwinger 1962)

$$\partial_i W^a_i = 0 \tag{3}$$

and

$$[W^a_i(\mathbf{x},t), {}^T G^b_{0j}(\mathbf{y},t)] = i\delta^{ab}\left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)\delta(\mathbf{x}-\mathbf{y}). \tag{4}$$

We take the field expansions for W^a_i and ${}^T G^a_{0i}$ at time $t = 0$ as (Schutte 1985)

$$W^a_i(\mathbf{x}) = (2\pi)^{-3/2} \int \frac{d\mathbf{k}}{(2\omega(\mathbf{k}))^{1/2}} (a^a_i(\mathbf{k}) + a^a_i(-\mathbf{k})^\dagger) \exp(i\mathbf{k}\cdot\mathbf{x}) \tag{5a}$$

and

$${}^T G^a_{0i}(\mathbf{x}) = (2\pi)^{-3/2} i \int d\mathbf{k} \left(\frac{\omega(\mathbf{k})}{2}\right)^{1/2} (-a^a_i(\mathbf{k}) + a^a_i(-\mathbf{k})^\dagger) \exp(i\mathbf{k}\cdot\mathbf{x}), \tag{5b}$$

which when substituted in (4) give the commutation relations for a^a_i and $a^{b\dagger}_j$ as

$$[a^a_i(\mathbf{k}), a^b_j(\mathbf{k}')^\dagger] = \delta^{ab} \Delta_{ij}(\mathbf{k}) \delta(\mathbf{k}-\mathbf{k}'), \tag{6}$$

where, $\omega(k)$ is arbitrary (Schutte 1985), and,

$$\Delta_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}. \tag{7}$$

In Coulomb gauge, the expression for the Hamiltonian density, $\mathscr{T}^{00}$ is given as

$$\begin{aligned} \mathscr{T}^{00} = {} &:\frac{1}{2} {}^T G^a_{0i} {}^T G^a_{0i} + \frac{1}{2} W^a_i(-\nabla^2) W^a_i + \frac{g^2}{4} f^{abc} f^{aef} W^b_i W^c_j W^e_i W^f_j \\ &+ \frac{1}{2}(\partial_i f^a)(\partial_i f^a):, \end{aligned} \tag{8}$$

where : : denotes the normal ordering with respect to the perturbative vacuum, say $|\mathrm{vac}\rangle$, defined through $a_i^a(\mathbf{k})|\mathrm{vac}\rangle = 0$.

To examine the stability of perturbative vacuum, $|\mathrm{vac}\rangle$, we construct a state with gluon condensates, say, $|\mathrm{vac}'\rangle$ and calculate its energy density by minimization of the expectation value of $\mathscr{T}^{00}$ with respect to $|\mathrm{vac}'\rangle$. For this purpose, we have to solve for f^a to be able to take the expectation value of $\mathscr{T}^{00}$ given by (8) with a given ansatz for $|\mathrm{vac}'\rangle$. From the equation of motion for the gauge field W_0^a, we obtain the constraint equation for f^a given as

$$(\nabla^2\delta^{ac} + gf^{abc}W_i^b\partial_i)f^c = -gf^{abc}W_i^{b\,T}G_{0i}^c \equiv J_0^a. \tag{9}$$

Since it is not possible to solve the above equation for f^a exactly, we could take a perturbative expansion for f^a in terms of the coupling constant g as

$$f^a = gf_1^a + g^2 f_2^a + \dots. \tag{10}$$

However, unless we can keep a reasonable sum of the series on the right hand side of (10), the solution will not be nonperturbative. It has been also our objective to attempt variational solution outside the summation of a subset of the perturbative series.

To do this we shall use a version of mean field approximation in the context of condensates. For this purpose, let us first take the space divergence of both sides of (2). This gives the relation between f^a and W_0^a as

$$f^a = -W_0^a - gf^{abc}(\nabla^2)^{-1}W_i^b(\partial_i W_0^c). \tag{11}$$

Substituting the expression above for f^a in (9), we have

$$\nabla^2 W_0^a + g^2 f^{abc}f^{cde}W_i^b\partial_i((\nabla^2)^{-1}(W_j^d(\partial_j W_0^e))) + 2gf^{abc}W_i^b(\partial_i W_0^c) = J_0^a. \tag{12}$$

Similar to (9), it is not possible to solve (12) for W_0^a. However, we proceed as stated, with a mean field type of approximation. What we shall do is to replace the operators in the left hand side of (12) by the corresponding expectation values in the condensate vacuum, $|\mathrm{vac}'\rangle$ for all the fields other than W_0^a. Then (12) gets replaced by

$$\nabla^2 W_0^a + g^2 f^{abc}f^{cde}\langle \mathrm{vac}'|W_i^b\partial_i((\nabla^2)^{-1}(W_j^d\partial_j)|\mathrm{vac}'\rangle W_0^e) = J_0^a. \tag{13}$$

To evaluate the above expression we first note that

$$\langle \mathrm{vac}'|W_i^a(\mathbf{x})W_j^b(\mathbf{y})|\mathrm{vac}'\rangle = f_{ij}(\mathbf{x}-\mathbf{y})\delta^{ab}, \tag{14}$$

where we substitute,

$$f_{ij}(\mathbf{x}-\mathbf{y}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})]F(\mathbf{k})\Delta_{ij}(\mathbf{k}). \tag{15}$$

The above expression is written down from translational invariance for $|\mathrm{vac}'\rangle$ and the transversality condition for the gluon fields. The function $F(\mathbf{k})$ will depend upon the particular construct one takes for $|\mathrm{vac}'\rangle$. For calculational convenience let us next take the Fourier transforms

$$\tilde{W}_0^a(\mathbf{k}) = \int W_0^a(\mathbf{x}) \exp[-i\mathbf{k}\cdot\mathbf{x}]\,d\mathbf{x}, \tag{16}$$

and

$$\tilde{J}_0^a(\mathbf{k}) = \int J_0^a(\mathbf{x}) \exp[-i\mathbf{k}\cdot\mathbf{x}]\,d\mathbf{x}. \tag{17}$$

Equation (13) then reduces to

$$(k^2 + \phi(\mathbf{k}))\,\tilde{W}_0^a(\mathbf{k}) = -\tilde{J}_0^a(\mathbf{k}), \tag{18}$$

where,

$$\phi(\mathbf{k}) = \frac{3g^2}{(2\pi)^3} \int d\mathbf{k}' F(\mathbf{k}') k^2 \frac{(1-(\hat{k}\cdot\hat{k}')^2)}{(\mathbf{k}-\mathbf{k}')^2}. \tag{19}$$

We may assume spherical symmetry for the function $F(\mathbf{k})$, in which case the angle integration in (19) can be carried out. Equation (19) then reduces to, with $k = |\mathbf{k}|$ and $k' = |\mathbf{k}'|$,

$$\phi(k) = \frac{3g^2}{8\pi^2} \int dk' F(k') \left(k^2 + k'^2 - \frac{(k^2 - k'^2)^2}{2kk'} \log\left|\frac{k+k'}{k-k'}\right| \right). \tag{20}$$

Thus, (18) and (20) give an approximation for W_0^a as modified by the gluon condensates in $|\text{vac}'\rangle$.

We now consider a trial state, $|\text{vac}'\rangle$ with gluon condensates over the perturbative vacuum, $|\text{vac}\rangle$ in a manner similar to Gross-Neveu model considered earlier (Mishra *et al* 1988; Misra 1987), given as

$$|\text{vac}'\rangle = U|\text{vac}\rangle, \tag{21}$$

where

$$U = \exp(B^\dagger - B), \tag{22}$$

with

$$B^\dagger = \frac{1}{2} \int f(\mathbf{k}) a_i^a(\mathbf{k})^\dagger a_i^a(-\mathbf{k})^\dagger d\mathbf{k}. \tag{23}$$

With the above transformation, the operators, say $b_i^a(\mathbf{k})$, which annihilate $|\text{vac}'\rangle$ are given as

$$b_i^a(\mathbf{k}) = U a_i^a(\mathbf{k}) U^{-1}. \tag{24}$$

We can explicitly evaluate that the operators $b_i^a(\mathbf{k})$ corresponding to the state, $|\text{vac}'\rangle$ are related to the operators corresponding to the state, $|\text{vac}\rangle$ through the Bogoliubov transformation given as

$$\begin{pmatrix} b_i^a(\mathbf{k}) \\ b_i^a(-\mathbf{k})^\dagger \end{pmatrix} = \begin{pmatrix} \cosh f(\mathbf{k}) & -\sinh f(\mathbf{k}) \\ -\sinh f(\mathbf{k}) & \cosh f(\mathbf{k}) \end{pmatrix} \begin{pmatrix} a_i^a(\mathbf{k}) \\ a_i^a(-\mathbf{k})^\dagger \end{pmatrix}, \tag{25}$$

where the function $f(\mathbf{k})$ is even in $\mathbf{k}$ and has been assumed to be real. The inverse transformation is given as

$$\begin{pmatrix} a_i^a(\mathbf{k}) \\ a_i^a(-\mathbf{k})^\dagger \end{pmatrix} = \begin{pmatrix} \cosh f(\mathbf{k}) & \sinh f(\mathbf{k}) \\ \sinh f(\mathbf{k}) & \cosh f(\mathbf{k}) \end{pmatrix} \begin{pmatrix} b_i^a(\mathbf{k}) \\ b_i^a(-\mathbf{k})^\dagger \end{pmatrix}. \tag{26}$$

Using (6) and (25), we obtain the same commutation relation for the operators b_i^a and $b_j^{b\dagger}$ given as

$$[b_i^a(\mathbf{k}), b_j^b(\mathbf{k})^\dagger] = \delta^{ab}\Delta_{ij}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}), \tag{27}$$

which merely reflects that the Bogoliubov transformation is a canonical transformation. To evaluate the expectation value of $\mathscr{T}^{00}$ with respect to $|\mathrm{vac}'\rangle$, we shall need the following formulae of the expectation values:

$$\langle \mathrm{vac}'|a_i^a(\mathbf{k})a_j^b(\mathbf{k}')|\mathrm{vac}'\rangle = \frac{\sinh 2f(\mathbf{k})}{2}\Delta_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\delta^{ab}, \tag{28a}$$

$$\langle \mathrm{vac}'|a_i^a(\mathbf{k})^\dagger a_j^b(\mathbf{k}')|\mathrm{vac}'\rangle = \sinh^2 f(\mathbf{k})\Delta_{ij}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')\delta^{ab}, \tag{28b}$$

$$\langle \mathrm{vac}'|a_i^a(\mathbf{k})^\dagger a_j^b(\mathbf{k}')^\dagger|\mathrm{vac}'\rangle = \frac{\sinh 2f(\mathbf{k})}{2}\Delta_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\delta^{ab}. \tag{28c}$$

Thus the function $F(\mathbf{k})$ in (15) is given by using (5) and (28),

$$F(k) = \frac{1}{\omega(k)}\left(\frac{\sinh 2f(k)}{2} + \sinh^2 f(k)\right). \tag{29}$$

Using (5), (8) and (28), we then obtain the expectation value of $\mathscr{T}^{00}$ with respect to $|\mathrm{vac}'\rangle$ as

$$\begin{aligned} \varepsilon_0 &\equiv \langle \mathrm{vac}'|{:}\mathscr{T}^{00}{:}|\mathrm{vac}'\rangle \\ &= C_1 + C_2 + C_3^2 + C_4, \end{aligned} \tag{30}$$

where

$$\begin{aligned} C_1 &= \langle {:}\tfrac{1}{2}{}^T G_{0i}^a\,{}^T G_{0i}^a{:}\rangle_{\mathrm{vac}'} \\ &= \frac{4}{\pi^2}\int \omega(k)k^2\,\mathrm{d}k\left(\sinh^2 f(\mathbf{k}) - \frac{\sinh 2f(\mathbf{k})}{2}\right), \end{aligned} \tag{31a}$$

$$\begin{aligned} C_2 &= \langle {:}\tfrac{1}{2}\,W_i^a(-\nabla^2)\,W_i^a{:}\rangle_{\mathrm{vac}'} \\ &= \frac{4}{\pi^2}\int \frac{k^4}{\omega(k)}\mathrm{d}k\left(\sinh^2 f(\mathbf{k}) + \frac{\sinh 2f(\mathbf{k})}{2}\right), \end{aligned} \tag{31b}$$

$$\begin{aligned} C_3^2 &= \langle {:}\tfrac{1}{4}g^2 f^{abc}f^{aef}\,W_i^b\,W_j^c\,W_i^e\,W_j^f{:}\rangle_{\mathrm{vac}'} \\ &= \left(\frac{2g}{\pi^2}\int \frac{k^2}{\omega(k)}\mathrm{d}k\left(\sinh^2 f(\mathbf{k}) + \frac{\sinh 2f(\mathbf{k})}{2}\right)\right)^2 \end{aligned} \tag{31c}$$

and

$$C_4 = \langle {:}\tfrac{1}{2}(\partial_i f^a)(\partial_i f^a){:}\rangle_{\mathrm{vac}'}. \tag{31d}$$

The evaluation of (31a), (31b), (31c) is straightforward. To evaluate equation (31d) we shall first eliminate f^a in favour of W_0^a using (11); write the mean field solution for W_0^a as given in (18) and then evaluate the expectation value. The contribution coming from the first term of (11) is

$$\begin{aligned} C_4^I &= \tfrac{1}{2}\langle \text{vac}'|: W_0^a(\mathbf{x})(-\nabla^2)\, W_0^a(\mathbf{x}):|\text{vac}'\rangle \\ &= \frac{1}{2}(2\pi)^{-6}\int \mathrm{d}\mathbf{k}' \frac{k'^2}{(k'^2+\phi(k'))^2} \\ &\quad \times \langle \text{vac}'|:\tilde{J}_0^a(k)\tilde{J}_0^a(k'):|\text{vac}'\rangle \exp[i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}] \\ &= 4(2\pi)^{-6}\int \mathrm{d}\mathbf{k} \frac{k^2 G(k)}{(k^2+\phi(k))^2}, \end{aligned} \tag{32}$$

where we have substituted

$$\langle \text{vac}'|:\tilde{J}_0^a(\mathbf{k})\tilde{J}_0^b(k'):|\text{vac}'\rangle = \delta^{ab}\delta(\mathbf{k}-\mathbf{k}')G(\mathbf{k}). \tag{33}$$

In the above $G(\mathbf{k})$ is given as

$$\begin{aligned} G(\mathbf{k}) &= 3g^2 \int \mathrm{d}\mathbf{q}\left(\frac{\sinh 2f(|\mathbf{q}|)}{2} + \sinh^2 f(|\mathbf{q}|)\right) \\ &\quad \times \left(\sinh^2 f(|\mathbf{k}+\mathbf{q}|) - \frac{\sinh 2f(|\mathbf{k}+\mathbf{q}|)}{2}\right)\frac{\omega(|\mathbf{k}+\mathbf{q}|)}{\omega(|\mathbf{q}|)} \\ &\quad \times \left(1 + \frac{(q^2+\mathbf{k}\cdot\mathbf{q})^2}{q^2(\mathbf{k}+\mathbf{q})^2}\right). \end{aligned} \tag{34}$$

Similarly for the second term in (11) one obtains

$$\begin{aligned} C_4^{II} &= \tfrac{1}{2} g^2 f^{abc} f^{aef} \langle \text{vac}'|: W_0^e\, W_i^b \partial_i((\nabla^2)^{-1} W_j^e \partial_j W_0^f:|\text{vac}'\rangle \\ &= 4(2\pi)^{-6}\int \mathrm{d}\mathbf{k} \frac{\phi(k)G(\mathbf{k})}{(k^2+\phi(k))^2}. \end{aligned} \tag{35}$$

Thus the expression for G_4 becomes

$$C_4 = C_4^I + C_4^{II} = 4(2\pi)^{-6}\int \mathrm{d}\mathbf{k} \frac{G(k)}{k^2+\phi(k)}, \tag{36}$$

where, $G(\mathbf{k})$ and $\phi(k)$ are as given in (34) and (20) respectively. We shall now approximate the unknown function $\omega(k)$ through an effective mass for gluons describing the condensate. Thus we approximate in (5), $\omega(k)$ as $(k^2+m^2)^{\frac{1}{2}}$, with m being an effective mass of gluon quanta in the condensate. We shall determine this mass parameter later from self-consistency requirements of the effective Lagrangian. $f(\mathbf{k})$ in the above equations is to be so determined that ε_0 in (30) is a minimum. We could achieve this in simple cases by functional differentiation of $\varepsilon_0\{f\}$ and equating the same to zero (Mishra *et al* 1988; Misra 1987). Since such a general programme is not practicable here, we adopt an alternative procedure of taking a reasonably simple ansatz for the gluon correlation function $f(\mathbf{k})$ and then calculating ε_0 through extremization with respect to the parameters of $f(\mathbf{k})$. The gluon correlation function, $f(\mathbf{k})$ should go to zero for large $\mathbf{k}$ as condensation is a long distance effect. Keeping this in mind, we choose a simple form for $\sinh f(\mathbf{k})$

$$\sinh f(\mathbf{k}) = A\exp[-Bk^2/2], \tag{37}$$

where the parameter A is to be determined through energy minimization and the dimensional parameter B gets determined by the SVZ parameter (Shifman *et al* 1979; Bertlmann 1981). Using the form (37) for $f(\mathbf{k})$, the energy density, ε_0 can be written in terms of the dimensionless quantities $x = \sqrt{B}k$ and $\mu = \sqrt{B}m$ as

$$\begin{aligned}\varepsilon_0 &= \frac{1}{B^2}(I_1 + I_2 + I_3^2 + I_4)\\ &\equiv \frac{1}{B^2}F(A),\end{aligned} \tag{38}$$

where

$$I_1 = \frac{4}{\pi^2}\int \omega(x)x^2\,\mathrm{d}x(A^2\exp[-x^2] - A\exp[-x^2/2](1 + A^2\exp[-x^2])^{\frac{1}{2}}), \tag{39a}$$

$$I_2 = \frac{4}{\pi^2}\int \frac{x^4}{\omega(x)}\mathrm{d}x(A^2\exp[-x^2] + A\exp[-x^2/2](1 + A^2\exp[-x^2])^{\frac{1}{2}}), \tag{39b}$$

$$I_3 = \frac{2g}{\pi^2}\int \frac{x^2}{\omega(x)}\mathrm{d}x(A^2\exp[-x^2] + A\exp[-x^2/2](1 + A^2\exp[-x^2])^{\frac{1}{2}}), \tag{39c}$$

and

$$\begin{aligned}I_4 = \frac{12g^2}{(2\pi)^6}\int &\frac{\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2}{x_1^2 + \phi(x_1)}(A^2\exp[-x_2^2] + A\exp[-x_2^2/2](1 + A^2\exp[-x_2^2])^{\frac{1}{2}})\\ &\times(A^2\exp[-(\mathbf{x}_1 + \mathbf{x}_2)^2] - A\exp[-(\mathbf{x}_1 + \mathbf{x}_2)^2/2]\\ &\times(1 + A^2\exp[-(\mathbf{x}_1 + \mathbf{x}_2)^2])^{\frac{1}{2}})\frac{\omega(|\mathbf{x}_1 + \mathbf{x}_2|)}{\omega(x_2)}\left(1 + \frac{(x_2^2 + \mathbf{x}_1\cdot\mathbf{x}_2)^2}{\mathbf{x}_2^2(\mathbf{x}_1 + \mathbf{x}_2)^2}\right),\end{aligned} \tag{39d}$$

with

$$\begin{aligned}\phi(x) = \frac{3g^2}{8\pi^2}\int &\frac{\mathrm{d}x'}{\omega(x')}\left(x^2 + x'^2 - \frac{(x^2 - x'^2)^2}{2xx'}\log\left|\frac{x + x'}{x - x'}\right|\right)\\ &\times(A^2\exp[-x'^2] + A\exp[-x'^2/2](1 + A^2\exp[-x'^2])^{\frac{1}{2}}).\end{aligned} \tag{40}$$

In the above, $\omega(x) = (x^2 + \mu^2)^{\frac{1}{2}}$, with μ as the effective gluon mass in $(1/B)^{\frac{1}{2}}$ units. We now also identify the gluon mass μ from the sum of the single contractions of the quartic interaction term of $\mathscr{T}^{00}$ in (8), the negative of which gives a mass term in the effective Lagrangian. We thus have the self consistency requirement that

$$\mu^2 = \frac{2g^2}{\pi^2}\int \frac{x^2\,\mathrm{d}x}{\omega(x)}(A^2\exp[-x^2] + A\exp[-x^2/2](1 + A^2\exp[-x^2])^{\frac{1}{2}}). \tag{41}$$

The right hand side above arises from the $|\text{vac}'\rangle$ expectation value and contains μ through $\omega(x)$, where as, the left hand side is the identification of μ^2 from the effective Lagrangian. μ is determined through an iterative procedure for any particular value of A so that (41) is satisfied with the input μ on the right hand side giving rise to the same output μ on the left hand side. The vacuum energy density ε_0 in (38) becomes a function $F(A)$ of A with B as only a scale parameter. We then extremise $F(A)$ with respect to A such that $F(A)$ has a minimum at $A = A_{\text{min}}$ for a given coupling constant. We find that there exists a critical value g_c of g with $g_c^2/4\pi \simeq 0{\cdot}39$, such that for $g > g_c$, $A_{\text{min}} \neq 0$ and ε_0 becomes negative. This demonstrates that for $g > g_c$, the perturbative vacuum is unstable and the physical vacuum contains gluon condensates. Below this value of the coupling constant, we see that A_{min} becomes zero and hence the gluon correlation function $f(\mathbf{k})$ vanishes, which means that for the present ansatz the

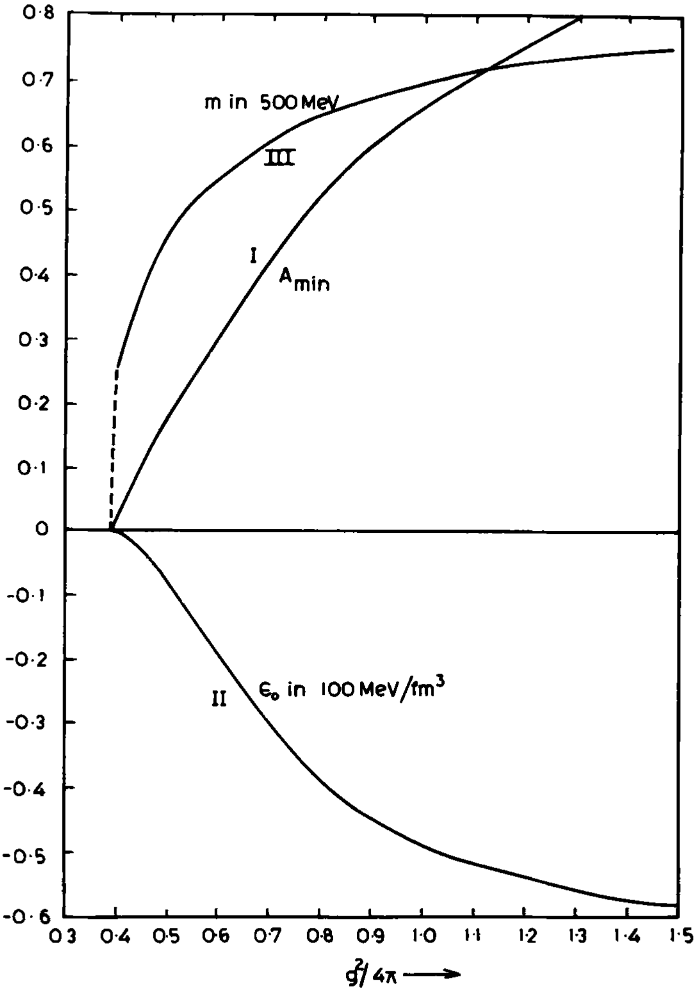

Figure 1. We plot here A_{min}, ε_0 and m respectively in curves I, II and III as functions of $g^2/4\pi$. In curve II, energy density ε_0 is expressed in units of 100 MeV/fm^3, and, in curve III, effective gluon mass m is expressed in units 500 MeV. Each of these curves leads to a critical coupling with $g_c^2/4\pi \simeq 0{\cdot}39$.

vacuum structure remains unaltered. We calculate the values of $A_{\min}$ for different values of coupling constants and plot the same as curve I in figure 1. The magnitude of $A_{\min}$ qualitatively reflects the degree of condensation as seen in the curve I above. The dependence of $A_{\min}$ on $g^2/4\pi$ through the succession of steps of calculations is highly nontrivial. In view of this, the final emergence of an almost linear graph as in curve I appears as surprising. The value of B, which is a scale parameter, is now determined by relating it with the SVZ parameter. In fact, the vacuum structure of QCD is given as (Shifman *et al* 1979; Bertlmann 1981)

$$\frac{g^2}{4\pi^2}\langle :G^a{}_{\mu\nu}G^{a\mu\nu}:\rangle_{\text{vac}'} = 0{\cdot}012\,\text{GeV}^4, \tag{42}$$

where $|\text{vac}'\rangle$ is the physical vacuum. This reduces to the equation

$$\frac{1}{B^2}\frac{g^2}{\pi^2}(-I_1(A)+I_2(A)+I_3(A)^2-I_4(A))|_{A=A_{\min}} = 0{\cdot}012\,\text{GeV}^4, \tag{43}$$

where $A_{\min}$ is the value of A corresponding to minimized energy density as function

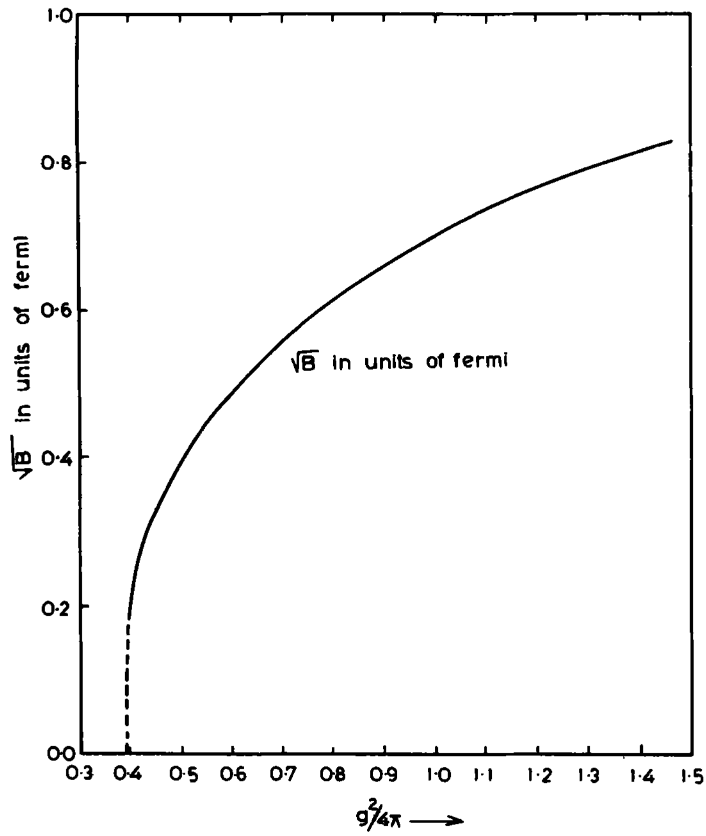

Figure 2. We plot here $\sqrt{B}$ in units of fermi for gluon condensates against $g^2/4\pi$. The correlation length between pairs of gluons is proportional to $\sqrt{B}$.

of coupling constant. Then from (43), we have the value of B given through the equation

$$\frac{1}{B^2}=\frac{0{\cdot}012\pi^2}{g^2}\left[\frac{1}{-I_1(A)+I_2(A)+I_3(A)^2-I_4(A)}\right]_{A=A_{\min}}\mathrm{GeV}^4. \tag{44}$$

In figure 2 we plot $\sqrt{B}$ in units of fermi against the coupling constant to indicate the length scales associated with gluon condensates for $g>g_c$. Since $A_{\min}\to 0$ as $g\to g_c$, the values of $\sqrt{B}$ at or below g_c have no meaning. The progressive increase of $\sqrt{B}$ with the coupling as in figure 2 may be noted.

We then calculate the value of ε_0 using (26) and (31) to obtain

$$\varepsilon_0=\frac{0{\cdot}012\pi^2}{g^2}\left[\frac{I_1(A)+I_2(A)+I_3(A)^2+I_4(A))}{-I_1(A)+I_2(A)+I_3(A)^2-I_4(A)}\right]_{A=A_{\min}}\mathrm{GeV}^4. \tag{45}$$

The above expression is clearly a function of g since as in curve I of figure 1 with $A_{\min}=A_{\min}(g)$. We shall now discuss the results, where we note that ε_0 is negative for $g^2/4\pi>0{\cdot}39$ as described below giving rise to a vacuum realignment.

3. Results

The energy density ε_0 for $|\mathrm{vac}'\rangle$ for different values of coupling constants is given by (45) and we plot the same as curve II in figure 1. We thus see that the perturbative gluon vacuum is unstable. Also, the effective gluon mass is determined from (40) as a function of $g^2/4\pi$ when we substitute A by $A_{\min}$ for that value of $g^2/4\pi$ with the scale parameter now known from (44). We plot m as a function of $g^2/4\pi$ as curve III in figure 1. Curve I clearly means that the condensate structure becomes stronger as we increase the coupling which is as one would expect. Curve II gives the energy density of $|\mathrm{vac}'\rangle$ as compared to $|\mathrm{vac}\rangle$ with the expected dependence on the coupling constant. The instability of gluon vacuum has been observed since quite some time (Savvidy 1977; Nielsen and Olesen 1978) through one loop radiative corrections compared to self consistent variational method used here. For $g^2/4\pi$ as equal to 0·5, 0·7, 1·0 as characteristic values for low energy coupling, the effective masses of the gluon field are calculated as 232, 294 and 346 MeV respectively as in curve III. We determine the mass scales here by estimating the SVZ parameter, as in (43), which determines the mass parameter of the ansatz. Such a dynamically generated gluon mass has been anticipated by Cornwall through approximate investigations of Schwinger Dyson equation in continuum QCD (Cornwall 1982) or through Monte Carlo calculations of gluon propagator (Mandula and Ogilvie 1987) in lattice QCD. We may note that Cornwall had a gluon mass of order of 300 MeV to 700 MeV (Cornwall 1982) corresponding to $\Lambda\simeq 300$ MeV to 700 MeV, the gluon mass being proportional to Λ. In the present context of $\Lambda\simeq 200$ MeV, the mass generated appears to be generally consistent with his estimates. The lattice QCD generated mass (Mandula and Ogilvie 1987) appears to be around 600 MeV, which is associated as usual, with the inverse of lattice spacing as the momentum or mass scale. Our result appears as an average between the two, when $\Lambda\simeq 200$ MeV. We feel further calculations may be needed both with our approach and lattice QCD or other approach to be able to have more reliable quantitative estimates. In our calculations, consistency of the critical coupling, $g_c^2/4\pi$ is seen since it is the same from any one of the three curves in figure 1, with as noted, $g_c^2/4\pi \simeq 0{\cdot}39$. The ansatz function includes a correlation length proportional to $\sqrt{B}$ for gluon condensates, which is plotted in figure 2. This length scale appears to be of the order of one fermi, which looks reasonable in the context of confinement. A feature of the present investigation is a self consistent determination of the gluon mass and its dependence on coupling constant and its link with the SVZ parameter (Bertlmann 1981). Global colour symmetry is maintained. QCD could still be an exact symmetry with some mechanism parallel to what was imagined by Cornwall (Cornwall 1982), where arbitrarily high order of fields were taken in an effective Lagrangian. Mass here is merely an effective parameter and is *not* the energy of *free* asymptotic particles which do not exist in view of confinement, and gluon mass has been used here in that sense. It may be worthwhile to consider this further in the light of some earlier discussions (Cornwall 1982; Mandula and Ogilvie 1987).

4. Discussion

Using a variational method, we have shown here the instability of the perturbative vacuum in QCD through an explicit construction of the vacuum state and a minimization of energy density. This is the parallel of the dynamical Higgs mechanism leading to the presence of gluon condensates discussed earlier (Mandula and Ogilvie 1987). Obtaining a finite gluon mass is the parallel of earlier results (Cornwell 1982) where an approximate solution of Schwinger Dyson equations was used. We have demonstrated here the same with the simple ansatz as in (37), showing that a phase transition takes place for $g^2/4\pi > 0{\cdot}39$. We should really vary over all possible functions for $f(\mathbf{k})$. In such a case the critical coupling could be smaller or even disappear, as in Gross-Neveu model (Mishra *et al* 1988) leading to an instability of vacuum for any coupling.

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