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Separability, gauge invariance and nonsuperluminality in direct interaction dynamics

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Abstract. We discuss the classical mechanics of relativistic systems with direct interaction. We show that various desiderata can all be accommodated in the single time approach by restricting the observables to the gauge invariant variables. We show how such observables can be constructed in general. We explicitly construct position observables in a general system and show that they lead to separable, invariant world lines. Nonsuperluminality is explicitly demonstrated for two body systems interacting via central forces of semibounded magnitude provided they ensure timelike canonical momenta. For two particles, our results reproduce the usual solution in covariant equal-time gauge.

Keywords. Separability; gauge invariance; direct interaction.

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1. Introduction

It is generally agreed that an acceptable classical theory of directly interacting particles should possess five desiderata, (a) a realization of the Poincare algebra by functions on the physical phase space of the system, (b) invariant physical world lines, (c) sufficiently general interaction, (d) nonsuperluminal propagation of particles, and (e) separability which requires the freedom to completely decouple the dynamics of widely separated clusters when they interact weakly. An old result (Currie *et al* 1963) proved it to be impossible in the instant form of dynamics of Dirac (1949) but there have been several attempts to construct a relativistic classical mechanics within the constrained Hamiltonian formalism by Komar, Droz-Vincent, Todorov and others (for extensive references, see Longhi and Lusanna 1986). The standard method is to start with an $8N$ dimensional phase space of covariant, individual particle variables (q_a^μ, p_a^μ), the canonical coordinates and momenta of the particle $a(a = 1$ to $N)$ and impose as many constraints as necessary to define the dynamics for each particle. (An alternative approach is also possible, see Thakur 1986). At the very least, one requires N Poincare invariant, separable constraints, one for each particle. (For an earlier discussion of separability, see Samuel 1982). These constraints being the generalization of the familiar energy-momentum relation of a free particle, are especially important and are called mass-shell constraints. These mass-shell constraints are assumed to be severely limited in form by the requirement that their Poisson brackets (PB) with each other vanish; in other words, the mass shell constraints must be first class. Such constraints are known to be generators of gauge changes (Dirac 1964). In such a theory, the only observables are the gauge invariant quantities i.e.

those dynamical variables whose PB with mass-shell constraints vanish. The canonical coordinate is not one of them; it may, however, be (weakly) equal to the physical coordinate for a restricted set of available states satisfying additional constraints called gauges. In general, gauges determine dynamics by picking out a particular Hamiltonian which preserves them and the problem is one of finding a satisfactory gauge which incorporates the conditions set forth above. It has been suggested (Balachandran *et al* 1982) that it is, in general, impossible to satisfy the above criteria especially (b) and (e) if the physical coordinates are strongly equal to the canonical coordinates in the chosen gauge. The formulation of the world line condition (WLC) given by Sudarshan *et al* (1981) and by Kihlberg *et al* (1981) is unduly restrictive in that it does not allow the physical coordinate and the canonical coordinate to differ by gauge constraints. With this freedom, the WLC becomes practically innocuous and one can accommodate the condition (d) of nonsuperluminality. In other words, a gauge invariant physical position vector can be defined in such a way as to meet the conditions listed above *and* be weakly equal to the canonical position vector for some gauge choice. The requirement of gauge invariance is natural in a theory such as this and it is gauge invariance which accommodates the conflicting requirements listed above. The construction of gauge invariant physical position four vectors satisfying the conditions listed above is, in fact, the chief new element in the present work.

The present construction was made possible by the realization that there may not exist one single gauge in which all the physical position vectors equal the canonical position vectors even weakly. We have, therefore, used different gauges to define the position vectors of different particles. This represents a major departure from the current uses in this field which, following Dirac, has followed the practice of attempting to find a single set of gauge constraints to make a second class set out of initial first class constraints. We believe that the earlier workers did not take full advantage of the flexibility of constrained Hamiltonian formalism and were unduly influenced by field theoretic analogy. For example, in electrodynamics, the gauge invariant quantities are known, from the beginning, to be given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and functions thereof. This is not true in the present case and as we show in § 3 gauge invariant quantities are best constructed by prescribing gauge conditions. This lends particular importance to the choice of gauge conditions for physical position vectors. Our choice satisfies all the requirements in the explicitly known cases. These are systems where each particle interacts with at most one other particle in the system; we call such systems monogamous. We insist that for a monogamous system with central (i.e. depending only on the spatial separation in the two body rest frame) potentials restricted to ensure time-like nature of canonical momenta, nonsuperluminality should hold for *all* available states. For more general i.e. momentum dependent potentials, the corresponding results are only a little more complicated. For nonmonogamous systems, our results show nonsuperluminality for infinitesimal central forces. We believe our result to be valid for all systems but we have not been able to prove it for nonmonogamous systems because of lack of information about the form of mass shell constraints.

The plan of the paper is as follows. In § 2 we give a brief introduction to the mass-shell constraints as we know them and show how the expectation value of a gauge invariant observable can be defined in terms of a probability distribution function. This is a general formulation but it shows, already at the classical level, how divergence difficulties can arise in a gauge theory. (At the classical level these difficulties are completely innocuous). In § 3 we devote ourselves to the construction of gauge invariant observables. Section 4 is devoted to the physical position observable and how it meets the requirements of separability, nonsuperluminality and WLC. In § 5 we illustrate our method with an explicit solution for a two body system interacting via harmonic forces. Section 6 is devoted to outlining a summary of the present work and making a few brief remarks.

2. General considerations

We describe a system of N particles by introducing an $8N$ dimensional phase space spanned by covariant, canonical coordinates and momenta (q_a^μ, p_a^μ), $a = 1$ to N, with the only non-vanishing PB being

$$[q_a^\mu, p_b^\nu] = -\delta_{ab} g^{\mu\nu} \tag{1}$$

where $g^{\mu\nu} = \mathrm{diag}(+, -, -, -)$. The motion of the particles is assumed to be subject to N mass-shell constraints of the form

$$K_a = m_a^2 - p_a^2 + V_a(q, p) \approx 0 \tag{2}$$

which are (a) Poincare invariant, (b) separable, and (c) strongly compatible in the sense of having (*strongly*) vanishing PB

$$[K_a, K_b] = 0 \tag{3}$$

It is important for our purpose that (3) holds strongly for in that case, separability can be used to fix K_a uniquely. Actually we require something even stronger. With the Lie-Liouville operator

$$L_{K_a} = \sum_b \left(\frac{\partial K_a}{\partial q_b^\mu} \frac{\partial}{\partial p_{b\mu}} - \frac{\partial K_a}{\partial p_{b\mu}} \frac{\partial}{\partial q_b^\mu} \right), \tag{4}$$

which satisfies

$$L_{K_a} \varphi = [\varphi, K_a] \tag{5}$$

we want $\exp(\Sigma_{i=1}^N \alpha_i L_{K_i})$ to be an N parameter Abelian group A_N of canonical transformations and we shall assume that this is so.

Several explicit solutions of (3) are known in the form of (2). For free particles, we can take $V_a = 0$. For two particles, we can take

$$V_1 = V_2 = V(q_T, p_1, p_2) \tag{6}$$

where we define $q_T = q - q \cdot \hat{P}\hat{P}$, $q = q_1 - q_2$, $\hat{P} = P/(P^2)^{\frac{1}{2}}$, $P = p_1 + p_2$. This is not the most general solution but is sufficient for our purpose. This solution can be generalized to the case of monogamous systems. For a general N-particle system ($N > 2$), no explicit physically acceptable solutions are known in closed form but the existence of such solutions is easily established. Let Φ be a Poincare invariant separable function of the form

$$\Phi(q_{ab}^T, p_c) \tag{7}$$

where $q_{ab}^T = q_{ab}^T - q_{ab}\cdot\hat{P}_{ab}\hat{P}_{ab}$, $P_{ab} = p_a + p_b$, $q_{ab} = q_a - q_b$. To be nontrivial, Φ must also have a complicated momentum dependence. Since a nontrivial Hamiltonian can be obtained in classical mechanics by canonical transformation of a trivial Hamiltonian we can take

$$K_a = \exp(L_\Phi)K_a^0, \tag{8}$$

where $K_a^0 = m_a^2 - p_a^2$ is the free particle constraint. Then (8) is a solution of (3) of the desired form. However, in general, this is only a local solution and conditions for obtaining global solutions are not known. Nevertheless, there is an approximate solution for infinitesimal potentials. It is (with $V_{ab} = V_{ba}$).

$$K_a = m_a^2 - p_a^2 + \varepsilon \sum_{b\neq a} V_{ab}(q_{ab}^T, p_a, p_b) + O(\varepsilon^2). \tag{9}$$

Since Dirac (1964) has shown that a theory with first class constraints is a gauge theory with the constraints being the generators of gauge changes, we are faced with the task of finding a physical interpretation for such a theory. A dynamical variable is gauge invariant if its PB with the mass-shell constraints vanishes. Such gauge invariant quantities may depend explicitly on the chosen evolution parameter θ. If it does not depend on θ explicitly we shall say that it is reparametrization invariant. It is easy to see from the equation of motion,

$$\frac{\mathrm{d}f}{\mathrm{d}\theta} = \frac{\partial f}{\partial\theta} + \sum_a U_a(\theta)[f, K_a], \tag{10}$$

that a gauge and reparametrization invariant quantity is necessarily a constant. We shall get this same result by using a representation of gauge invariant observables in terms of a standard set of canonical variables that we shall often use. Define a set of canonical variables such that $K_a = P_a$ are N of the new momenta. Let the conjugate variables be Q_a with $[Q_a, P_b] = \delta_{ab}$. Let the remaining variables, all constants of motion, be denoted by Q^* and P^*. Then a gauge invariant variable is necessarily of the form

$$f(Q^*, P^*, P_a, \theta). \tag{11}$$

It is reparametrization invariant if it does not depend on θ and it is then necessarily a constant. (Here and in § 3 we are using ideas similar in spirit to Faddeev and Popov (1967) method in gauge field theory.)

We shall formulate classical mechanics in the general way common in statistical mechanics. We introduce a probability distribution function $\rho(p, q, \theta)$ subject to two conditions. Firstly, ρ is gauge invariant,

$$L_{K_a}\rho = [\rho, K_a] = 0. \tag{12}$$

Secondly, the support of ρ is limited to regions where (2) holds. A standard form of

ρ satisfying these criteria is

$$\rho = \tilde{\rho}(P^*, Q^*, \theta) \prod_{a=1}^{N} \delta(P_a), \tag{13}$$

where, for a physically acceptable probability measure, we also require

$$\int (\mathrm{d}P^* \mathrm{d}Q^*) \tilde{\rho}(P^*, Q^*, \theta) < \infty. \tag{14}$$

The observables are assumed to be one-to-one with gauge invariant quantities. We define the expectation value of an observable $f(P^*, Q^*, P_a, \theta)$ to be

$$\langle f \rangle = \frac{\int (\mathrm{d}P^* \mathrm{d}Q^*) f(P^*, Q^*, 0, \theta) \tilde{\rho}(P^*, Q^*, \theta)}{\int (\mathrm{d}P^* \mathrm{d}Q^*) \tilde{\rho}(P^*, Q^*, \theta)} \tag{15}$$

Equation (15) is unexceptionable but from a practical point of view, the canonical variables (P^*, Q^*) are hard to obtain. It is more common to define $\langle f \rangle$ as the ratio of two formally divergent quantities.

$$\langle f \rangle = \frac{\int (\mathrm{d}p \, \mathrm{d}q) f \rho(p, q, \theta)}{\int (\mathrm{d}p \, \mathrm{d}q) \rho(p, q, \theta)}. \tag{16}$$

If we use $(\mathrm{d}p \, \mathrm{d}q) = (\mathrm{d}P^* \mathrm{d}Q^*)(\Pi_{a=1}^{N} \mathrm{d}P_a \mathrm{d}Q_a)$ and cancel the formally divergent factor $\int \Pi_{a=1}^{N} \mathrm{d}Q_a$ in the numerator and denominator on the right of (16), we see that (16) reduces to (15) for the case of a gauge invariant function f. Actually, (15) is, perhaps, the only unambiguous meaning that one can give to (16).

With (16) as our basic assumption, we notice that the expectation value of a gauge variant dynamical variable with compact support in at least one of the Q_a's is always zero. Thus only gauge invariant variables can be observables (i.e. quantities whose measurement can be used to gain information about the state of the system).

3. Construction of gauge invariant observables

Our next task is the construction of gauge invariant observables. These is a general way in which we can construct a gauge invariant variable which takes the same values in any gauge as a particular gauge variant quantity does in a specified gauge. Let the variable be $f(q, p, \theta)$ and let us consider the guage,

$$\chi_a(p, q) - \theta \approx 0, \tag{17}$$

for $a = 1$ to N. It is acceptable as a gauge if $[\chi_a, K_b]$ is a nonsingular matrix i.e. if

$$\det [\chi_a, K_b] \neq 0. \tag{18}$$

In terms of our standard canonical variables this reads

$$\det\left[\frac{\partial \chi_a}{\partial Q_b}\right] \neq 0, \tag{19}$$

so we can solve (17) for Q_a:

$$Q_a = \bar{Q}_a(P^*, Q^*, P_a, \theta), \tag{20}$$

The function f then takes the form

$$f = f(P_a, \bar{Q}_a(P^*, Q^*, P_a, \theta), Q^*, P^*, \theta) \tag{21}$$

in this gauge. If one thinks of this explicitly as a function of P_a, P^* and Q^*, one has a gauge invariant function. We shall now give an explicit representation of such a function in a form which has obvious quantum mechanical analogy. First, we note that the following integral exists:

$$(\Delta)^{-1} = \int \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_N \exp\left(\sum_{i=1}^{N} \alpha_i L_{K_i}\right) \prod_{a=1}^{N} \delta(\chi_a - \theta). \tag{22}$$

We use

$$\prod_{a=1}^{N} \delta(\chi_a - \theta) = \left[\det \frac{\partial \chi_a}{\partial Q_b}\right]_{Q_b = \bar{Q}_b} \prod_{c=1}^{N} \delta(Q_c - \bar{Q}_c) \tag{23}$$

and

$$\exp\left(\sum_{i=1}^{N} \alpha_i L_{K_i}\right) \prod_{c=1}^{N} \delta(Q_c - \bar{Q}_c) = \prod_{c=1}^{N} \delta(Q_c + \alpha_c - \bar{Q}_c). \tag{24}$$

This also shows that Δ is gauge invariant. Now consider

$$\hat{f} = \int \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_N \Delta \exp\left(\sum_{i=1}^{N} \alpha_i L_{K_i}\right) f \prod_{c=1}^{N} \delta(\chi_c - \theta). \tag{25}$$

Using (22)–(24) we see that this reduces to (21). Actually we can prove something more specific also. Let us write

$$\exp\left(\sum_{i=1}^{N} \alpha_i L_{K_i}\right) f = f(\alpha_1, \alpha_2, \cdots \alpha_N). \tag{26}$$

We work in the gauge (17). Let C be the matrix inverse of $[\chi_a, K_b]$ i.e.

$$\sum_b [\chi_a, K_b] C_{bc} = \delta_{ac},$$

$$\sum_b C_{ab} [\chi_b, K_c] = \delta_{ac}, \tag{27}$$

In (25), the α_i's are fixed by the vanishing of the argument of the delta function. Since

$$\exp\left(\sum_{i=1}^{N} \alpha_i L_{K_i}\right) \delta(\chi_a - \theta) = \delta\left(\chi_a - \theta + \sum_{i=1}^{N} \alpha_i [\chi_a, K_i] \cdots\right). \tag{28}$$

we get in the gauge (17)

$$\alpha_i \approx -\sum_a C_{ia}(\chi_a - \theta). \tag{29}$$

Hence

$$\hat{f} \approx f - \sum_{i,a} [f, K_i] C_{ia}(\chi_a - \theta) \tag{30}$$

in this gauge. The important thing to notice is that $\hat{f}$ weakly equals f in this gauge but the PB of $\hat{f}$ with an arbitrary variable φ does not equal $[f, \varphi]$ even weakly; the PB of the gauge constraints $\chi_a - \theta$ with φ also contribute.

4. Physical position vector: separability, non-superluminality and world line condition

The most important observable in classical mechanics is unquestionably the physical coordinate x_a^μ since it is the quantity directly observed. We want x_a^μ to be gauge invariant and to be weakly equal to the canonical coordinate q_a^μ in some gauge. If the gauge is chosen properly, x_a^μ will automatically satisfy WLC. Another important requirement is nonsuperluminality which must hold for all initial states subject only to (2). Also we want x_a^μ to possess separability in the sense made precise below.

We shall concentrate on x_1^μ the coordinates of the particle number 1. We must choose the gauge in which x_1^μ is weakly equal to q_1^μ in such a way that $\mathrm{d}q_1^\mu/\mathrm{d}\theta$ is a time-like vector in this gauge for all admissible initial states (i.e. those satisfying (2)). Considering the constraint (9) and the equation of motion (10) we find (the arbitrary functions $U_a(\theta)$ become specified when the gauge is fixed).

$$\frac{\mathrm{d}q_1^\mu}{\mathrm{d}\theta} = 2U_1(\theta)p_1^\mu - \sum_b U_b(\theta) \sum_{c \neq b} \varepsilon \frac{\partial V_{bc}}{\partial p_{1\mu}} + O(\varepsilon^2). \tag{31}$$

Restricting our considerations for the present to central potentials (i.e. to $V_{ab}(q_{ab}^{T2})$, we find

$$\frac{\partial V_{b1}}{\partial p_{1\mu}} = V'_{b1}(q_{b1}^{T2})(-2q_{b1}\cdot P_{b1})\frac{q_{b1}^{T\mu}}{\sqrt{(P_{b1})^2}} \tag{32}$$

where prime denotes differentiation with respect to the argument. Now this is a space-like vector which can become arbitrarily large for some initial conditions since q_{b1}, P_{b1} ranges between $-\infty$ and $+\infty$ unless it is made to vanish by a suitable choice of gauge. Such a term can destroy non-superluminality by making $\mathrm{d}q_1^\mu/\mathrm{d}\theta$ space-like. Therefore, we shall choose our gauge to be ($b = 2$ to N)

$$\chi_b \equiv q_{b1}\cdot \hat{P}_{b1} \approx 0. \tag{33}$$

This is the special gauge (with one other gauge condition to be specified) in which x_1^μ equals q_1^μ weakly. (Notice that the choice of gauge depends on the particle being considered and is different for different particles). We must choose the final constraint to be Poincare variant and explicitly dependent on θ (because it sets the scale for the time variables q_a^0). We choose the constraint

$$\chi_1 - \theta \approx 0 \tag{34}$$

where $[\chi_1, K_b] = 0$ for $b \neq 1$. By assumption χ_a and K_b are a second class set for $a, b = 1$ to N. Then, necessarily, χ_a and K_b are also a second class set for $a, b = 2$ to N. We now define a dynamical variable X_1^μ by

$$X_1^\mu = \int \mathrm{d}\alpha_2 \cdots \mathrm{d}\alpha_N \exp\left(\sum_{i=2}^{N} \alpha_i L_{K_i} \right) q_1^\mu \Delta_1 \prod_{a=2}^{N} \delta(q_{1a} \cdot \hat{P}_{1a}), \tag{35}$$

where

$$(\Delta_1)^{-1} = \int \mathrm{d}\alpha_2 \cdots \mathrm{d}\alpha_N \exp\left(\sum_{i=2}^{N} \alpha_i L_{K_i} \right) \prod_{a=2}^{N} \delta(q_{1a} \cdot \hat{P}_{1a}). \tag{36}$$

Note that $[X_1^\mu, K_b] = 0$ for $b \neq 1$. X_1^μ is as close to gauge invariance as a reparametrization invariant nonconstant quantity can get. It is not an observable by our criterion (only gauge invariant quantities are observables by our criterion) but it takes the same set of values in any gauge because its equation of motion is gauge independent within reparametrization and can be written as

$$\frac{\mathrm{d}X_1^\mu}{\mathrm{d}\tau_1} = [X_1^\mu, K_1] \tag{37}$$

by defining $\mathrm{d}\tau_1 = U_1(\theta)\mathrm{d}\theta$. We shall use this variable often. We now define the gauge invariant position vector x_1^μ by choosing a final parametrization gauge. Two particularly simple choices are:

$$\text{(1) Synchronous parametrization } X_1^0 = \theta \tag{38}$$

$$\text{(2) Covariant parametrization } X_1 \cdot \hat{P} = \theta \tag{39}$$

Both choices define the same world line and differ only in parametrization (i.e. τ_1 of eq. (37)). Considering the former first, we write

$$x_1^\mu = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1 \delta_1 \exp(\alpha_1 L_{K_1}) X_1^\mu \delta(X_1^0 - \theta) \tag{40}$$

where δ_1 is defined by

$$(\delta_1)^{-1} = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1 \exp(\alpha_1 L_{K_1}) \delta(X_1^0 - \theta). \tag{41}$$

Similar equations hold for the covariant parametrization. Although the correct position observable is x_1^μ for many purposes X_1^μ is easier to discuss. In terms of our standard canonical variables,

$$X_1^\mu = X_1^\mu(P_a, P^*, Q^*, Q_1) \tag{42}$$

The parametrization constraints (38) or (39) fix Q_1 as a function of θ. This freedom of choosing the parametrization constraints or gauges is just the freedom of reparametrization because θ is an arbitrary parameter. Since Q_1 varies between $-\infty$

and $+\infty$ monotonically as θ varies in the same range, X_1^μ takes the same values in all parametrization gauges and contains all the information on dynamics. We notice that X_1^μ is a four vector, has the right behaviour under translations and is separable as we now prove. Notice that the separability is required of X_1^μ and not of x_1^μ. In some parametrization gauges (e.g. eq. (39)), x_1^μ is not separable but the non-separability, resides only in the parametrization and is not significant.

4.1 *Separability*

Let us assume that the system consists of two subsystems $S_{\rm I}$ and $S_{\rm II}$ which do not have any interaction between them. Let the particles 1 to N_1 belong to $S_{\rm I}$ and the remaining particles belong to $S_{\rm II}$. Let us adopt the notation

$$\exp\left(\sum_{i=2}^{N}\alpha_i L_{K_i}\right)f = f(,\alpha_2,\alpha_3\cdots\alpha_N), \tag{43}$$

then

$$X_1^\mu = \int \mathrm{d}\alpha_2\cdots\mathrm{d}\alpha_N \Delta_1 q_1^\mu(,\alpha_2\cdots\alpha_N)\delta(\chi_2(,\alpha_2\cdots\alpha_N))\cdots\delta(\chi_N(,\alpha_2\cdots\alpha_N)). \tag{44}$$

By the assumed separability of the mass shell constraints $[q_1^\mu, K_b]=0$ if $b>N_1$ so $q_1^\mu(,\alpha_2\cdots\alpha_N)$ is actually independent of $\alpha_{N_1+1},\cdots\alpha_N$. This is also true of $\chi_b(,\alpha_2\cdots\alpha_N)$ for $b\leqslant N_1$. Thus the α's are to be determined by solving the equations

$$\begin{aligned}
&\chi_2(,\alpha_2\cdots\alpha_{N_1})=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\chi_{N_1}(,\alpha_2\cdots\alpha_{N_1})=0\\
&\chi_{N_1+1}(,\alpha_2\cdots\alpha_N)=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\chi_N(,\alpha_2\cdots\alpha_N)=0.
\end{aligned} \tag{45}$$

In (45), the first N_1-1 equations fix $\alpha_2\cdots\alpha_{N_1}$ and hence X_1^μ since the remaining equations merely serve to pick up a value of $\alpha_i(i>N_1)$ and give unity after integration of the delta function. The solution for $\alpha_i(i\leqslant N_1)$ involve the variables of $S_{\rm I}$ only and the variables of $S_{\rm II}$ which enter the expressions for $\alpha_i(i>N_1)$ disappear on integration. It is clear that X_1^μ does not contain the variables of $S_{\rm II}$ and is, in fact, exactly what it would be if $S_{\rm II}$ did not exist. Notice that $X_1^\mu = q_1^\mu$ if the particle 1 is completely free.

4.2 *Nonsuperluminality*

Our construction of the gauge invariant position vector x_1^μ depends crucially on the choice of the gauge (33). One can check that this gauge is sufficient for monogamous systems interacting via central forces; nonsuperluminality is ensured if the constraints guarantee time-like nature of the canonical momentum of particle 1 i.e. if

$$m_1^2 + V_1 \geqslant 0. \tag{46}$$

The relevant gauge for the particle a is $q_{ba}\cdot\hat{P}_{ba}=0$ (all $b\neq a$). One gauge does not work for all particles and one must have for the purpose of definition, the freedom of choosing different gauges for different particles as far as the construction of physical position vector is concerned. It is for this reason that one has to introduce gauge invariant physical position vectors. We emphasize that nonsuperluminality has not been established for cases more general than one obtaining in monogamous systems. Even for monogamous systems if the potential V explicitly depends on momenta, a separate investigation is called for. We shall say that a potential is admissible if it ensures nonsuperluminal propagation of particles in the gauge (33). The class of admissible interactions certainly includes 'central potentials' in monogamous systems. Whether a sufficiently wide class of admissible interactions exists for more general systems remains to be found out.

4.3 *World line condition*

We shall explicitly establish WLC for x_1^μ which states that under infinitesimal canonical transformations generated by a generator of the Poincare group x_1^μ should transform exactly as expected of a position four vector apart from a translation along its world line. Let us choose the synchronous parametrization (38) (similar result holds for other cases). Let G denote an infinitesimal generator of the Poincare group

$$G=\varepsilon_\mu P^\mu+\tfrac{1}{2}\varepsilon_{\mu\nu}M^{\mu\nu}, \tag{47}$$

where

$$P^\mu=\sum_a p_a^\mu,\ M^{\mu\nu}=\sum_a(q_a^\mu p_a^\nu-q_a^\nu p_a^\mu).$$

We check that

$$[G,q_1^\mu]=\varepsilon^\mu-\varepsilon^{\mu\nu}q_{1\nu}. \tag{48}$$

It is obvious by inspection of (35) that

$$\begin{aligned}\overline{X_1^\mu}&=\exp(-L_G)X_1^\mu=X_1^\mu+[G,X_1^\mu]\\&=X_1^\mu+\varepsilon^\mu-\varepsilon^{\mu\nu}X_{1\nu}.\end{aligned} \tag{49}$$

Then

$$\begin{aligned}x_1^\mu&=\int\mathrm{d}\alpha_1\exp(\alpha_1 L_{K_1})\overline{X_1^\mu}\,\delta(\overline{X_1^0}-\theta)\bar{\delta}_1\\&=x_1^\mu+\varepsilon^\mu-\varepsilon^{\mu\nu}x_{1\nu}+\delta I^\mu,\end{aligned} \tag{50}$$

where

$$\delta I^\mu=\int\mathrm{d}\alpha_1\exp(\alpha_1 L_{K_1})X_1^\mu[\delta(\overline{X_1^0}-\theta)\bar{\delta}_1-\delta(X_1^0-\theta)\delta_1]. \tag{51}$$

Let us write $\exp(\alpha_1 L_{K_1})X_1^\mu=X_1^\mu(\alpha_1)$ and note that

$$\delta(X_1^0(\alpha_1)-\theta)\delta_1=\delta(\alpha_1-\alpha_1^0(\theta)), \tag{52}$$

and

$$\delta(\overline{X_1^0}-\theta)\bar{\delta}_1=\delta(\alpha_1-\alpha_1^0(\theta)-\delta\alpha_1^0(\theta)), \tag{53}$$

so

$$\delta I^\mu = \frac{-\delta\alpha_1^0(\theta)}{(\mathrm{d}\alpha_1^0/\mathrm{d}\theta)}\frac{\mathrm{d}x_1^\mu}{\mathrm{d}\theta}. \tag{54}$$

Equations (50) and (54) constitute the world line condition. We note that as a consequence of the world line condition the classical expression for the electromagnetic current to order e

$$J^\mu(y) = \sum_{a=1}^{N} e_a \int_{-\infty}^{\infty} \frac{\mathrm{d}x_a^\mu}{\mathrm{d}\theta}\delta(y - x_a)\mathrm{d}\theta, \tag{55}$$

is a gauge invariant (in our sense of particle dynamics) four vector.

5. An explicit solution

Let us find out an explicit solution for the two particle case with the harmonic oscillator potential $V = -\frac{1}{4}\Omega^2 q_T^2$. First we try to find X_1^μ defined by (35) which reads in this case

$$X_1^\mu = \int \mathrm{d}\alpha_2 \exp(\alpha_2 L_{K_2}) q_1^\mu \delta(q\cdot\hat{P})\Delta_1. \tag{56}$$

It is convenient to go to the following set of centre of mass and relative variables,

$$\left.\begin{aligned} P &= P_1 + P_2 \\ p &= \beta p_1 - \alpha p_2, \alpha + \beta = 1 \\ q &= q_1 - q_2 \end{aligned}\right\}. \tag{57}$$

We do not need to define Q which is canonically conguate to P. We take

$$\alpha = \frac{1}{2}\left(1 + \frac{m_1^2 - m_2^2}{P^2}\right) \tag{58}$$

to ensure

$$K_1 - K_2 = m_1^2 - p_1^2 - m_2^2 + p_2^2 = -2\, p\cdot P, \tag{59}$$

and

$$K_2 = m_2^2 - \beta^2 P^2 + 2\beta p\cdot P - (p\cdot\hat{P})^2 - p_T^2 + V. \tag{60}$$

Defining $f(,\alpha_2) = \exp(\alpha_2 L_{K_2})f$, we write

$$X_1^\mu = \int \mathrm{d}\alpha_2 q_1^\mu(,\alpha_2)\delta(q\cdot\hat{P}(,\alpha_2))\Delta_1. \tag{61}$$

Now $q\cdot\hat{P}(,\alpha_2) = q\cdot\hat{P} - 2\alpha_2(\beta M - p\cdot\hat{P})$ where we have set $M = (P^2)^{\frac{1}{2}}$. Also

$$\frac{\mathrm{d}q_1^\mu}{\mathrm{d}\alpha_2}(,\alpha_2) = [q_1^\mu(,\alpha_2), K_2]. \tag{62}$$

The bracket is the canonical transform of

$$[q_1^\mu, V] = -\frac{\partial V}{\partial p_{1\mu}} = V'(q_T^2)2q\cdot\hat{P}\frac{q_T^\mu}{M}, \tag{63}$$

assuming a central potential but not explicitly using the form harmonic oscillator. We also have

$$[p_T^\mu, K_2] = [p_T^\mu, V] = V'(q_T^2)2q_T^\mu \tag{64}$$

Using (61)–(64) we get

$$X_1^\mu = q_1^\mu + \frac{1}{M}\int_0^{q\cdot\hat{P}} \mathrm{d}\alpha_2 (p_T^\mu(,\alpha_2/2(\beta M - p\cdot\hat{P})) - p_T^\mu. \tag{65}$$

Let us introduce a fake parameter σ by calling $p_T^\mu = p_T^\mu(\sigma)$ and $q_T^\mu = q_T^\mu(\sigma)$. Then $p_T^\mu(,\alpha_2) = \exp(\alpha_2 L_{K_2})p_T^\mu = \exp(\alpha_2 L_h)p_T^\mu = p_T^\mu(\sigma + \alpha_2)$ where we have set $h = -p_T^2 + V$. Thus, we may write

$$X_1^\mu = q_1^\mu + \frac{1}{M}\int_0^{q\cdot\hat{P}} \mathrm{d}\alpha_2 \left(p_T^\mu\left(\sigma + \frac{\alpha_2}{2(\beta M - p\cdot\hat{P})}\right) - p_T^\mu(\sigma)\right). \tag{66}$$

Now, for the harmonic oscillator $V = -\frac{1}{4}\Omega^2 q_T^2$ we have

$$\frac{\mathrm{d}p_T^\mu}{\mathrm{d}\sigma} = [p_T^\mu, h] = -\tfrac{1}{2}\Omega^2 q_T^\mu$$

$$\frac{\mathrm{d}q_T^\mu}{\mathrm{d}\sigma} = [q_T^\mu, h] = 2p_T^\mu. \tag{67}$$

The solution can be written as

$$q_T^\mu(\sigma + \tau) = q_T^\mu \cos\Omega\tau + \frac{2p_T^\mu}{\Omega}\sin\Omega\tau$$

$$p_T^\mu(\sigma + \tau) = p_T^\mu \cos\Omega\tau - \tfrac{1}{2}\Omega q_T^\mu \sin\Omega\tau, \tag{68}$$

Using this solution in (66) and integrating, we get

$$X_1^\mu = q_1^\mu + \frac{p_T^\mu}{M}\left[\frac{2(\beta M - p\cdot\hat{P})}{\Omega}\sin\frac{\Omega q\cdot\hat{P}}{2(\beta M - p\cdot\hat{P})} - q\cdot\hat{P}\right]$$
$$- (\beta M - p\cdot\hat{P})\frac{q_T^\mu}{M}\left(\cos\frac{\Omega q\cdot\hat{P}}{2(\beta M - p\cdot\hat{P})} - 1\right). \tag{69}$$

Let us now calculate x_1^μ. We use the covariant parametrization. We write

$$x_1^\mu = \int \mathrm{d}\alpha_1 \exp(\alpha_1 L_{K_1}) X_1^\mu \delta(X_1\hat{P} - \theta)\delta_1 \tag{70}$$

and use

$$\exp(\alpha_1 L_{K_1}) f(X_1^\mu) = \exp(\alpha_1 L_{K_1 - K_2}) f(X_1^\mu). \tag{71}$$

Now

$$\exp(\alpha_1 L_{K_1-K_2}) q_1^\mu = q_1^\mu + 2\alpha_1 p_1^\mu \tag{72}$$

and

$$\exp(\alpha_1 L_{K_1-K_2}) q\cdot\hat{P} = q\cdot\hat{P} + 2M\alpha_1. \tag{73}$$

while all other quantities commute with $K_1 - K_2$. Collecting everything, we get

$$x_1^\mu = q_1^\mu + 2p_1^\mu\alpha_1^0 + \frac{1}{M}p_T^\mu\left[\frac{2\gamma}{\Omega}\sin\frac{\Omega\eta}{2\gamma} - \eta\right] + \frac{\gamma q_T^\mu}{M}\left[\cos\frac{\Omega\eta}{2\gamma} - 1\right] \tag{74}$$

where

$$\gamma = \beta M - p\cdot\hat{P}, \quad \eta = q\cdot\hat{P} + 2M\alpha_1^0, \quad \alpha_1^0 = (\theta - q_1\cdot\hat{P})/2p_1\cdot\hat{P}. \tag{75}$$

The combinations of q and p occurring in (74) can be evaluated at any θ for a pure state; in particular, they can be evaluated for $\theta = 0$ without, however, changing the explicit dependence on θ. We now restrict initial states to $q_{a0}\cdot\hat{P} \approx 0$ at $\theta = 0$ and also put $p\cdot\hat{P} \approx 0$ (more general initial states give rise to phases in the sinusoidal factors.) We then get

$$x_1^\mu \approx q_{10}^\mu + \frac{P^\mu}{M}\theta + p_{T0}^\mu\frac{2\beta}{\Omega}\sin\frac{\Omega\theta}{2\alpha\beta M} + \beta q_{T0}^\mu\left(\cos\frac{\Omega\theta}{2\alpha\beta M} - 1\right) \tag{76}$$

a result that agrees with a direct calculation from the equations of motion (10) in the gauge

$$q_a\cdot\hat{P} \approx \theta. \tag{77}$$

6. Summary

Relativistic classical mechanics can be set up as a single time formalism with N first class constraints whose PB vanishes strongly. These constraints are the generators of an N-parameter Abelian group of canonical transformations A_N. The observables are gauge invariant (i.e. A_N invariant) quantities which may in general depend on the evolution parameter θ. In particular a position observable can be constructed which meets the requirements of world line condition and separability in all cases and of nonsuperluminality for monogamous systems. This is, in fact, the chief new element of the present work. Three points remain to be taken up. First, we would like to know how to construct physically acceptable constraints for nonmonogamous systems. Second, one would like to verify nonsuperluminality in this case. Finally, we need to know the full electromagnetic current and not just the electromagnetic current to order e that we have constructed in (55).

References

Balachandran A P, Dominici D, Marmo G, Mukunda N, Nilsson J, Samuel J, Sudarshan E C G and Zaccaria F 1982 *Phys. Rev.* **D26** 3492

Currie D G, Jordan T F and Sudarshan E C G 1963 *Rev. Mod. Phys.* **35** 350

Dirac P A M 1949 *Rev. Mod. Phys.* **21** 392

Dirac P A M 1964 *Lectures on quantum mechanics* (New York: Yeshiva University)

Faddeev L D and Popov V N 1967 *Phys. Lett.* **B25** 29
Kihlberg A, Marnelius R and Mukunda N 1981 *Phys. Rev.* **D23** 2201
Longhi G and Lusanna L 1986 *Phys. Rev.* **D34** 3707
Samuel J 1982 *Phys. Rev.* **D26** 3482
Sudarshan E C G, Mukunda N and Goldberg J 1981 *Phys. Rev.* **D23** 2218
Thakur J 1986 *Pramana – J. Phys.* **27** 731