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Dirac equation in time dependent electric field and Robertson–Walker space-time

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Abstract. We show that the motion of a Dirac electron in a time dependent electromagnetic field can be considered as a motion in a dielectric medium with time dependent dielectric function. We find that this electromagnetic case is analogous to the description in Robertson-Walker (RW) space-time. We solve the Dirac equation is such a simulated space-time.

Keywords. Dirac equation; Robertson-Walker space-time; time-varying electric-field.

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1. Introduction

Recently there have been attempts to understand the origin of confined phase of QED (Cornwall and Tiktopoulos 1989; Cea Paolo 1989; Biswas and Das 1990b, d) from different angles. It is found that strong classical fields with certain kinds of time variations can create e^+e^- or $\gamma\gamma$ pair which are strongly resonant in energy and momentum. The confined phase, termed as confined phase of QED, produced in heavy ion collision process of heavy ions is supposed to arise from the interference of different amplitudes needed in the description of time dependent electric field produced by the heavy ions. The usual field-theoretic based explanation is unable to explain the origin of e^+e^- bound pair observed in U + Th, Th + Th, Th + Cm collisions. On the other hand, a time dependent gravitational background has an inbuilt structure of particle creation, a well posed problem in cosmology. The unusual time dependent strong electric field produced by heavy nuclei prompted many authors to investigate the solutions of Dirac equation in a time dependent gauge. Knowing the mode solutions, one tries to understand the mode of pair production in a time varying electric field. The study of the solution of Dirac equation is not only important in strong field QED but it has also importance in cosmology dealing with pair production in some models of universe (Parker 1969; Hartle and Hawking 1976; Duru and Unal 1986; Barut and Duru 1989; Lotze 1989). It has been suggested that the spontaneous pair production in e.m. case is parallel to some models of expanding universe. In this paper we investigate this problem. In §2 we show that a time dependent e.m. field $A_u(t)$ can be considered to simulate a background that resembles a RW space-time. We have shown elsewhere that the time dependent Dirac equation in an electromagnetic field $A_u(t)$, in Minkowski space-time, is equivalent to the motion of Dirac electron in RW space-time. This analogy is substantiated and justified in this paper. In § 3 we obtain the exact solutions of Dirac equation for the background obtained in § 2. Section 4 deals with creation of particles. The concluding section discusses some other aspects of QED.

It has been shown (Landau and Lifshitz 1975) that when one writes down e.m. field equations in a gravitational background, the equation resembles very much with the motion in a dielectric medium; the gravitational background simulates the effect of a dielectric medium. In some earlier works we used this idea (Biswas and Kumar 1989a, b; Biswas *et al* 1990a) to discuss the confinement of quarks, gluon and photon. We follow mostly our work (Biswas *et al* 1990a) to generate a time dependent dielectric function from postulating a Lagrangian density for the dielectric function field treated as a scalar. It has been suggested by Dicke (Dicke 1957) many years ago that the e.m. vacuum can be considered as a dielectric medium having space dependent dielectric function. Now the Dirac vacuum in strong e.m. field, particularly in a time varying field, is very unstable. Actually one does not know what the vacuum is (remember the Klein paradox) when the gap $2m_e C^2$ is closed due to strong electric field at $eA \gg 2m_e$. Moreover, one can deal with Klein paradox-like situation, not in space but in time (Cornwall and Tiktopoulos 1989; Biswas and Das 1990b, d) to discuss the pair production in a time dependent electromagnetic field. To justify the approach carried out in this paper we recall the Volkov solution of Dirac equation (Landau and Lifshitz 1982) in a strong e.m. background. There one finds an effective mass m^* for the electron given by

$$m^* = m_e\left(1 - \frac{e^2}{m_e^2}\bar{A}^2\right) \tag{1}$$

where $\bar{A}^2$ is the time average of A^2. Equation (1) suggests that the mass m^* will also be dependent on position signalling a gravitation-like background.

2. Formulation of the model

Let the background created by a time dependent e.m. field be described by a RW space-time

$$ds^2 = \frac{1}{\varepsilon(t)}dt^2 - \varepsilon(t)(dx^2 + dy^2 + dz^2). \tag{2}$$

The coulomb force law in a dielectric medium can be viewed as a transformation $r^2 \to \varepsilon(t) r^2$. We incorporate the dependence of electric field of heavy ions nuclei through $\varepsilon(t)$ term. Instead of working with (2) we use a flat Newtonian coordinate system

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2), \tag{3}$$

$$= \eta_{\mu\nu} dx^\mu dx^\nu, \tag{4}$$

with $\mu, \nu = 0, 1, 2, 3$ and $\eta_{00} = 1$, $\eta_{ii} = 1$ and $\eta_{ij} = 0$ for $i \neq j$. Equation (3) in view of (2) necessitates a rescaling, at every point of space-time. We take (Biswas and Kumar 1989; Biswas *et al* 1990a; Dicke 1957) for length and time the scaling

$$L = L_0 \varepsilon^{-\frac{1}{2}}$$

$$\omega = \omega_0 \varepsilon^{-\frac{1}{2}} \tag{5}$$

with L_0, ω_0 as constants. We rewrite (2) as

$$ds^2 = f_{\mu\nu} dx^\mu dx^\nu, \tag{6}$$

where,

$$f_{00} = \frac{1}{\varepsilon(t)}, \quad f_{11} = f_{22} = f_{33} = -\varepsilon(t),$$

$$f_{ij} = 0, \quad \text{for } i \neq j \tag{7}$$

To describe the motion of a Dirac particle in (2), we first determine the form of $\varepsilon(t)$ for a given $A_\mu(t)$ and then solve the Dirac equation in the background (2). We take the Lagrangian density as

$$L = \frac{1}{2k} f^{\mu\nu} \partial_\mu \varepsilon \partial_\nu \varepsilon - \frac{\varepsilon}{16\pi} F_{\mu\nu} F^{\mu\nu}. \tag{8}$$

For a more general description we take $\varepsilon = \varepsilon(r, t)$. In view of (5) and (7),

$$F^{\mu\nu} = f^{\mu\alpha} F_{\alpha\beta} f^{\beta\nu}. \tag{9}$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual e.m. field tensor. The variational principle

$$\delta \int L(-\eta)^{\frac{1}{2}} \quad d^4x = 0$$

will now determine the time dependent scalar field ε.

We choose the electromagnetic potential to be time dependent as

$$A_\mu = (0, 0, 0, Et), \tag{10}$$

corresponding to a constant field **E** in the Z-direction. The more familiar potential

$$A'_\mu = (-Ez, 0, 0, 0) \tag{11}$$

is related to A_μ by a gauge transformation

$$A'_\mu = A_\mu + \partial_\mu \Lambda \tag{12}$$

with

$$\Lambda = -Ezt. \tag{13}$$

The field equation corresponding to (8) for a general $\varepsilon = \varepsilon(r, t)$ and $A_\mu(t) = (0, 0, 0, A_3)$ is given by

$$\nabla^2 \varepsilon - \varepsilon^2 \frac{\partial^2 \varepsilon}{\partial t^2} = -K \left\{ \frac{1}{8\pi} \varepsilon E^2 - \frac{1}{2k} \left[\varepsilon \left(\frac{\partial \varepsilon}{\partial t} \right)^2 + \frac{1}{\varepsilon} (\nabla \varepsilon)^2 \right] \right\} \tag{14}$$

For a more general choice of $A_\mu(t)$, (14) will also contain B^2/ε term. The coupling constant K of the scalar field is a parameter in our model. Equation (14) for $\varepsilon = \varepsilon(t)$ and A_μ given by (11) now reduces to

$$\frac{\partial^2 \varepsilon}{\partial t^2} + \frac{(\partial \varepsilon / \partial t)^2}{2\varepsilon} = \frac{k}{8\pi} E^2 / \varepsilon. \tag{15}$$

We take the solution of (15) as

$$\varepsilon = (1 + \Lambda t), \tag{16}$$

where $\Lambda = (k/4\pi)^{\frac{1}{2}} E$. The solution (16) is also important from cosmological point of view. We can calculate the energy momentum tensor from the expression

$$T^{\mu}_{\nu} = \varepsilon_{,\mu} \frac{\partial L}{\partial \varepsilon_{,\nu}} - \delta^{\mu}_{\nu} L,$$

with

$$T_{00}(\varepsilon) = \frac{1}{2k} \varepsilon \left(\frac{\partial \varepsilon}{\partial t} \right)^2, \tag{17}$$

$$T_{00}(A_{\mu}) = \frac{\varepsilon E^2}{8\pi}. \tag{18}$$

Equation (15) then takes the form

$$\frac{\partial^2 \varepsilon}{\partial t^2} = -k \left\{ \frac{E^2}{8\pi\varepsilon} - \frac{1}{2k} \varepsilon \left(\frac{\partial \varepsilon}{\partial t} \right)^2 \right\}. \tag{19}$$

The first term is $1/\varepsilon^2$ times the energy density of e.m. field whereas the second term in the r.h.s. is the gravitational energy density. So at a finite interval of infinitesimal time, the energy densities, if be large, will cancel among themselves to reproduce a proper ε. Moreover we may interpret (19) as follows. A fraction $1/\varepsilon^2$ of e.m. energy density is converted into gravitational energy density to simulate the gravitational background. The form (14) as well as (19) is an indication that a confined phase of QED might arise in our approach due to cancellation of energy density terms. The r-dependence of $\varepsilon(r, t)$ will be responsible for such a confined phase formation. This is dealt in a subsequent paper. The solution of non-linear equation (14) or (15) is not easy for a general $A_{\mu}(t)$. However, the emergence of a RW type gravitational background seems to be basic outcome of our approach. In case of $\varepsilon = \varepsilon(r)$ only, the space-time will be anti-desitter like (Biswas *et al* 1990a) and the existence of confined phase (solitonic solutions) in such a background is now well established (Salam and Strathdee 1976; Biswas *et al* 1990a; Sivaram and Sinha 1979).

Now to solve the Dirac equation in our Newtonian co-ordinate system (3) we have to take into account (5) in writing down the Lagrangian density. We discussed elsewhere (Biswas *et al* 1990a, see also Gasperini 1987) that such a method is an approximation to the exact equation in a curved space-time. In order to take into account the coupling of spin with curvature, the spin connection must be taken into the formalism. So we proceed with the solution of Dirac equation in curved space-time.

3. Solution of Dirac equation

The space-time for our problem is now described by

$$ds^2 = \frac{1}{a^2(t)} dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \tag{20}$$

Here

$$a(t) = (1 + \Lambda t)^{\frac{1}{4}} \tag{21}$$

The covariant Dirac equation in (20) is given by

$$[i\tilde{\gamma}^{\mu}(\partial_{\mu} - \Gamma_{\mu}) - m]\psi = 0 \tag{22}$$

The space-time dependent $\tilde{\gamma}^{\mu}$ matrices now satisfy the relation

$$\tilde{\gamma}^{\mu}\tilde{\gamma}^{\nu} + \tilde{\gamma}^{\nu}\tilde{\gamma}^{\mu} = 2f^{\mu\nu}. \tag{23}$$

The connection with the flat time Dirac matrices is given by

$$\begin{aligned} \tilde{\gamma}^{0} &= a(t)\gamma^{0}, \\ \tilde{\gamma}^{i} &= -(1/a(t))\gamma^{i}, \quad i = 1, 2, 3. \end{aligned} \tag{24}$$

The spin connection Γ_{μ} are defined by the relation

$$[\Gamma_{\nu}, \tilde{\gamma}^{\mu}(x)] = \frac{\partial \tilde{\gamma}^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\nu\rho}\tilde{\gamma}^{\rho} \tag{25}$$

where $\Gamma^{\mu}_{\nu\rho}$ are the Christoffel symbols. We calculate $\Gamma^{\mu}_{\nu\rho}$ for the matrices (20) and find

$$\Gamma^{0}_{\mu\nu} = \begin{pmatrix} -aa' & 0 & 0 & 0 \\ 0 & a^3a' & 0 & 0 \\ 0 & 0 & a^3a' & 0 \\ 0 & 0 & 0 & a^3a' \end{pmatrix}, \quad \Gamma^{1}_{\mu\nu} = \begin{pmatrix} 0 & a'/a & 0 & 0 \\ a'/a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^{2}_{\mu\nu} = \begin{pmatrix} 0 & 0 & a'/a & 0 \\ 0 & 0 & 0 & 0 \\ a'/a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Gamma^{3}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & a'/a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a'/a & 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

In (26), a' means differentiation w.r.t. time variable. Henceforth prime for the variable will indicate differentiation w.r.t. its arguments i.e., $a'(\eta) = \partial a/\partial\eta$. To calculate Γ_{ν}, we first put $\nu = 0$ in (25) and obtain four equations for $\mu = 0, 1, 2, 3$. It is found that

$$[\Gamma_0, \gamma^0] = [\Gamma_0, \gamma^1] = [\Gamma_0, \gamma^2] = [\Gamma_0, \gamma^3] = 0. \tag{27}$$

So we take $\Gamma_0 = 0$. Now taking $\nu = 1$ and $\mu = 0, 1, 2, 3$ as before, we get

$$[\Gamma_1, \gamma^0] = aa'\gamma_1, \quad [\Gamma_1, \gamma^1] = -aa'\gamma_0. \tag{28}$$

Solving (28) we evaluate Γ_1 to be

$$\Gamma_1 = (aa'(t)/2)\gamma^0\gamma^1. \tag{29}$$

The symmetry of the problem then allows us to write the other two connections easily. We have

$$\Gamma_0 = 0, \Gamma_i = (aa'(t)/2)\gamma^0\gamma^i, \quad (i = 1, 2, 3). \tag{30}$$

Using (24) and (30) we find

$$\tilde{\gamma}^{\mu}\Gamma_{\mu} = -\tfrac{3}{2}a'(t)\gamma^{0}. \tag{31}$$

Multiplying both sides of (22) by $(-i\gamma^0)$ we get

$$\left(a\frac{\partial}{\partial t} - \frac{1}{a}\boldsymbol{\alpha}\cdot\nabla + \frac{3}{2}a'(t) + i\gamma_0 m\right)\psi = 0. \tag{32}$$

Let us make a change of variable as

$$\mathrm{d}\eta = \mathrm{d}t/a(t), \tag{33}$$

to reduce (32) as

$$\left[\frac{\partial}{\partial\eta} - \frac{1}{a(\eta)}\boldsymbol{\alpha}\cdot\nabla + \frac{3}{2}\frac{a'(\eta)}{a(\eta)} + i\gamma^0 m\right] = 0. \tag{34}$$

Here $a\partial/\partial t = \partial/\partial\eta$ and $a'(t) = \partial a/\partial t = \partial a/\partial\eta\cdot\partial\eta/\partial t = a'(\eta)/a(\eta)$. To solve (34) let us put

$$\psi(\mathbf{x},\eta) = \frac{\exp(i\mathbf{p}\cdot\mathbf{x})}{(2\pi)^{3/2}}\begin{pmatrix} f_{\mathrm{I}}(\mathbf{P},\eta) \\ f_{\mathrm{II}}(\mathbf{P},\eta)\end{pmatrix}, \tag{35}$$

in (34). The two first order equations are

$$\left(\frac{\partial}{\partial\eta} + \frac{3}{2}\frac{a'(\eta)}{a(\eta)} + im\right)f_I - \frac{i}{a}\boldsymbol{\sigma}\cdot\mathbf{P}f_{\mathrm{II}} = 0, \tag{36a}$$

$$\left(\frac{\partial}{\partial\eta} + \frac{3}{2}\frac{a'(\eta)}{a(\eta)} - im\right)f_{\mathrm{II}} - \frac{i}{a}\boldsymbol{\sigma}\cdot\mathbf{P}f_{\mathrm{I}} = 0. \tag{36b}$$

Let us put

$$\begin{pmatrix} f_{\mathrm{I}} \\ f_{\mathrm{II}}\end{pmatrix} = a(\eta)^{-3/2}\begin{pmatrix} \bar{f}_{\mathrm{I}} \\ \bar{f}_{\mathrm{II}}\end{pmatrix} \tag{37}$$

in (36) to get

$$\left(\frac{\partial}{\partial\eta} + im\right)\bar{f}_{\mathrm{I}} - \frac{i}{a(\eta)}\boldsymbol{\sigma}\cdot\mathbf{P}\bar{f}_{\mathrm{II}} = 0, \tag{38}$$

$$\left(\frac{\partial}{\partial\eta} - im\right)\bar{f}_{\mathrm{II}} - \frac{i}{a(\eta)}\boldsymbol{\sigma}\cdot\mathbf{P}\bar{f}_{\mathrm{I}} = 0. \tag{39}$$

Using standard techniques we reduce (39) into a second order differential equation

$$\left[\frac{\partial^2}{\partial\eta^2} + \frac{a'(\eta)}{a(\eta)}\frac{\partial}{\partial\eta} + \left(\frac{P^2}{a^2} + m^2 - im(a'/a)\right)\right]\bar{f}_{\mathrm{II}} = 0. \tag{40}$$

Further substitution

$$\bar{f}_{\mathrm{II}} = \exp\left[\left(-\frac{1}{2}\right)\int^{\eta}\frac{a'(\eta')}{a(\eta')}\mathrm{d}\eta'\right]u, \tag{41}$$

reduces (40) as

$$\left\{\frac{\partial}{\partial\eta^2}+\left[-\frac{1}{2}\frac{a''(\eta)}{a(\eta)}+\frac{1}{4}\left(\frac{a'}{a}\right)^2-im\frac{a'}{a}+\frac{P^2}{a^2}+m^2\right]\right\}h_{\text{II}}=0, \tag{42}$$

where $h_{\text{II}}=f_{\text{II}}/a^2$. Using (41) and (37) we have gone back to f_{II} to get (42). Let us now evaluate $a(\eta)$ for our problem

$$\eta=\int^t\frac{\mathrm{d}t}{(1+\Lambda t)^{\frac{1}{2}}}=\frac{1}{(\Lambda/2)}(1+\Lambda t)^{\frac{1}{2}}.$$

Hence with $\Lambda/2=a_0$

$$a(\eta)=a_0\eta. \tag{43}$$

Using this value in (42) we have

$$\left[\partial_\eta^2+\left(\frac{P^2/a_0^2+\frac{1}{4}}{\eta^2}-\frac{im}{\eta}\right)+m^2\right](f_{\text{II}}/a^2)=0 \tag{44}$$

The model of expansion given by (43) was also considered by Schrödinger (Schrödinger 1932). Recently the solution of (44) is also obtained by Barut and Duru (1987) with the variable η replaced by t. We just mention the steps. Putting $z=\mp 2imt$, (44) is reduced to Whittaker differential equation with the solution

$$f_{\text{II}}^{\pm}/a^2(\eta)=W_{\pm 1/2,iP/a_0}(\mp 2im\eta). \tag{45}$$

Now using (39) and identities

$$W'_{k,\mu}=\left(\frac{1}{2}+\mu-k\right)\left(\frac{1}{2}-\mu-k\right)\frac{1}{z}W_{k-1,\mu}(z)+\left(\frac{k}{2}-\frac{1}{2}\right)W_{k,\mu}(z),$$

for $k=\frac{1}{2}$ and

$$W'_{k,\mu}(z)=-\frac{1}{z}W_{k+1,\mu}(z)-\left(\frac{k}{z}-\frac{1}{2}\right)W_{k,\mu}(z),$$

for $k=-\frac{1}{2}$, four independent solutions of ψ are obtained. For completeness we mention them. For a wrong sign before m in Barut's paper, our solutions correspond to an interchange of the f_{I} and f_{II} components in Barut's paper. For our problem the solutions are

$$\psi_1=N_1\frac{\exp(i\mathbf{p}\cdot\mathbf{x})}{(2\pi)^{3/2}}\frac{1}{(2a_0t+1)}\begin{pmatrix}-\dfrac{i}{a_0}\begin{pmatrix}P_3\\P_+\end{pmatrix}W_{-1/2,iP/a_0}(-2im\eta)\\ \begin{pmatrix}1\\0\end{pmatrix}W_{1/2,iP/a_0}(-2im\eta)\end{pmatrix} \tag{46}$$

$$\psi_2=N_2\frac{\exp(i\mathbf{p}\cdot\mathbf{x})}{(2\pi)^{3/2}}\frac{1}{(2a_0t+1)}\begin{pmatrix}\begin{pmatrix}P_-\\-P_3\end{pmatrix}W_{-1/2,iP/a_0}(-2im\eta)\\ \begin{pmatrix}0\\1\end{pmatrix}W_{1/2,iP/a_0}(-2im\eta)\end{pmatrix}, \tag{47}$$

$$\psi_3 = N_3 \frac{\exp(i\mathbf{p}\cdot\mathbf{x})}{(2\pi)^{3/2}} \frac{1}{(2a_0 t + 1)} \begin{pmatrix} \frac{ia_0}{P^2}\begin{pmatrix} P_3 \\ P_+ \end{pmatrix} W_{1/2, iP/a_0}(2im\eta) \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} W_{-1/2, iP/a_0}(2im\eta) \end{pmatrix}, \tag{48}$$

$$\psi_4 = N_4 \frac{\exp(i\mathbf{p}\cdot\mathbf{x})}{(2\pi)^{3/2}} \frac{1}{(2a_0 t + 1)} \begin{pmatrix} \begin{pmatrix} P_- \\ -P_3 \end{pmatrix} ia_0/P^2 \, W_{1/2, iP/a_0}(2im\eta) \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} W_{-1/2, iP/a_0}(2im\eta) \end{pmatrix}. \tag{49}$$

To deal with creation of free Dirac particles it is convenient to find a suitable set of mode solutions that will ease the calculations of Bogolubov co-efficients.

4. Creation for free Dirac particles

Making a further change of variable

$$\mathrm{d}\xi = \mathrm{d}\eta/a(\eta), \quad (f_{\mathrm{I}}, f_{\mathrm{II}}) = a(\xi)^{-3/2}(F_{\mathrm{I}}, F_{\mathrm{II}}), \tag{50}$$

the Dirac equation reduces to the form

$$\left[\gamma_0 \frac{\partial}{\partial \xi} - \gamma^i \partial_i + ima(\xi)\right]\psi(\xi) = 0 \tag{51}$$

In first order form we have

$$\left(\frac{\partial}{\partial \xi} + ima(\xi)\right)F_{\mathrm{I}} - i\boldsymbol{\sigma}\cdot\mathbf{P}F_{\mathrm{II}} = 0, \tag{52a}$$

$$\left(\frac{\partial}{\partial \xi} - ima(\xi)\right)F_{\mathrm{II}} - i\boldsymbol{\sigma}\cdot\mathbf{P}F_{\mathrm{I}} = 0. \tag{52b}$$

As before the pair (52a) and (52b) reduces to the form

$$F''^{(\varepsilon)}(\xi) + [P^2 + m^2 a^2(\xi) + i\varepsilon m a'(\xi)]F^{(\varepsilon)}(\xi) = 0. \tag{53}$$

Here $\varepsilon = \pm 1$ and $F^{(+)} = F_{\mathrm{I}}$, $F^{(-)} = F_{\mathrm{II}}$. Knowing the solution of (53) we calculate the solutions $\hat{U}(\mathbf{P}, d; \xi)$ and $\hat{V}(\mathbf{P}, d; \xi)$ of (51) as follows:

$$\hat{U}(\mathbf{P}, d; \xi) = [(\partial_\xi - \gamma^i \partial_i - ima(\xi)F^{(-)}u_d] N^{(-)} \tag{54}$$

$$= N^{(-)}[D_- F^{(-)}u_d + F^{(-)}\tilde{u}_d],$$

$$\hat{V}(\mathbf{P}, d; \xi) = [(\partial_\xi + \gamma^i \partial_i + ima(\xi))F^{(+)}v_d] N^{(+)}$$

$$= N^{(+)}[(D_+ F^+)v_d + F^{(+)}\tilde{v}_d]. \tag{55}$$

In (54) and (55) $d = \pm 1$ denotes the spin projections, u_d and v_d are eigen bispinors of

γ^0 matrix, $N^{(\varepsilon)}$ are normalization constants and

$$\tilde{u}_d = i\gamma^i P_i u_d$$

$$\tilde{v}_d = -i\gamma^i P_i v_d$$

$$D_\pm = \partial_\xi \pm ima(\xi).$$

To verify that $\hat{U}$ given in (54) is a solution of Dirac equation (51) we find

$$\left[\gamma_0 \frac{\partial}{\partial \xi} - \gamma^i \partial_i + ima(\xi)\right](\partial_\xi - \gamma^i \partial_i - ima(\xi))F^{(-)}u_d$$

$$= \left[\gamma_0 \frac{\partial^2}{\partial \xi^2} - \gamma^0\gamma^i\partial_i\partial_\xi - \gamma_0 ima' - \gamma_0 ima\partial_\xi - \gamma_i\partial_i\partial_\xi + (\gamma^i\partial_i)^2\right.$$

$$\left. + (\gamma^i\partial_i ima(\xi)) + ima(\xi)\partial/\partial\xi - ima(\xi)\gamma^i\partial_i + m^2a^2\right]F^{(-)}u_d.$$

Now using $\gamma^0\gamma^i = -\gamma^i\gamma^0$ and $\gamma^0 u_d = u_d$ and $(\gamma^i)^2 = -1$, we get

$$[\partial_\xi^2 + (P^2 + m^2 a(\xi) - ima'(\xi)]F^{(-)}u_d = 0$$

by virtue of (53). Hence $\hat{U}$ given by (54) is a solution of Dirac equation. Similarly we can also prove that $\hat{V}(\mathbf{p}, d; \xi)$ is also a solution of Dirac equation. The most general solution of Dirac equation is then written as

$$\psi = (2\pi)^{-3/2}\int d^3P[a(\mathbf{p}, d)\hat{U}(\mathbf{p}, d; \xi)\exp(i\mathbf{p}\cdot\mathbf{x})$$

$$+ b^+(\mathbf{p}, d)\,\hat{V}(\mathbf{p}, d; \xi)\exp(-i\mathbf{p}\cdot\mathbf{x})]. \tag{56}$$

Now we assume that there exists 'in' and 'out' regions for $|\xi| \to \infty$. In these regions Minkowski vacuum exists and particle and antiparticle solutions are defined according to WKB prescription (Parker 1969; Lotze 1989)

$$\lim_{\xi \to \mp\infty} U_{\text{in/out}} \simeq \exp[iS^{(\text{in/out})}] \tag{57}$$

$$\lim_{\xi \to \mp\infty} V_{\text{in/out}} \simeq \exp[-iS^{(\text{in/out})}] \tag{58}$$

with $\lim_{\xi\to\mp\infty} S = S^{(\text{in/out})}$ where S is the classical action. In our case

$$S = \mathbf{P}\cdot\mathbf{x} - \int [P^2 + m^2a^2(\xi) + i\varepsilon ma'(\xi)]^{\frac{1}{2}}\,\mathrm{d}\xi. \tag{59}$$

We have, with $a_0 = 2b$

$$a(\xi) = 2b\exp(2b\xi), \tag{60}$$

so that

$$S^{(\text{in})} = \mathbf{P}\cdot\mathbf{x} - \omega_{\text{in}}\xi \tag{61}$$

$$S^{(\text{out})} = \mathbf{P}\cdot\mathbf{x} - \frac{\tilde{m}}{2b}\exp(2b\xi), \tag{62}$$

where

$$\omega_{\text{in}}^2 = P^2 + \tilde{m}^2 a_{\text{in}}^2.$$

We modify (60) as $a(\xi) = 2b(a_{\text{in}} + \exp 2b\xi)$ and take $\tilde{m} = m2b$. At the end of the calculation we will put $a_{\text{in}} = 0$. Making the substitutions

$$z = i(\tilde{m}/b)\exp(2b\xi),$$

$$f^{(\varepsilon)} = \exp(-b\xi)g^{(\varepsilon)}, \tag{63}$$

in (53) with ω_{in} and $a(\xi)$ defined as before the equation

$$f''^{(\varepsilon)} + [\omega_{\text{in}}^2 + 2\tilde{m}(\tilde{m}a_{\text{in}} + i\varepsilon b)\exp(2b\xi) + \tilde{m}^2\exp(4b\xi)]f^{(\varepsilon)} = 0 \tag{64}$$

reduces into Whittaker's differential equation. The solutions are then taken as (Lotze 1989)

$$f_{\text{in}}^{(-)} = f_{\text{in}}^{(+)*} = \exp(-b\xi)M_{-i\mu_{\text{in}}-\frac{1}{2},-i\nu}(z), \tag{65}$$

$$f_{\text{out}}^{(-)} = f_{\text{out}}^{(+)*} = \exp(-b\xi)W_{-i\mu_{\text{in}}-\frac{1}{2},-i\nu}(z), \tag{66}$$

with normalization constants

$$N_{\text{in}}^{(\pm)} = \frac{1}{2|p|}a_{\text{in}}[\nu - \mu_{\text{in}})/\nu\mu_{\text{in}}]^{\frac{1}{2}}\exp\left(-\frac{\pi}{2}\nu\right), \quad \nu = \omega_{\text{in}}/2b, \tag{67}$$

$$N_{\text{out}}^{(\pm)} = \frac{1}{2(\tilde{m}b)^{\frac{1}{2}}}\exp\left(-\frac{\pi}{2}\mu_{\text{in}}\right), \quad \mu_{\text{in}} = \tilde{m}a_{\text{in}}/2b. \tag{68}$$

To look at pair production we take (51) as the Dirac equation with as conformal time. Equations (65) and (66) are then the mode solutions. The substitutions and the change of variables carried out in this section have been made to get (64) for which there are standard results (Birrel and Davies 1983; Lotze 1989) for pair production amplitude. We follow Lotze (1989) in this paper. The mode of particle production in a time dependent background is formulated through the method of Bogolubov transformation (Parker 1977, 1982; Birrel and Davies 1982). The technique of Bogolubov transformation is directly related to the violation of Poincare invariance by the external field. This technique has been used to show the simultaneous creation of pairs and quartets of massive scalar particles (Birrel and Davies), massive with massless scalar particles (Kuroda 1983; Audrestsch and Sphangehl 1985, 1986, 1987) etc. from vacuum. We outline briefly the method of Bogolubov transformation.

To define the vacuum of a system one needs mode solutions corresponding to the equation of motion. Initially or at $t \to -\infty$, let $\omega = \omega_{\text{in}}$ and the mode solutions are defined such that

$$\frac{\partial}{\partial t}X_j(t) = -i\omega_{\text{in}}X_j(t). \tag{69}$$

Let the vacuum defined by these mode solutions be $|0_{\text{in}}\rangle$ i.e.,

$$\hat{a}_j|0_{\text{in}}\rangle = 0 \tag{70}$$

where

$$\psi = \sum_j [\hat{a}_j X_j(t) + \hat{a}_j^+ X_j^*(t)] \tag{71}$$

with $\hat{a}_j^+$ and $\hat{a}_j$ as creation and annihilation operators. In a time dependent field there is a mixing of positive energy and negative energy solutions and the meaning of (20) is lost. In Minkowski space-time $\partial/\partial t$ is a killing vector orthogonal to the space-like hypersurfaces $t = \text{constant}$ and the vacuum is invariant under the action of poincare group. If due to some reasons the poincare symmetry is lost (this situation occurs in curved space-time), one has to define another set of mode solutions $\bar{X}_j$. Particularly when $\omega_{\text{in}} \neq \omega_{\text{out}}$, where $\omega_{\text{out}} = \lim_{t\to+\infty}\omega$, we take

$$\psi(t) = \sum_j [\bar{\hat{a}}_j(t)\bar{X}_j + \bar{\hat{a}}_j^+ \bar{X}_j(t^*)] \tag{72}$$

and define another vacuum $|\bar{0}_{\text{out}}\rangle$ corresponding to $\bar{X}_j$. Obviously

$$|0_{\text{in}}\rangle \neq |\bar{0}_{\text{out}}\rangle$$

as $\omega_{\text{in}} \neq \omega_{\text{out}}$ signalling a particle production. As both sets are complete one writes

$$\bar{X}_j = \sum_i (\alpha_{ji} X_i + \beta_{ji} X_i^*), \tag{73}$$

$$X_i = \sum_j (\alpha_{ji}^* X_i + \beta_{ji} \bar{X}_j^*). \tag{74}$$

Here α_{ij} and β_{ij} are known as Bogolubov co-efficients. Whenever $\beta_{ij} \neq 0$, there is a particle creation and the number of particles created is given by (Birrel and Davies 1982)

$$\langle \bar{0}|N_i|\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \tag{75}$$

The Bogolubov co-efficient for the present case is given by

$$\beta_{dd'}(P) = -N_{\text{out}}^{(-)} N_{\text{in}}^{(+)} [(D_- f_{\text{out}}^{(-)})^* f_{\text{in}}^{(+)} + D_+ f_{\text{in}}^{(+)} f_{\text{out}}^{(-)*})] \cdot u_d^\psi \bar{v}_d. \tag{76}$$

The co-efficient $\beta_{dd'}(P)$ is then given by

$$\beta_{dd'}(P) = \frac{2^{\frac{1}{2}}}{2|P|}\left(\frac{\nu - \mu_{\text{in}}}{\nu}\right)^{1/2} \exp\left[-\frac{\pi}{2}(\nu + \mu_{\text{in}})\right] \frac{\Gamma(1 + 2i\nu)}{(1 + i(\nu - \mu_{\text{in}}))} u_d^\psi \bar{v}_d.$$

Setting $\mu_{\text{in}} = 0$ i.e., $a_{\text{in}} = 0$ in this expression, the number of electrons $N(-, \mathbf{P})$ and positron $N(+, \mathbf{P})$ are given by

$$N(-, \mathbf{P}) = N(+, \mathbf{P}) = 2\frac{1 - \exp(2\pi\nu)}{1 - \exp(4\pi\nu)} \tag{77}$$

with $\nu = \omega_{\text{in}}/2b = P/a_0 = 2P/(k/4\pi)^{\frac{1}{2}} E$, i.e., $\nu = (4\pi^{\frac{1}{2}}/k)(P/E)$.

5. Conclusion

The simulation of a time dependent gravitational background by a general $A_\mu(t)$ is not easy to find out from the nonlinear equation (19). However, in view of the calculations done on RW space-time, the form of $a(\xi)$ or $a(\eta)$ may be prescribed a priori. The corresponding $E(t)$ or $A_\mu(t)$ may be obtained from (15). Thus the production of e^+e^- pair or $\gamma\gamma$ pair in heavy ions scatterings finds a proper explanation in our approach. In cosmological particle creation examples, in Minkowski space one has to define 'in' and 'out' vacuum so that at a limited portion of time the space-time is RW like. In heavy ion scatterings the coulomb field is practically constant (see Cornwall 1989; Caldi and Chodos 1987) during a finite time of the order of Compton time $\sim 1/m_e$. After that the space dependence part of coulomb field is operative. Perhaps this will give rise to the confined phase of QED. This is discussed in another paper (Biswas and Das 1990d). The production of particle pairs and the existence of confined phase has also been noted by Dicke (1957) in a classic paper. The present work is motivated along these lines. The simulation of a medium with dielectric-like behaviour seems to be a very effective approach both in QED, weak interaction (dealing with charmed meson decays) and strong interaction (with MIT bag-like picture) dealing with QCD confined phase.

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