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Relativistic quantum mechanics of spin-zero and spin-half particles

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Abstract. We consider the quantum mechanics of directly interacting relativistic particles of spin-zero and spin-half. We introduce a scalar product in the vector space of physical states which is finite, positive definite and relativistically invariant and keeps orthogonal eigenstates of total four momentum belonging to different eigenvalues. This allows us to show that the vector space of physical states is, in fact, a Hilbert space. The case of two particles is explicitly considered and the Cauchy problem of physical wave function illustrated. The problem of a spin-1/2 particle interacting with a spin-zero particle is considered and a new equation is proposed for two spin-1/2 particles interacting via the most general form of interaction possible. The restrictions due to Hermiticity, space inversion and time reversal invariance are also considered.

Keywords. Relativistic quantum mechanics; directly interacting particles; spin-zero; spin-half.

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1. Introduction

There has recently been considerable interest in the classical and quantum theories of directly interacting relativistic particles (For extensive references, see Longhi and Lusanna 1986). We have considered elsewhere (Thakur 1991) the relativistic classical mechanics of directly interacting particles. The basic elements of this quantum mechanics are well-known at least for spin-zero particles. To the particles we associate mutually commuting mass-shell constraints, one for each particle. These constraints are operators which operate in some vector space. The physically acceptable vectors in this vector space are those vectors which are annihilated by the operator mass-shell constraints. It is reasonable to assume that the operator mass shell constraints have purely continuous eigenvalues on the whole real line; otherwise, the spectrum will depend on the mass chosen. (This is certainly true for free particles). As a consequence, physical vectors are not normalizable in the L^2-norm. This problem has been looked at by various workers (Droz Vincent 1984; Rizov *et al* 1985; Sazdjian 1986a, 1988). Nevertheless, we believe that the problem deserves a fresh look because the solutions previously proposed are *ad hoc* since they are not directly related to the gauge invariance of a theory with commuting constraints. They are also limited in scope both in the number of particles considered ($N = 2$) and in some cases, in the limited nature of potentials considered (they do not allow for the fact that relativistic potentials can depend on the square of total four momentum of the interacting particles). The method used by Sazdjian (1986, 1988) based on the construction of tensorial conserved currents is in our opinion not very suitable because it leads to a rather complicated condition on the reality of expectation values. With the ordinary scalar product, every

Hermitian operator has a real expectation value. This is not true of the scalar product introduced by Sazdjian which, in addition, is not manifestly positive definite. We give a general way of constructing the vector space of physical states with a relativistically invariant, positive definite scalar product which is such that physical eigenstates of total four momentum with different eigenvalues are orthogonal despite the generality of potentials considered. Our method is analogous to the Faddeev-Popov trick (Faddeev and Popov 1967) of factoring out the integral over the gauge group in the path integral method. As a consequence, we are able to prove that the physical states constitute a Hilbert space with the scalar product we have considered. We construct the S-matrix in the "interaction picture" and show that a unitary S-matrix is obtained with a Hermitian interaction potential. It should, however, be pointed out that the S-matrix restricted to the sector of fixed number of particles need not be unitary since particles can be produced and annihilated. Our methods must be further generalized if one has to describe a regime where such processes are important.

We then consider the case of one spin-zero particle interacting with one spin-1/2 particle. This problem shows the difficulties in the treatment of spin. Even for free particles and even with an indefinite metric, the requirement of pseudohermiticity of Dirac-type constraints is incompatible with the requirement of a positive definite scalar product. We, therefore, propose that for spin-1/2 particles the square of Dirac type constraint be considered the generator of gauge group rather than the constraint itself. One amusing consequence of our construction is that there are no *zitterbewegung* type effects for free particles. The constraint itself then restricts the initial states.

We then consider the case of two spin-1/2 particles. This problem has been considered by Crater and Van Alstine (Crater *et al* 1987, 1988) and by Sazdjian (Sazdjian 1986) but we believe that their construction is not general enough (Crater and Van Alstine are unable to accommodate sufficiently many form factors and Sazdjian is unable to accommodate strong compatibility that we prefer). Also the supersymmetric methods used by Crater and Van Alstine are perhaps unfamiliar in this context. In contrast, we use the ordinary operator algebra, and are able to construct the most general constraint possible and give conditions for Hermiticity, space inversion and time reversal invariance. One intriguing aspect of our construction is the emergence of everywhere bounded potentials signalled by the appearance of hyperbolic tangent in the form of the potential. In all this both gauge invariance and separability play a crucial role. The problem was considered earlier without separability (Thakur 1986). There is no question, however, that separability is a very desirable requirement.

The plan of the paper is the following. In § 2 we consider the quantum mechanics of spinless particles in direct interaction. The chief new element in this section is the definition of a scalar product of two physical state vectors which is positive definite, relativistically invariant and maintains the orthogonality of eigenstates of total four momentum belonging to different eigenvalues. In § 3 we consider the explicit form of the metric operator for the two-body case. In § 4 we consider the Cauchy problem for two interacting particles (for simplicity we have only considered equal mass particles; the general case is analogous). In § 5 we construct the scattering matrix in the interaction representation. In § 6 we consider a spin-1/2 particle interacting with a spin-zero particle and in § 7 we consider the problem of two spin-1/2 particles interacting via the most general two-body interaction possible subject to the usual discrete symmetries of space inversion and time reversal.

Throughout this paper, we use time favoured metric ($g^{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$). The basic commutation relation is $[q^{\mu}_{a}, p^{\nu}_{b}] = -i\delta_{ab}g^{\mu\nu}$.

2. Quantum mechanics of N spinless particles

To quantize any classical system, we have to make the usual transition from classical dynamical variables to operators. To each particle we associate a Hermitian, Poincare invariant and separable mass-shell constraint operator K_a

$$K_a = m_a^2 - p_a^2 + V_a \tag{1}$$

satisfying the commutation relation

$$[K_a, K_b] = 0. \tag{2}$$

It is assumed that the action of the constraints, and more particularly, of the potential V_a on the test functions belonging to $S(R^{4N})$, the space of infinitely differentiable functions of fast decrease, in known. Explicit solutions of (2) are known only for systems where each particle interacts with atmost one other particle (we call such systems monogamous). Some positivity constraints must also be imposed which ensure in the classical limit $p_a^0 > 0$, $p_a^2 > 0$ which in turn ensure $P^2 > 0$, $P^0 > 0$. We shall confine ourselves to such solutions even though it is not yet clear to us as to how to state these constraints generally in the unambiguous language of operators. The constraint (1) is imposed on the physical states

$$K_a \Psi = 0 \tag{3}$$

in the sense that the ordinary scalar product

$$(K_a \Phi, \Psi) = 0 \tag{4}$$

for all $\Phi \in S$. The set of physical states would form a subspace H of the dual S^* of S consisting of all linear functionals on S. The acceptable solutions of (4) must further be restricted by the need for a probability interpretation. In other words, we would like to define a scalar product and hence a norm for H. This scalar product must be positive over H, be relativistically invariant and should keep orthogonal eigenstates of total four momentum belonging to different eigenvalues. For free particles, the mass shell constraint operator has purely continuous eigenvalues in the range $-\infty$ to $+\infty$ and we shall confine ourselves to those interaction potentials V_a which preserve this property of the mass shell constraint operator. Ordinary or $L^2(R^{4N})$ norm of vectors Ψ satisfying (4) is then infinite ($+\infty$ to be precise). It is convenient to think of the (ordinary) scalar product as the (ordinary) matrix element of the gauge invariant operator $\mathbb{1}$ (unity). We introduce the resolution of the identity (*with the operator θ independent of the α's*),

$$\mathbb{1} = \int \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_N \exp\left(i\sum_i \alpha_i K_i\right)\theta \exp\left(-i\sum_i \alpha_i K_i\right). \tag{5}$$

(It is sufficient that this equation holds in the weak sense of matrix elements between physical states. We shall not need to define θ for unphysical states). Then for physical states Ψ, Φ satisfying (3), we have

$$(\Psi, \Phi) = \int d\alpha_1 \cdots d\alpha_N (\Psi, \theta\Phi). \tag{6}$$

By assumption, the observables are all gauge invariant operators. Every gauge invariant observable has the form

$$\hat{f} = \int d\alpha_1 \cdots d\alpha_N \exp\left(i \sum_i \alpha_i K_i \right) f \exp\left(-i \sum_i \alpha_i K_i \right) \tag{7}$$

provided the integral exists. The matrix elements of $\hat{f}$ then have the form

$$(\Psi, \hat{f}\Phi) = \int d\alpha_1 \cdots d\alpha_N (\Psi, f\Phi). \tag{8}$$

Now, the observable quantities always involve ratios of matrix elements. For instance, for calculating the expectation value of $\hat{f}$ we need $(\Psi, \hat{f}\Psi)/(\Psi, \Psi)$; for calculating transition matrix we need

$$(\Psi, \hat{f}\Phi)/[(\Psi, \Psi)(\Phi, \Phi)]^{\frac{1}{2}}. \tag{9}$$

These expressions are ill-defined because both the numerator and denominator diverge. We shall interpret them by using (6) and (8) and cancelling the infinite factor $\int d\alpha_1 \cdots d\alpha_N$. This leads to the definition of H as the set of states satisfying (3) for which

$$((\Psi, \Psi)) \equiv (\Psi, \theta\Psi) < \infty. \tag{10}$$

It is easy to understand why this scalar product defines a positive definite norm over H. If the physical norm $((\Psi, \Psi))$ were zero, the mathematical norm (Ψ, Ψ) would also be zero and this will imply that Ψ is a null vector. Similarly a negative value for $(\Psi, \theta\Psi)$ would mean that (Ψ, Ψ) is minus infinity which is absurd. We can also show the positive definite nature of $((\Psi, \Psi))$ directly. Introduce a basis $|\lambda_1 \cdots \lambda_N; \omega\rangle$ labelled by the eigenvalues of K_a and other necessary operators Ω which commute with K_a and among themselves. Equation (5) can then be written as

$$\begin{aligned}(2\pi)^N \delta(\lambda'_1 - \lambda_1) \cdots \delta(\lambda'_N - \lambda_N) \langle \lambda_1 \cdots \lambda_N, \omega' | \theta | \lambda_1 \cdots \lambda_N, \omega \rangle \\ = \delta(\lambda'_1 - \lambda_1) \cdots \delta(\lambda'_N - \lambda_N) \delta(\omega', \omega).\end{aligned} \tag{11}$$

This gives on factoring out the delta functions

$$\langle \lambda_1 \cdots \lambda_N, \omega' | \theta | \lambda_1 \cdots \lambda_N, \omega \rangle = (2\pi)^{-N} \delta(\omega', \omega). \tag{12}$$

Now let

$$|\Psi\rangle = \int c(\omega) |0; \omega\rangle, \quad |\Phi\rangle = \int d(\omega) |0; \omega\rangle. \tag{13}$$

Then

$$(\Psi, \theta\Phi) = \int (2\pi)^{-N} c^*(\omega) d(\omega) \tag{14}$$

and this leads to positive definite norm for physical states. Another desirable property of H is completeness and we would like to indicate the proof of completeness of H. Let Ψ_n be a strongly Cauchy sequence of physical vectors in H in the norm of (10):

$$\lim_{n,m\to\infty} (\Psi_n - \Psi_m, \theta(\Psi_n - \Psi_m)) = 0 \tag{15}$$

which implies

$$\lim_{n,m\to\infty} (\Psi_n - \Psi_m) = \text{null vector} \tag{16}$$

and hence Ψ_n is also a weak Cauchy sequence in S^*. By the completeness of S^* (Bohm, unpublished), there is a vector Ψ in S^* to which Ψ_n must converge. Also

$$(K_i\Psi, f) = (\Psi, K_i f) = \lim_{n\to\infty} (\Psi_n, K_i f) = 0 \tag{17}$$

for every $f \in S$, so Ψ satisfies (3). Finally, $(\Psi_n, \theta\Psi_n)$ is easily shown to converge to $(\Psi, \theta\Psi)$ and since the former is bounded by uniform boundedness principle (Amrein *et al* 1977), the latter must also be bounded. These considerations show that H is, in fact, a Hilbert space.

Let us go back to (5) and discuss the question of uniqueness of θ and the scalar product. It is easy to see that θ is not unique. First we show that if Q_a are canonically conjugate to K_a with $[Q_a, K_b] = i\delta_{ab}$ and $[Q_a, Q_b] = 0$ then

$$\theta = \delta(Q_1) \cdots \delta(Q_N) \tag{18}$$

is a standard solution. As can be checked, this is consistent with (5) because $\exp(i\alpha_1 K_1)\delta(Q_1)\exp(-i\alpha_1 K_1) = \delta(Q_1 + \alpha_1)$ and $\int \delta(Q_1 + \alpha_1)\mathrm{d}\alpha_1 = 1$. A trickier problem is the need to reconcile $K_i|\psi\rangle = 0$ and $[Q_a, K_b] = i\delta_{ab}$ with $\langle\psi|[Q_a, K_b]|\psi\rangle = 0(?)$ that one obtains by naively treating K_b as a Hermitian operator. The partial integration needed to verify $(\Psi, K_a Q_a \Psi) = (K_a\Psi, Q_a\Psi)$ leaves out an infinite integrated part and is illegitimate. Nevertheless it is permissible to treat $\exp(i\Sigma_i \alpha_i K_i)$ as a unitary operator and go from (5) to (6) because in this the integrated part will vanish since θ is localized in Q_a. Delicacy is called for when using these unbounded operators and we have usually checked our results in special cases. To generate more general solutions we notice that we have a whole class of solutions $\theta(f)$ given by

$$\theta(f) = \int_{-\infty}^{\infty} \mathrm{d}\beta_1 \cdots \mathrm{d}\beta_N f(\beta_1 \cdots \beta_N) \exp\left(i\sum_i \beta_i K_i\right)\theta \exp\left(-i\sum_i \beta_i K_i\right) \tag{19}$$

provided only

$$\int_{-\infty}^{\infty} \mathrm{d}\beta_1 \cdots \mathrm{d}\beta_N f(\beta_1 \cdots \beta_N) = 1. \tag{19a}$$

The validity of (19) is easily checked by a shift of integration variable $\alpha_i \to \alpha_i + \beta_i$. However, $\theta(f)$ defines the same scalar product as θ for physical states as is easily checked. For this reason it is not necessary to worry about the separability of θ. It is also seen that if θ is one solution then so is

$$\theta' = U^\dagger(a, \Lambda)\theta U(a, \Lambda) \tag{20}$$

because of the Poincare invariance of mass shell constraints (1). Equation (20) implies

$$(U\Psi, \theta U\Phi) = (\Psi, \theta\Phi) \tag{21}$$

and expresses the relativistic invariance of the scalar product. As a consequence, if G is an infinitesimal generator of Poincare group

$$(\Psi, [G, \theta]\Phi) = 0. \tag{22}$$

Let $G = \varepsilon_\mu P^\mu$ and Ψ, Φ be eigenstates of P^μ belonging to different eigenvalues P_1^μ and P_2^μ respectively. Then

$$(P_1^\mu - P_2^\mu)(\Psi, \theta\Phi) = 0. \tag{23}$$

If follows that

$$(\Psi, \theta\Phi) \propto \delta(P_1 - P_2). \tag{24}$$

This result is perfectly general and in deriving this *we have not assumed the potentials to be independent* of P^2 (the square of the total four momentum).

3. Explicit form for the two-body case

We consider two particles described by coordinates q_a^μ and momenta p_a^μ. We go to the centre of mass (CM) and relative variables defined by

$$\left.\begin{aligned} P &= p_1 + p_2 \\ p &= \beta p_1 - \alpha p_2, \;\; \alpha + \beta = 1 \\ q &= q_1 - q_2 \end{aligned}\right\} \tag{25}$$

and a CM coordinate which we need not write down. Here α is a function of P^2 only. Explicitly, we choose (see below)

$$\alpha = \frac{1}{2}\left(1 + \frac{m_1^2 - m_2^2}{P^2}\right). \tag{26}$$

We consider the general two particle potential of the form

$$V_a = V = V(P^2, p_T, q_T) \tag{27}$$

where we have defined for any four vector, a,

$$a_T = a - a\cdot\hat{P}\hat{P} \tag{28}$$

with $\hat{P} = P/(P^2)^{1/2}$ being a unit four vector along the total four momentum. For potentials of the form (27), the difference of mass-shell constraints, eq (1), is independent of interaction and is proportional to $p\cdot P$. (This is the reason of the choice made in (26)). Our task then is to look for wavefunctions satisfying the constraints

$$(-b(P^2) - p_T^2 + V)\Psi = 0 \tag{29}$$

$$p\cdot\hat{P}\Psi = 0 \tag{30}$$

where

$$b(P^2) = \frac{1}{4}P^2 + \frac{(m_1^2 - m_2^2)^2}{4P^2} - \frac{m_1^2 + m_2^2}{2}. \tag{31}$$

Let us introduce a complete orthonormal set of eigenfunctions of $-p_T^2 + V$. In the CM frame these eigenfunctions satisfy

$$(\mathbf{p}^2 + V)\varphi_{nlm}^{P^2}(\mathbf{p}) = E_{nl}(P^2)\varphi_{nlm}^{P^2}(\mathbf{p}) \tag{32}$$

$$\int \frac{d^3p}{(2\pi)^3}\varphi_{nlm}^{P^2*}(\mathbf{p})\varphi_{n'l'm'}^{P^2}(\mathbf{p}) = \delta_{nn'}\delta_{ll'}\delta_{mm'} \tag{33}$$

$$\sum_{nlm}\varphi_{nlm}^{P^2}(\mathbf{p})\varphi_{nlm}^{P^2*}(\mathbf{p}') = (2\pi)^3\delta(\mathbf{p}-\mathbf{p}'). \tag{34}$$

We shall now construct an eigenfunction of $-p_T^2 + V$ in an arbitrary frame using relativistically invariant construction. Let $L(P, \mathring{P})$ be the standard boost operator from the CM frame where $\mathring{P}^\mu = ((P^2)^{1/2}, 0, 0, 0)$ to the actual frame where the total four momentum is P^μ. Explicitly, L can be taken as (Longhi and Lusanna 1986)

$$L^\mu_\nu = \delta^\mu_\nu + 2\frac{P^\mu \mathring{P}_\nu}{M^2} - \frac{(P^\mu + \mathring{P}^\mu)(P_\nu + \mathring{P}_\nu)}{(P\cdot\mathring{P} + \mathring{P}^2)} \tag{35}$$

where $M^2 = P^2 = \mathring{P}^2$. We then define the polarization vectors,

$$e^\mu_a(P) = L^\mu_a \tag{36}$$

for $a = 1, 2, 3$. Then an eigenfunction of $-p_T^2 + V$ in an arbitrary frame is $\varphi_{nlm}^{P^2}(-e_a\cdot p)$ and a physical state with total four momentum k and internal quantum numbers n, l, m is up to an overall constant factor

$$\Psi_{knlm} = \delta^4(P-k)\varphi_{nlm}^{k^2}(-e_a\cdot p)\delta(p\cdot\hat{k}) \tag{37}$$

where k^2 is fixed by the constraint

$$E_{nl}(k^2) - b(k^2) = 0. \tag{38}$$

When E_{nl} does not depend on k^2 this equation becomes a quadratic in k^2. Our assumed positivity constraints require us to take

$$(k^2)^{1/2} = [m_1^2 + E_{nl}(k^2)]^{1/2} + [m_2^2 + E_{nl}(k^2)]^{1/2}. \tag{39}$$

We shall assume that this equation has a unique solution giving a positive $(k^2)^{1/2}$ for each n and l.

We shall now try to explicitly obtain θ as an integral operator. We write

$$\theta(P, p; P', p') = \sum_{nlm} A_{nl}(P^2, p'^2)\varphi_{nlm}^{P^2}(p_T)\varphi_{nlm}^{*P'^2}(p'_T)\delta(\hat{P} - \hat{P}') \tag{40}$$

where the three-dimensional delta function $\delta(\hat{P} - \hat{P}')$ is defined by the following construction. For time-like P we write

$$P_0 = (P^2)^{1/2}\cosh\chi, \quad |\mathbf{P}| = (P^2)^{1/2}\sinh\chi \tag{41}$$

with the components of **P** being obtained by the usual construction in terms of polar angles θ, ϕ. We then find

$$d^4 P = \tfrac{1}{2} P^2 \, dP^2 \, d\hat{P} \tag{42}$$

where

$$d\hat{P} = \sinh^2\chi \sin\theta \, d\chi \, d\theta \, d\phi. \tag{43}$$

Consequently, we can write

$$\delta(P - P') = \frac{2}{P^2} \delta(P^2 - P'^2) \delta(\hat{P} - \hat{P}'). \tag{44}$$

Let us now calculate

$$\theta\Psi = \int \frac{d^4 P' d^4 p'}{(2\pi)^8} \theta(P, p; P', p') \Psi(P', p'). \tag{45}$$

We use (33) in the convariant form

$$\int \frac{d^4 p'}{(2\pi)^3} \varphi^{P'^2 *}_{n'l'm'}(p'_T) \varphi^{P'^2}_{nlm}(p'_T) \delta(p' \cdot \hat{P}') = \delta_{nn'} \delta_{ll'} \delta_{mm'}. \tag{46}$$

We get

$$\theta\Psi = (2\pi)^{-5} \delta(\hat{P} - \hat{k}) A_{nl}(P^2, k^2) \varphi^{P^2}_{nlm}(p_T). \tag{47}$$

We evaluate A_{nl} by using (5) to get

$$A_{nl}(P^2, P'^2) = v_{nl}(P^2) v_{nl}(P'^2) \tag{48}$$

where (a prime after a function denotes derivative with respect to its argument)

$$v_{nl}(P^2) = 2(2\pi)^{3/2} (P^2)^{-1/4} |b'(P^2) - E'_{nl}(P^2)|^{1/2}. \tag{49}$$

We can then normalize (37). The normalized eigenfunctions are

$$\begin{aligned} \tilde{\Psi}_{knlm}(P, p) = {} & (32\pi^7)^{1/2} (k^2)^{1/4} |b'(k^2) - E'_{nl}(k^2)|^{-1/2} \\ & \times \delta^4(P - k) \varphi^{P^2}_{nlm}(-e_a(k) \cdot p) \delta(p \cdot \hat{k}). \end{aligned} \tag{50}$$

These satisfy

$$((\tilde{\Psi}_{knlm}, \tilde{\Psi}_{k'n'l'm'})) = \delta(\hat{k} - \hat{k}') \delta_{nn'} \delta_{ll'} \delta_{mm'}. \tag{51}$$

The vector $\tilde{\Psi}_{knlm}$ does not, strictly speaking, belong to H. Elements of H have the form

$$\tilde{\Psi} = \int \frac{d\hat{k}}{(2\pi)^{3/2}} \sum_{nlm} g_{nlm}(\hat{k}) \tilde{\Psi}_{knlm}(P, p) \tag{52}$$

with

$$((\tilde{\Psi}, \tilde{\Psi})) = \int \frac{d\hat{k}}{(2\pi)^3} \sum_{nlm} |g_{nlm}(\hat{k})|^2 < \infty. \tag{53}$$

4. Cauchy problem

We have discussed so far the wavefunction in momentum representation. By the usual transformation theory we can obtain the wavefunction in coordinate space. For simplicity we consider the equal mass case when $\alpha = \beta = 1/2$ in (26). In this section we shall denote the coordinates by x_1, x_2 with $X = (1/2)\,(x_1 + x_2)$ and $x = x_1 - x_2$ being conjugate to P and p respectively. The wavefunction in coordinate space corresponding to (52) is

$$\tilde{\Psi}(x_1, x_2) = \int \frac{\mathrm{d}\hat{k}}{(2\pi)^{3/2}} \sum_{nlm} g_{nlm}(\hat{k}) C(k_{nl}^2) \int \frac{\mathrm{d}^4 P \mathrm{d}^4 p}{(2\pi)^8} \exp(-iP\cdot X - ip\cdot x) \times \delta(P-k)\delta(p\cdot\hat{P})\varphi_{nlm}^{P^2}(-e_a\cdot p). \tag{54}$$

(We shall use the same letter to denote the same wavefunction in either coordinate or momentum representation). In (54) we have written

$$C(k_{nl}^2) = (32\pi^7)^{1/2}(k_{nl}^2)^{1/4}|b'(k_{nl}^2) - E'_{nl}(k_{nl}^2)|^{-1/2}. \tag{55}$$

We define

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^3} \varphi_{nlm}^{k^2}(-e_a(k)\cdot p)\exp(-ip\cdot x)\delta(p\cdot\hat{k}) = \varphi_{nlm}^{k^2}(-e_a\cdot x). \tag{56}$$

It is orthonormalized with

$$\int \mathrm{d}^4 x\,\delta(k\cdot x)\varphi_{nlm}^{*}(-e_a\cdot x)\varphi_{n'l'm'}(-e_a\cdot x) = \delta_{nn'}\delta_{ll'}\delta_{mm'} \tag{57}$$

Thus

$$\tilde{\Psi}(x_1, x_2) = \int \frac{\mathrm{d}\hat{k}}{(2\pi)^{3/2}} \sum_{nlm} g_{nlm}(\hat{k}) C(k_{nl}^2) \frac{\exp(-ik\cdot X)}{(2\pi)^5} \varphi_{nlm}^{k^2}(-e_a(k)\cdot x). \tag{58}$$

The Cauchy problem is the reconstruction of the full wave function $\tilde{\Psi}(x_1, x_2)$ if $\tilde{\Psi}$ is known at the instant $x_1^0 = x_2^0 = 0$, say. It has rightly been emphasized (Longhi and Lusanna 1986) that this is the key to the existence of dynamics in the true sense. The conclusion there was that Cauchy problem can be solved only in particular frames. This is not the case with the present theory and Cauchy problem is well posed as we now show. Let the wave function at $x_1^0 = x_2^0 = 0$ be known to be $\Phi(\mathbf{X}, \mathbf{x})$. Then

$$\Phi(\mathbf{X}, \mathbf{x}) = \int \frac{\mathrm{d}\hat{k}}{(2\pi)^{3/2}} \sum_{nlm} g_{nlm}(\hat{k}) C(k_{nl}^2) \frac{\exp(i\mathbf{k}\cdot\mathbf{X})}{(2\pi)^5} \varphi_{nlm}^{k^2}(\mathbf{e}_a\cdot\mathbf{x}). \tag{59}$$

We Fourier transform both sides with respect to X and write

$$\Phi(\mathbf{k}', \mathbf{x}) = \int \mathrm{d}\mathbf{X} \exp(-i\mathbf{k}'\cdot\mathbf{X})\Phi(\mathbf{X}, \mathbf{x}). \tag{60}$$

We also use

$$\delta^3(\mathbf{P} - \mathbf{k}) = \frac{1}{\omega k^2}\delta(\hat{P} - \hat{k}) \tag{61}$$

where $\omega = k^0 = (k^2 + \mathbf{k}^2)^{1/2}$. We then get

$$\Phi(\mathbf{k}, x) = \frac{1}{(2\pi)^{7/2}} \sum_{nlm} g_{nlm}(\hat{k}) C(k_{nl}^2) \frac{1}{\omega_{nl} k_{nl}^2} \varphi_{nlm}(\mathbf{e}_a(\mathbf{k}) \cdot \mathbf{x}). \tag{62}$$

Finally, we use

$$\int d^3 x \varphi^*_{n'l'm'}(\mathbf{e}_a \cdot \mathbf{x}) \varphi_{nlm}(\mathbf{e}_a \cdot \mathbf{x}) = \frac{1}{|\det e_a^b|} \delta_{nn'} \delta_{ll'} \delta_{mm'} \tag{63}$$

to get g_{nlm} and the reconstruct $\tilde{\Psi}(x_1, x_2)$. We have thus established that the knowledge of $\tilde{\Psi}(x_1, x_2)$ at the instant $x_1^0 = x_2^0 = 0$ allows one to construct the full wavefunction at all times.

5. Scattering matrix

In setting up the constraint, eq. (1), we assumed it (and hence the interaction V_a) to be Hermitian. This is necessary for the present formalism because the integrals such as the one occurring in (5) are unlikely, without further restrictions, to exist for non-Hermitian constraints. We would now show that a Hermitian V_a leads, at least for two bodies, to a unitary S-matrix. This simple result is not obvious by nonrelativistic intuition because the relativistic potential V_a also depends on total four-momentum squared. Also, we shall show that as in nonrelativistic mechanics and unlike in field theory, scattering takes place only via positive energy intermediate states, a result that remains valid when spin is included.

Let us go back to (1) and (27). We look for a solution of the equations

$$(m_1^2 - p_1^2 + V)\Psi = 0 \tag{64}$$

$$(m_a^2 - p_2^2 + V)\Psi = 0. \tag{65}$$

Equation (64) can be converted into an integral equation (in coordinate space)

$$\Psi = \Psi_0 + \frac{1}{p_1^2 - m_1^2 + i\varepsilon} V\Psi \tag{66}$$

where Ψ_0 is the free particle wave function and (65) is equivalent to the differential equation

$$(m_1^2 - p_1^2 - m_2^2 + p_2^2)\Psi = 0. \tag{67}$$

Although the Green's function in front of V in (66) looks like the Feynman prescription, it is different in actuality because the negative energy pole does not contribute to scattering. To see this note that with V defined by (27) $p_1 \cdot \hat{P}$ and $p_2 \cdot \hat{P}$ both commute with V and are, therefore, constants of motion. Since these are positive for the incident wavefunction Ψ_0 which obeys (64) and (65) with $V = 0$, they remain positive after scattering and the negative energy pole cannot contribute. Let us write (66) as

$$\Psi(\alpha) = \Psi_0(\alpha) + \frac{\exp(-i\alpha(p_1^2 - m_1^2 + i\varepsilon))}{p_1^2 - m_1^2 + i\varepsilon} V\Psi \tag{68}$$

where we shall write

$$f(\alpha) = \exp(-i\alpha(p_1^2 - m_1^2))f\exp(i\alpha(p_1^2 - m_1^2)) \tag{69}$$

for an operator f and

$$\Phi(\alpha) = \exp(-i\alpha(p_1^2 - m_1^2))\Phi \tag{70}$$

for a wavefunction. Thus we get

$$\Psi(\alpha) = \Psi_0 - i\int_{-\infty}^{\alpha} \mathrm{d}\alpha' \exp(-i\alpha'(p_1^2 - m_1^2 + i\varepsilon))\, V\Psi. \tag{71}$$

Equation (71) can be written as the integral equation

$$\Psi(\alpha) = \Psi_0 - i\int_{-\infty}^{\alpha} \mathrm{d}\alpha'\, V(\alpha')\Psi(\alpha'). \tag{72}$$

This representation shows that $\Psi(-\infty) = \Psi_0$ and $\Psi(\infty) = S\Psi_0$ are both free particle states and gives the S matrix in the interaction representation

$$S = 1 - i\int_{-\infty}^{\infty} \mathrm{d}\alpha'\, V(\alpha') + (-i)^2 \int_{-\infty}^{\infty} \mathrm{d}\alpha' \int_{-\infty}^{\alpha'} \mathrm{d}\alpha''\, V(\alpha')\, V(\alpha'') + \cdots \tag{73}$$

It shows that (a) the S-matrix is unitary if V is Hermitian (b) the S-matrix is free particle gauge invariant so that the elements of the S-matrix with respect to free particle states can be defined by the method of (7) and (8). There is no reason, however, to expect the S-matrix restricted to the sector of a fixed number of particles to be unitary in the relativistic domain because of the possibility of production and annihilation processes, but this case is excluded from the domain of the present theory.

6. A spin-1/2 and spin-0 particle

For a spin-1/2 particle we adopt the Dirac trick of taking the square root of the spin-0 constraint. Thus we choose the free particle constraint to be

$$D_1 = \gamma_5(m_1 - \gamma\cdot p_1). \tag{74}$$

The overall sign is arbitrary and meaningless but the sign in front of $\gamma\cdot p_1$ is conventional for particles. (It will be a plus sign for antiparticles). Notice that D_1 is neither Hermitian nor skew-Hermitian but satisfies

$$(i\beta\gamma_5)D_1 = -D_1^\dagger(i\beta\gamma_5). \tag{75}$$

Thus D_1 is skew-Hermitian for indefinite metric $i\beta\gamma_5$. But $i\beta\gamma_5$ is not suitable at all as a metric operator as $\bar{u}(p)i\gamma_5 u(p)$ vanishes. For this reason we shall assume that gauge transformations are generated by D_1^2 and D_1 merely restricts the initial states. In the presence of interaction the modified constraint is taken to be

$$\mathscr{D}_1 = \gamma_5\left(m_1 - \gamma\cdot p_1 + \sum\right) \tag{76}$$

where Σ is Poincare invariant and satisfies

$$\left[\sum, p_1^2 - p_2^2\right] = 0. \tag{77}$$

We take the other constraint to be

$$K_2 = m_2^2 - p_2^2 - m_1^2 + p_1^2 + \mathscr{D}_1^2. \tag{78}$$

Obviously

$$[K_2, \mathscr{D}_1] = 0 \tag{79}$$

We now consider the different issues separately

6.1 *The S-matrix*

The integral equation for the physical wavefunction is

$$\Psi = \Psi_0 + \frac{(m_1 + \gamma \cdot p_1)}{p_1^2 - m_1^2 + i\varepsilon} \sum \Psi \tag{80}$$

With

$$(m_1^2 - p_1^2 - m_2^2 + p_2^2)\Psi = 0. \tag{81}$$

Comparing (80) with (66) and following the same reasoning as in § 5 we get the S-matrix

$$S = 1 - i \int_{-\infty}^{\infty} \mathrm{d}\alpha'(m_1 + \gamma \cdot p_1) \sum(\alpha') + (-i)^2 \int_{-\infty}^{\infty} \mathrm{d}\alpha' \int_{-\infty}^{\alpha'} \mathrm{d}\alpha''(m_1 + \gamma \cdot p_1) \sum(\alpha')(m_1 + \gamma \cdot p_1) \sum(\alpha'') + \cdots. \tag{82}$$

Because of the appearance of what is obviously a projection operator, scattering takes place only through positive energy spinor states. For this reason it is sufficient to consider, as is well-known, no more than two form factors in Σ.

6.2 *Scalar product*

Our next task is to define a relativistically invariant positive definite scalar product. We first consider free particles. We define the usual boost matrix, denoted here by A, by

$$A = A(\mathbf{P}) = \frac{1}{\sqrt{2M(E + M)}}(P \cdot \gamma\gamma_0 + M), \tag{83}$$

where $M = (P^2)^{1/2}, E = (\mathbf{P}^2 + M^2)^{1/2}$. We find

$$D_1 = A\gamma_5 \beta(m_1 \beta + \boldsymbol{\alpha} \cdot \mathring{\mathbf{p}}_1 - \mathring{p}_{10})A^{-1} \tag{84}$$

where $\mathring{p}_{10} = p_1 \cdot \hat{P}$ and $\mathring{\mathbf{p}}_1$ is the momentum of the first particle in the CM frame. The free particle wavefunctions which are momentum eigenstates have the form

$$\Psi_{k,q,\lambda}(P, p) = 16\pi^5 (k^2)^{1/4} |b'(k^2)|^{-1/2} \times \delta(P - k)(E_q/m_1)^{-1/2} \delta(p_T - q)\delta(p \cdot \hat{k}) A u_\lambda(\mathring{\mathbf{q}}) \tag{85}$$

where $\lambda = \pm\frac{1}{2}$ is a spin or helicity label. We take the metric operator to be

$$\eta = (AA^{+})^{-1}\theta_0 \tag{86}$$

where θ_0 is the metric operator for free spinless particles (cf. eq. 40)

$$\langle P, p|\theta_0|P', p'\rangle = 4(2\pi)^6|b'(P^2)|^{1/2}|b'(P'^2)|^{1/2} \times (P^2P'^2)^{-1/4}\delta(p_T - p'_T)\delta(\hat{P} - \hat{P}'). \tag{87}$$

We easily check that

$$((\Psi_{k',q',\lambda'}, \Psi_{k,q,\lambda})) = (\Psi_{k',q',\lambda'}, \eta\Psi_{k,q,\lambda}) = \delta(\hat{k} - \hat{k}')\delta^3(q_T - q'_T)\delta_{\lambda\lambda'} \tag{88}$$

(The delta function involving q is three-dimensional because q has only three independent components ($q\cdot k = q'\cdot k' = 0$).

6.3 *Hermiticity*

We write the full constraint (76) as in (84). We get

$$\mathscr{D}_1 = A\gamma_5\beta(m_1\beta + \boldsymbol{\alpha}\mathring{\mathbf{p}}_1 + \sigma - \mathring{p}_{10})A^{-1} \tag{89}$$

i.e. we have written, by definition of $\sigma, \gamma_5\Sigma = A\gamma_5\beta\sigma A^{-1}$. Let us write $\sigma = \sigma_+ + \sigma_-$ where

$$[\sigma_+, \beta\gamma_5] = 0 \tag{90}$$

$$\{\sigma_-, \beta\gamma_5\} = 0. \tag{91}$$

Squaring (89), we get

$$\mathscr{D}_1^2 = A[(m_1\beta + \boldsymbol{\alpha}_1\cdot\mathring{\mathbf{p}}_1 + \sigma_-)^2 - [\sigma_+, m_1\beta + \boldsymbol{\alpha}\cdot\mathring{\mathbf{p}}_1 + \sigma_-] - (\mathring{p}_{10} - \sigma_+)^2]A^{-1}. \tag{92}$$

This is pseudo-Hermitian if (1) $\sigma_+ = 0$, (ii) σ_- is Hermitian. A possible form for Σ is

$$\sum = V_0 - \{V_1, \gamma\cdot p_T\} \tag{93}$$

$$\sigma = \beta V_0 + \{V_1, \boldsymbol{\alpha}\cdot\mathring{\mathbf{p}}\} \tag{94}$$

where V_0 and V_1 are Hermitian scalar form factors which are identities in the spinor space.

6.4 *Space inversion*

For free particles, the space inverted constraint is

$$(D_1)_P = A(-\mathbf{P})\gamma_5(m_1 - \beta\boldsymbol{\alpha}\cdot\mathring{\mathbf{p}}_1 - \beta\mathring{p}_{10})A^{-1}(-\mathbf{P}). \tag{95}$$

We see that with

$$\mathscr{P} = A(\mathbf{P})\beta A^{-1}(-\mathbf{P}) \tag{96}$$

we find

$$\mathscr{P}(D_1)_P\mathscr{P}^{-1} = -D_1. \tag{97}$$

Under space inversion the interaction potential transforms as

$$\gamma_5\sum \rightarrow A(-\mathbf{P})\gamma_5[V_0 - \{V_1, \beta\boldsymbol{\alpha}\cdot\mathring{\mathbf{p}}\}]A^{-1}(-\mathbf{P}) \equiv \left(\gamma_5\sum\right)_P. \tag{98}$$

Indeed we find

$$\mathscr{P}\left(\gamma_5\sum\right)_P\mathscr{P}^{-1} = -\gamma_5\sum. \tag{99}$$

6.5 *Time reversal*

For free particles, under time reversal, which must be an antilinear operation, we have

$$D_1 \to (D_1)_T = A^*(-\mathbf{P})\gamma_5(m_1 - \beta\boldsymbol{\alpha}^*\cdot\hat{\mathbf{p}}_1 - \beta\hat{p}_{10})A^{*-1}(-\mathbf{P}) \tag{100}$$

where it is understood that we are working in a representation in which β and γ_5 are real. We look for an operator T such that

$$T(D_1)_T T^{-1} = (+\text{ or }-)D_1 \tag{101}$$

we find

$$T = A(\mathbf{P})(i\sigma_y)A^{*-1}(-\mathbf{P}) \tag{102}$$

and with this T,

$$T(D_1)_T T^{-1} = D_1. \tag{103}$$

Considering now the case of interacting particles we can check that time reversal invariance requires V_0 and V_1 in (94) to be real.

7. Two spin-1/2 particles

The constraint formalism acquires its full complexity when one considers two interacting spin-1/2 particles. We begin by writing the constraint for free spin-1/2 particles.

$$D_1 = \gamma_{51}(m_1 - \gamma_1\cdot p_1) \tag{104}$$

$$D_2 = \gamma_{52}(m_2 - \gamma_2\cdot p_2). \tag{105}$$

There are two distinct general ways of generating commuting constraints by operations performed on free particle constraints. One is the canonical flow, this would lead to unitarily equivalent constraints and hence to no scattering. We do not consider this. The other is an anticanonical flow which we define by

$$\frac{\mathrm{d}\mathscr{D}_1}{\mathrm{d}\lambda} = \{X, \mathscr{D}_2\} \tag{106}$$

$$\frac{\mathrm{d}\mathscr{D}_2}{\mathrm{d}\lambda} = \{X, \mathscr{D}_1\} \tag{107}$$

where λ is a parameter denoting the strength of interaction ($\lambda = 0$ means free particle) and X is a λ-independent manifestly covariant kernel. From (106) and (107) we see that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}[\mathscr{D}_1, \mathscr{D}_2] = [\mathscr{D}_1^2 - \mathscr{D}_2^2, X]. \tag{108}$$

Let $D_a = \mathscr{D}_a$ for $\lambda = 0$ and let us assume that

$$[D_1^2 - D_2^2, X] = [p_2^2 - p_1^2, X] = 0 \tag{109}$$

finally

$$\frac{d}{d\lambda}[\mathscr{D}_1^2 - \mathscr{D}_2^2] = \{[\mathscr{D}_1, \mathscr{D}_2], X\}. \tag{110}$$

Equations (108) and (110) together show that $[\mathscr{D}_1, \mathscr{D}_2]$ obeys a second order differential equation and, therefore, vanishes because the solution of a second order differential equation vanishes identically everywhere if it vanishes, together with its derivative, at any point. Actually (106) and (107) can be solved directly by forming sums and differences and integrating the resulting first order differential equation. Finally, putting $\lambda = 1$, we get

$$2\mathscr{D}_1 = e^X(D_1 + D_2)e^X + e^{-X}(D_1 - D_2)e^{-X} \tag{111}$$

$$2\mathscr{D}_2 = e^X(D_1 + D_2)e^X - e^{-X}(D_1 - D_2)e^{-X}. \tag{112}$$

The physical wavefunction satisfies

$$(D_1 + D_2)e^X\Psi = 0 \tag{113}$$

$$(D_1 - D_2)e^{-X}\Psi = 0. \tag{114}$$

These are equivalent to

$$(D_1 \cosh X + D_2 \sinh X)\Psi = 0 \tag{115}$$

$$(D_1 \sinh X + D_2 \cosh X)\Psi = 0 \tag{116}$$

or defining $\Phi = \cosh X\Psi$, we get

$$(D_1 + D_2 \tanh X)\Phi = 0 \tag{117}$$

$$(D_1 \tanh X + D_2)\Phi = 0. \tag{118}$$

Notice the hyperbolic tangent appearing in front of X. The effective interaction is automatically bounded if X is real. Otherwise this equation bears some resemblance to equations for two spin-1/2 particles considered by Sazdjian (Sazdjian 1986).

7.1 *The S-matrix*

We write (117) and (118) as the integral equation

$$\Phi = \Phi_0 + \frac{D_1 D_2}{p_1^2 - m_1^2 + i\varepsilon} \tanh X\Phi \tag{119}$$

and the differential equation

$$(D_1^2 - D_2^2)\Phi = 0. \tag{120}$$

Comparing with the spinless case, we see that this leads to the S-matrix

$$S = 1 - i\int_{-\infty}^{\infty} d\alpha_1'(m_1 + \gamma_1\cdot p_1)(m_2 + \gamma_2\cdot p_2)\gamma_{51}\gamma_{52} \tanh X(\alpha_1')$$

$$+(-i)^2\int_{-\infty}^{\infty}d\alpha'_1\int_{-\infty}^{\alpha'_1}d\alpha''_1(m_1+\gamma_1\cdot p_1)(m_2+\gamma_2\cdot p_2)\gamma_{51}\gamma_{52}$$
$$\times\tanh X(\alpha'_1)(m_1+\gamma_1\cdot p_1)(m_2+\gamma_2\cdot p_2)\gamma_{51}\gamma_{52}\tanh X(\alpha''_1)+\cdots \quad (121)$$

We see that contribution to the S-matrix arises only from positive energy spinor states.

7.2 *Hermiticity*

We rewrite (111) and (112) as

$$\mathscr{D}_1=\cosh XD_1\cosh X+\sinh XD_1\sinh X+\cosh XD_2\sinh X$$
$$+\sinh XD_2\cosh X \quad (122)$$

and similarly for $\mathscr{D}_2$. We write

$$D_1=A(\mathbf{P})\gamma_{51}\beta_1(m_1\beta_1+\boldsymbol{\alpha}_1\cdot\hat{\mathbf{p}}_1-\hat{p}_{10})A^{-1}(\mathbf{P}) \quad (123)$$

where in this section

$$A(\mathbf{P})=A_1(\mathbf{P})A_2(\mathbf{P}) \quad (124)$$

and similarly for D_2. If we write

$$\mathscr{D}_1=A\gamma_{51}\beta_1(m_1\beta_1+\boldsymbol{\alpha}_1\cdot\hat{\mathbf{p}}_1-\hat{p}_{10}+Z_1)A^{-1} \quad (125)$$

then $\mathscr{D}_1^2$ is seen to be pseudo-Hermitian (i.e. $A^{-1}\mathscr{D}_1^2A$ is Hermitian) if (i) $Z_1\gamma_{51}\beta_1=-\gamma_{51}\beta_1Z_1$ and (ii) Z_1 is Hermitian. Let us now look for sufficient conditions on X that ensure the pseudo-Hermiticity of $\mathscr{D}_1^2$. Let

$$X=A(\mathbf{P})\gamma_{51}\gamma_{52}\beta_1\beta_2\,YA^{-1}(\mathbf{P}) \quad (126)$$

and let

$$\{Y,\gamma_{51}\beta_1\}=\{Y,\gamma_{52}\beta_2\}=0. \quad (127)$$

This ensures

$$[Y,\gamma_{51}\beta_1\gamma_{52}\beta_2]=0 \quad (128)$$

and

$$[Y^2,\gamma_{51}\beta_1]=[Y^2,\gamma_{52}\beta_2]=0 \quad (129)$$

we then get on working out (122)

$$\mathscr{D}_1=A\gamma_{51}\beta_1[\cosh Y(m_1\beta_1+\boldsymbol{\alpha}_1\cdot\hat{\mathbf{p}}_1)\cosh Y$$
$$+\sinh Y(m_1\beta_1+\boldsymbol{\alpha}_1\cdot\hat{\mathbf{p}}_1)\sinh Y-\hat{p}_{10}$$
$$+\sinh Y(m_2\beta_2+\boldsymbol{\alpha}_2\cdot\hat{\mathbf{p}}_2)\cosh Y$$
$$+\cosh Y(m_2\beta_2+\boldsymbol{\alpha}_2\cdot\hat{\mathbf{p}}_2)\sinh Y]A^{-1}. \quad (130)$$

This has the right form (cf. (125)) if Y is Hermitian. We shall call a constraint like (130) Dirac form of constraint. There is another form of constraint which we call Schrödinger-Pauli form of constraint in which with the help of projection operator

Table 1. Six suggested forms of invariant interaction terms in the kernel X. The form factors U_1 through U_6 are Hermitian and real. Both space inversion and time reversal invariance are valid. For identical particles only the symmetrical combination of (5) and (6) can occur. $l_{\mu\nu}$ is the covariant generalization of orbital angular momentum operator.

Number	X	Y
1	$U_1\gamma_{51}\gamma_{52}$	$U_1\beta_1\beta_2$
2	$U_2\gamma_{51}\gamma_{52}\gamma^{\mu}_{1T}\gamma_{2T\mu}$	$-U_2\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2$
3	$U_3\gamma_{51}\gamma_{52}\sigma^{\mu\nu}_{1T}\sigma_{2T\mu\nu}$	$2U_3\beta_1\beta_2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$
4	$U_4\gamma_1\cdot\hat{P}\gamma_2\cdot\hat{P}$	$U_4\gamma_{51}\gamma_{52}$
5	$U_5\gamma_{51}\gamma_{52}\sigma_{1\mu\nu}l^{\mu\nu}$	$2U_5\beta_1\beta_2\boldsymbol{\sigma}_1\cdot\mathbf{l}$
6	$U_6\gamma_{51}\gamma_{52}\sigma_{2\mu\nu}l^{\mu\nu}$	$2U_6\beta_1\beta_2\boldsymbol{\sigma}_2\cdot\mathbf{l}$

$(1+\beta_1)(1+\beta_2)/4$ in the Foldy-Wouthuysen representation we ensure $X^2 = 0$. This form may be more convenient for purely phenomenological applications because in this form the spin is decoupled from dynamics. Since the most general scattering amplitude with space inversion and time reversal invariance contains six independent invariant forms, we need the same number of terms in constructing X. A possible choice for these six invariant terms in X is given in table 1.

7.3 *Space inversion*

For free particles, under space inversion $D_1 \rightarrow (D_1)_P$ when $(D_1)_P$ is obtained by changing the sign of all three momenta. It is easy to check that

$$\mathscr{P}(D_i)\mathscr{P}^{-1} = -D_i \tag{131}$$

if

$$\mathscr{P} = A(\mathbf{P})\beta_1\beta_2 A^{-1}(-\mathbf{P}). \tag{132}$$

A sufficient condition for space inversion invariance in the presence of interaction is then

$$\beta_1\beta_2 Y = Y\beta_1\beta_2. \tag{133}$$

7.4 *Time reversal*

We get the time reversed constraint $(D_i)_T$ for free particles by changing the sign of all three momenta and taking the complex conjugate of the gamma matrices occurring in it. We look for an operator T such that

$$T(D_i)_T T^{-1} = D_i. \tag{134}$$

We find

$$T = A(\mathbf{P})i\sigma_{1y}i\sigma_{2y}A^{*-1}(-\mathbf{P}) \tag{135}$$

in the representation in which β_i, γ_{5i} are real. A sufficient condition for time reversal invariance in the presence of interaction is

$$T(Y)_T T^{-1} = Y. \tag{136}$$

8. Conclusions and comments

In the previous sections we have tried to construct, as far as possible, relativisitic quantum mechanics of spin-zero and spin-1/2 particles. The general outline of such a construction has been known for quite some time. What was missing was a proper construction of the vector space for these particles. Although there have previously been attempts in this direction, we found them unsatisfactory for reasons listed in the introduction. We believe that the method outlined in § 2 is the natural way of constructing the vector space with scalar product needed for a probability interpretation and is valid for any number of particles with any type of interaction. We have explicitly constructed the metric operator for two particles and discussed the Cauchy problem for two equal mass particles. Beyond this we have discussed the case of spin-1/2 and spin-zero particles in mutual interaction and finally the case of two interacting spin-1/2 particles with the most general possible interaction. This last work in § 7 is also new and together with § 2 represents the most important result of the present work. We have also looked at the external field problem and have constructed elsewhere (Thakur, 1991) the classical electromagnetic current to order *e*. However, we have not been able to set up the quantum mechanical current because of factor ordering ambiguities.

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