Pramana – J. Phys., Vol. 36, No. 3, March 1991, pp. 289–304. © Printed in India.

Half-shell T matrix for Coulomb-modified Graz separable potential

U LAHA and B TALUKDAR*

Department of Physics, Regional Institute of Technology, Jamshedpur 831 014, India
*Department of Physics, Visva-Bharati University, Santiniketan 731 235, India

MS received 7 May 1990; revised 27 November 1990

Abstract. We construct a closed form expression for the off-shell Jost function for scattering by the Coulomb-distorted Graz separable potential and express it in the 'maximal reduced form'. Our result is particularly suitable for numerical computation. We present a case study in support of this and examine the role of Coulomb interaction in the $p-p$ half-shell scattering in the ${}^1S_0$ channel.

Keywords. Coulomb-distorted separable potential; off-shell Jost function; half-shell $p-p$ T matrix.

PACS Nos 03·65; 24·10

1. Introduction

Based on a differential equation approach (van Leeuwen and Reiner 1961) to the T matrix, Fuda and Whiting (1973) have introduced the concept of off-shell Jost function $f_l(k,q)$. It is a function of the wave number k and an off-shell momentum q. For scattering on a short range potential $f_l(k,q)$ is a continuous function of the off-shell momentum so that $f_l(k,q)$ becomes the ordinary Jost function (Jost 1947) $f_l(k)$ when $q=k$. This is, however, not true for Coulomb (C) and Coulomb plus short range (CS) potentials (van Haeringen 1979) and the off-shell Jost function $f_l^{\mathrm{C}}(k,q)$ or $f_l^{\mathrm{CS}}(k,q)$ exhibits a discontinuity at the energy shell. The characteristic Coulomb discontinuity arises from the fact that a long range potential distorts not only the scattered wave but also the incident plane wave (Ford 1964). The on-shell limiting behaviour of $f_l^{\mathrm{C}}(k,q)$ or $f_l^{\mathrm{CS}}(k,q)$ is given by the singular factor $(q-k)^{-i\eta}$, where η is the Sommerfeld parameter. The functions $f_l^{\mathrm{C}}(k)$ and $f_l^{\mathrm{CS}}(k)$ can be obtained from the corresponding off-shell quantities by using the relation (van Haeringen 1979)

$$f_l^t(k)=\lim_{q\to k}\omega f_l^t(k,q),\quad k>0. \tag{1}$$

Here

$$\omega=[(q-k)/(q+k)]^{i\eta}\frac{\exp(\pi\eta/2)}{\Gamma(1+i\eta)} \tag{2}$$

and the superscript t stands for either C or CS.

In the present paper we use the Graz separable potential (Crepinsek *et al* 1975) for the short range interaction and obtain $f_l^{\mathrm{CS}}(k)$ and $f_l^{\mathrm{CS}}(k,q)$ for the Coulomb-distorted Graz potential in closed analytic form. We also demonstrate their usefulness by means of a model calculation. Our construction procedure for $f_l^{CS}(k,q)$ is based on a suitable modification of the intergral representation of the off-shell Jost function in terms of the regular solution of the Schrödinger equation. The on-shell Jost function is then obtained by using the prescription given in (1). In the course of our study we shall see that our results for $f_l^{CS}(k)$ and $f_l^{CS}(k,q)$ serve as an illustration of "being in maximal reduced form". Such a nice result could earlier be constructed for the s-wave only (van Haeringen 1983) and higher partial wave results exhibit inordinate complications. It is important to note that one of us (Talukdar *et al* 1984) treated the off-shell scattering by Coulomb-distorted Graz potential and derived an expression for $f_l^{CS}(k,q)$ which involve an infinite sum of ${}_3F_2(\cdot)$ functions. Naturally, it was unsuitable for any useful application. Here we shall remove this infinite sum as well as write the result in terms of ${}_2F_1(\cdot)$ functions.

In § 2 we rewrite the integral representation of the off-shell Jost function in a form which does not involve the potential explicitly. This form is very suitable for dealing with potentials which have a r^{-1} singularity at the origin (Talukdar *et al* 1977). For scattering on short range potentials half-off-shell T matrix can be written directly in terms of on- and off-shell Jost functions. We demonstrate that this is also true for Coulomb plus short range potentials. Our proof in respect of this is based on a straightforward fiddling with the interacting Green function and does not suffer from any ambiguity. In § 3 we construct the expression for $f_l^{CS}(k,q)$ and make certain useful checks. In § 4 we specialize to the s-wave case and present numerical results for the half-off-shell T matrix for $p-p$ scattering in the ${}^1S_0$ channel.

2. Off-shell Jost function and half-shell T matrix

The Schrödinger equation for the Coulomb plus rank N separable potential is written as

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-V_c(r)\right]\psi_l^{CS}(k,r)=\sum_{i=1}^{N}\lambda_{li}g_{li}(\beta_{li},r)$$
$$\times\int_0^\infty ds\, g_{li}(\beta_{li},s)\psi_l^{CS}(k,s), \qquad (3)$$

where

$$V_c(r)=2k\eta r^{-1} \qquad (4)$$

and $g_{li}(\beta_{li},r)$s the form factors of the potential. The quantities λ_{li} and β_{li} stand for the strength and inverse range parameters. We shall denote the regular solution of (3) by $\varphi_l^{CS}(k,r)$. In terms of $\varphi_l^{CS}(k,r)$ the integral representation (Fuda and Whiting 1973; Talukdar *et al* 1984) of the off-shell Jost function $f_l^{CS}(k,q)$ is

$$f_l^{CS}(k,q)=1+\frac{q^l}{(2l+1)!!}\int_0^\infty qr\,h_l^{(+)}(qr)\left[V_c(r)\varphi_l^{CS}(k,r)\right.$$
$$\left.+\sum_{i=1}^{N}\lambda_{li}g_{li}(\beta_{li},r)\int_0^\infty ds\, g_{li}(\beta_{li},s)\varphi_l^{CS}(k,s)\right]dr \qquad (5)$$

with $h_l^{(+)}(qr)$, the spherical Hankel function in the phase convention of Messiah (1961, 1962). From (3) and (5) we write

$$f_l^{CS}(k,q)=1+\frac{q^l}{(2l+1)!!}\int_0^\infty qr h_l^{(+)}(qr)\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}\right]\varphi_l^{CS}(k,r)\,dr. \quad (6)$$

Integrating (6) twice by parts and using the fact that

$$\lim_{r\to 0} r^{-(l+1)}\varphi_l^{CS}(k,r)=1 \quad (7a)$$

and

$$x h_l^{(+)}(x) \underset{x\to 0}{\sim} x^{-l}(2l-1)!! \quad (7b)$$

we obtain

$$f_l^{CS}(k,q)=\frac{(k^2-q^2)q^l}{(2l+1)!!}\int_0^\infty dr\, qr h_l^{(+)}(qr)\varphi_l^{CS}(k,r). \quad (8)$$

This integral representation will be useful for our future development if $\varphi_l^{CS}(k,r)$ is known in analytic form. In the following we show that the half-off-shell T matrix $T_l^{CS}(k,q,k^2)$ can be written directly in terms of $f_l^{CS}(k,\pm q)$ and $f_l^{CS}(k)$.

The off-shell Jost and physical solutions $f_l^{CS}(k,q,r)$ and $\psi_l^{(+)CS}(k,q,r)$ satisfy the inhomogeneous differential equations (Talukdar *et al* 1984)

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-V_c(r)\right]f_l^{CS}(k,q,r)-\sum_{i=1}^N \lambda_{li}g_{li}(\beta_{li},r) \times\int_0^\infty ds\, g_{li}(\beta_{li},s)f_l^{CS}(k,q,s)=(k^2-q^2)\exp(il\pi/2)\hat{h}_l^{(+)}(qr) \quad (9)$$

and

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-V_c(r)\right]\psi_l^{(+)CS}(k,q,r)-\sum_{i=1}^N \lambda_{li}g_{li}(\beta_{li},r) \times\int_0^\infty ds\, g_{li}(\beta_{li},s)\psi_l^{(+)CS}(k,q,s)=(k^2-q^2)\hat{j}_l(qr) \quad (10)$$

with

$$\hat{h}_l^{(+)}(x)=x h_l^{(+)}(x) \quad (11a)$$

$$\hat{j}_l(x)=\frac{1}{2i}[\hat{h}_l^{(+)}(x)-\hat{h}_l^{(-)}(x)] \quad (11b)$$

and

$$\hat{h}_l^{(-)}(x)=[\hat{h}_l^{(+)}(x)]^*. \quad (11c)$$

We introduce the Green functions for (9) and (10) as (Newton 1982)

$$G_l^{(+)CS}(r,r')=-\varphi_l^{CS}(k,r_<)f_l^{CS}(k,r_>)/\mathscr{F}_l^{CS}(k) \quad (12)$$

and

$$G_l^{(I)CS}(r,r')=-[\varphi_l^{CS}(k,r)f_l^{CS}(k,r')-\varphi_l^{CS}(k,r')f_l^{CS}(k,r)]/\mathscr{F}_l^{CS}(k). \quad (13)$$

Here $\varphi_l^{CS}(k,r)$ and $f_l^{CS}(k,r)$ are the regular and irregular solutions of the Schrödinger equations corresponding to (9) and (10), and $r_<$ and $r_>$ stand for the smaller and larger values of r and r'. Also

$$\mathscr{F}_l^{CS}(k)=(2l+1)!!\,k^{-l}\exp(il\pi/2)f_l^{CS}(k). \quad (14)$$

In terms of these Green functions the solutions of (9) and (10) are given by

$$f_l^{\mathrm{CS}}(k,q,r)=\frac{(k^2-q^2)\exp(il\pi/2)}{\mathscr{F}_l^{\mathrm{CS}}(k)}\left[f_l^{\mathrm{CS}}(k,r)\int_r^\infty \mathrm{d}r'\varphi_l^{\mathrm{CS}}(k,r')\hat{h}_l^{(+)}(qr')\right.$$
$$\left.-\varphi_l^{\mathrm{CS}}(k,r)\int_r^\infty \mathrm{d}r' f_l^{\mathrm{CS}}(k,r')\hat{h}_l^{(+)}(qr')\right] \tag{15}$$

and

$$\psi_l^{(+)\mathrm{CS}}(k,q,r)=-\frac{(k^2-q^2)}{2i\mathscr{F}_l^{\mathrm{CS}}(k)}\left[f_l^{\mathrm{CS}}(k,r)\int_0^r \mathrm{d}r'\varphi_l^{\mathrm{CS}}(k,r')[\hat{h}_l^{(+)}(qr')-\hat{h}_l^{(-)}(qr')]\right.$$
$$\left.+\varphi_l^{\mathrm{CS}}(k,r)\int_r^\infty \mathrm{d}r' f_l^{\mathrm{CS}}(k,r')[\hat{h}_l^{(+)}(qr')-\hat{h}_l^{(-)}(qr')]\right]. \tag{16}$$

Equations (15) and (16) can be combined to write

$$\psi_l^{(+)\mathrm{CS}}(k,q,r)=-\frac{(k^2-q^2)}{2i\mathscr{F}_l^{\mathrm{CS}}(k)}f_l^{\mathrm{CS}}(k,r)\int_0^\infty \mathrm{d}r'\varphi_l^{\mathrm{CS}}(k,r')[\hat{h}_l^{(+)}(qr')-\hat{h}_l^{(-)}(qr')]$$
$$+\frac{1}{2i}[\exp(-il\pi/2)f_l^{\mathrm{CS}}(k,q,r)-\exp(il\pi/2)f_l^{\mathrm{CS}}(k,-q,r)]. \tag{17}$$

In writing (17) we have used

$$\hat{h}_l^{(-)}(x)=\exp(i\pi l)\hat{h}_l^{(+)}(-x). \tag{18}$$

We now make use of (8) and (14) to write (17) in the form

$$\psi_l^{(+)\mathrm{CS}}(k,q,r)=\frac{1}{2}\pi q\left[\left(\frac{k}{q}\right)^l\left\{\frac{f_l^{\mathrm{CS}}(k,q)-f_l^{\mathrm{CS}}(k,-q)}{i\pi q f_l^{\mathrm{CS}}(k)}\right\}\right]\exp(-il\pi/2)f_l^{\mathrm{CS}}(k,r)$$
$$+\frac{1}{2i}[\exp(-il\pi/2)f_l^{\mathrm{CS}}(k,q,r)-\exp(il\pi/2)f_l^{\mathrm{CS}}(k,-q,r)]. \tag{19}$$

Using the prescribed asymptotic boundary conditions on $\psi_l^{(+)\mathrm{CS}}(k,q,r)$ we write the half-off-shell T matrix as

$$T_l^{\mathrm{CS}}(k,q,k^2)=\left(\frac{k}{q}\right)^l\left[\frac{f_l^{\mathrm{CS}}(k,q)-f_l^{\mathrm{CS}}(k,-q)}{i\pi q f_l^{\mathrm{CS}}(k)}\right]. \tag{20}$$

3. Coulomb-distorted Graz potential

In the above we have seen that $T_l^{\mathrm{CS}}(k,q,k^2)$ can be written in terms of appropriate on- and off-shell Jost functions. Here we shall derive a closed form expression for $f_l^{\mathrm{CS}}(k,q)$ by using the integral representation (8). The corresponding on-shell result will be obtained with the help of the limiting eq. (1).

For the regular solution $\varphi_l^{CS}(k,r)$ the integral equation corresponding to (3) can be written in the form

$$\varphi_l^{CS}(k,r)=\varphi_l^{C}(k,r)+\sum_{i=1}^{N}\lambda_{li}d_{li}^{CS}(k)\int_0^r dr'\,g_{li}(\beta_{li},r')G_l^{(R)C}(r,r'), \tag{21}$$

where the Coulomb function (Newton 1982)

$$\varphi_l^{C}(k,r)=r^{l+1}\exp(ikr)\Phi(l+1+i\eta,2l+2;-2ikr) \tag{22}$$

and

$$d_{li}^{CS}(k)=\int_0^\infty ds\,g_{li}(\beta_{li},s)\varphi_l^{CS}(k,s). \tag{23}$$

The regular Coulomb Green function is given by

$$\begin{aligned}G_l^{(R)C}(r,r')=\frac{1}{(2l+1)}(2ik)^{2l+1}(rr')^{l+1}\exp(ik(r+r'))[\bar{\Phi}(l+1+i\eta,2l+2;\\ -2ikr)\Phi(l+1+i\eta,2l+2;-2ikr')-\bar{\Phi}(l+1+i\eta,2l+2;\\ -2ikr')\Phi(l+1+i\eta,2l+2;-2ikr)]\end{aligned} \tag{24}$$

for $r'<r$ and zero elsewhere. Here the function $\bar{\Phi}(a,c;Z)$ is related to the regular confluent hypergeometric function $\Phi(a,c;Z)$ by

$$\bar{\Phi}(a,c;Z)=Z^{1-c}\Phi(a-c+1,2-c;Z). \tag{25}$$

Since (21) represents an inhomogeneous integral equation with a degenerate kernel, it can easily be solved to write

$$d_{li}^{CS}(k)=\frac{1}{\det{}_N A_l^{CS}(k)}\sum_{i=1}^{N}a_{lij}^{CS}(k)\,Y_{lj}^{CS}(k), \tag{26}$$

where

$$A_{lij}^{CS}(k)=\delta_{ij}-\lambda_{lj}\int_0^\infty\int_0^r dr\,dr'\,g_{li}(\beta_{li},r)G_l^{(R)C}(r,r')g_{li}(\beta_{lj},r') \tag{27}$$

and

$$Y_{lj}^{CS}(k)=\int_0^\infty dr\,\varphi_l^{C}(k,r)g_{lj}(\beta_{lj},r). \tag{28}$$

Here $a_{lij}^{CS}(k)s$ stand for the cofactors of $A_{lij}^{CS}(k)s$. From (21), (22) and (26) we get

$$\varphi_l^{CS}(k,r)=\varphi_l^{C}(k,r)+\frac{1}{\det{}_N A_l^{CS}(k)}\sum_{i,j=1}^{N}\lambda_{li}a_{lij}^{CS}(k)\,Y_{lj}^{CS}(k)I_{li}^{CS}(\beta_{li},k,r) \tag{29}$$

with

$$I_{li}^{CS}(\beta_{li},k,r)=\int_0^\infty dr'\,g_{li}(\beta_{li},r')G_l^{(R)C}(r,r'). \tag{30}$$

The form factors of the rank 2 Graz-I potential are given by

$$g_{l1}(\beta_{l1},r)=2^{-l}(l!)^{-1}r^l\exp(-\beta_{l1}r) \tag{31a}$$

and

$$g_{l2}(\beta_{l2},r)=2^{-l}(l!)^{-1}r^{l}\left[1+\frac{\beta_{l2}}{2(l+1)}\frac{\partial}{\partial\beta_{l2}}\right]\exp(-\beta_{l2}r). \tag{31b}$$

Using these form factors we have constructed analytical expressions for $Y_{lj}^{\mathrm{CS}}(k)$, $I_{li}^{\mathrm{CS}}(\beta_{li},k,r)$ and elements of $\det{}_N A_l^{\mathrm{CS}}(k)[i,j=1,2;N=2]$. These results are written as

$$Y_{l1}^{\mathrm{CS}}(k)=2^{-l}(l!)^{-1}\frac{\Gamma(2l+2)}{(\beta_{l1}^2+k^2)^{l+1}}\left(\frac{\beta_{l1}-ik}{\beta_{l1}+ik}\right)^{i\eta}, \tag{32}$$

$$Y_{l2}^{\mathrm{CS}}(k)=2^{-l}(l!)^{-1}\frac{\Gamma(2l+2)}{(\beta_{l2}^2+k^2)^{l+1}}\left(\frac{\beta_{l2}-ik}{\beta_{l2}+ik}\right)^{i\eta}\left[1-\frac{\beta_{l2}\{(l+1)\beta_{l2}+k\eta\}}{(l+1)(\beta_{l2}^2+k^2)}\right], \tag{33}$$

$$I_{l1}^{\mathrm{CS}}(\beta_{l1},k,r)=-\frac{1}{2ik}2^{-l}(l!)^{-1}r^{l+1}\exp(ikr)\sum_{n=0}^{\infty}\frac{\rho_{l1}^n}{n!}\times\theta_{n+1}(l+1+i\eta,2l+2;-2ikr), \tag{34}$$

$$\begin{aligned}I_{l2}^{\mathrm{CS}}(\beta_{l2},k,r)&=[I_{l1}^{\mathrm{CS}}(\beta,k,r)]_{\beta=\beta_{l2}}+\frac{\beta_{l2}}{2(l+1)}\left[\frac{\partial}{\partial\beta}I_{l1}^{\mathrm{CS}}(\beta,k,r)\right]_{\beta=\beta_{l2}}\\&=-\frac{1}{2ik}2^{-l}(l!)^{-1}r^{l+1}\exp(ikr)\sum_{n=0}^{\infty}\frac{\rho_{l2}^n}{n!}\bigg[\theta_{n+1}(l+1+i\eta,2l+2;\\&\quad-2ikr)+\frac{\beta_{l2}}{4ik(l+1)}\theta_{n+2}(l+1+i\eta,2l+2;-2ikr)\bigg],\end{aligned} \tag{35}$$

$$A_{l11}^{\mathrm{CS}}(k)=1-[\bar{G}_l^{(R)C}(\alpha,\beta)]_{\alpha=\beta=\beta_{l1}}, \tag{36}$$

$$A_{l12}^{\mathrm{CS}}(k)=-\lambda_{l2}\left\{[\bar{G}_l^{(R)C}(\alpha,\beta)]_{\alpha=\beta_{l1},\beta=\beta_{l2}}+\frac{\beta_{l2}}{2(l+1)}\times\left[\frac{\partial}{\partial\beta}\bar{G}_l^{(R)C}(\alpha,\beta)\right]_{\alpha=\beta_{l1},\beta=\beta_{l2}}\right\}, \tag{37}$$

$$A_{l21}^{\mathrm{CS}}(k)=-\lambda_{l1}\left\{[\bar{G}_l^{(R)C}(\alpha,\beta)]_{\alpha=\beta_{l2},\beta=\beta_{l1}}+\frac{\beta_{l2}}{2(l+1)}\times\left[\frac{\partial}{\partial\alpha}\bar{G}_l^{(R)C}(\alpha,\beta)\right]_{\alpha=\beta_{l2},\beta=\beta_{l1}}\right\} \tag{38}$$

and

$$\begin{aligned}A_{l22}^{\mathrm{CS}}(k)=1-\lambda_{l2}\bigg\{&[\bar{G}_l^{(R)C}(\alpha,\beta)]_{\alpha=\beta=\beta_{l2}}+\frac{\beta_{l2}}{2(l+1)}\bigg[\frac{\partial}{\partial\alpha}\bar{G}_l^{(R)C}(\alpha,\beta)\\&+\frac{\partial}{\partial\beta}\bar{G}_l^{(R)C}(\alpha,\beta)\bigg]_{\alpha=\beta=\beta_{l2}}+\frac{\beta_{l2}^2}{4(l+1)^2}\left[\frac{\partial^2}{\partial\alpha\partial\beta}\bar{G}_l^{(R)C}(\alpha,\beta)\right]_{\alpha=\beta=\beta_{l2}}\bigg\},\end{aligned} \tag{39}$$

where the function $\theta_\sigma(a,c;Z)$ is related to the ${}_2F_2(\cdot)$ function by (Babister 1967)

$$\theta_\sigma(a,c;Z)=\frac{Z^\sigma}{\sigma(\sigma+c-1)}{}_2F_2(1,c+a;\sigma+1,\sigma+c;Z). \tag{40}$$

In (34) and (35) we have used

$$\rho_{li} = (\beta_{li} + ik)/2ik, \quad i = 1, 2. \tag{41}$$

The quantity $\bar{G}_l^{(R)C}(\alpha, \beta)$ stands for the integral

$$\begin{aligned}\bar{G}_l^{(R)C}(\alpha,\beta) &= 2^{-2l}(l!)^{-2}\int_0^\infty\int_0^r \mathrm{d}r\,\mathrm{d}r'(rr')^l G_l^{(R)C}(r,r')\exp(-\alpha r)\exp(-\beta r')\\ &= \frac{2^{-2l}(l!)^{-2}\Gamma(2l+2)}{(l+1+i\eta)(\beta-ik)}\bigg[(\alpha^2+k^2)^{-l-1}\left(\frac{\alpha-ik}{\alpha+ik}\right)^{i\eta}\\ &\quad\times {}_2F_1\left(1, i\eta-l; l+2+i\eta; \frac{\beta+ik}{\beta-ik}\right) - \frac{(\alpha+\beta)^{-2l-1}}{(\alpha-ik)}\\ &\quad\times {}_2F_1\left(1, i\eta-l; l+2+i\eta; \frac{(\beta+ik)(\alpha+ik)}{(\beta-ik)(\alpha-ik)}\right)\bigg].\end{aligned} \tag{42}$$

In writing the results in (32)–(35) we have made use of the standard integrals (Babister 1967)

$$\int_0^\infty \exp(-\lambda Z)Z^\nu\Phi(a,c;pZ)\mathrm{d}Z = \frac{\Gamma(\nu+1)}{\lambda^{\nu+1}}\,{}_2F_1(a,\nu+1;c;p/\lambda) \tag{43}$$

and

$$\begin{aligned}&\frac{1}{(c-1)}\bigg[\Phi(a,c;Z)\int^Z \mathrm{d}Z'\,Z'^{\sigma+c-2}\exp(-Z')\bar{\Phi}(a,c;Z') - \bar{\Phi}(a,c;Z)\\ &\quad\times\int^Z \mathrm{d}Z'\,Z'^{\sigma+c-2}\exp(-Z')\Phi(a,c;Z)\bigg] = \theta_\sigma(a,c;Z).\end{aligned} \tag{44}$$

The result for $\bar{G}_l^{(R)C}(\alpha,\beta)$ can be obtained by expressing the regular Green function $G_l^{(R)C}(r,r')$ in terms of the physical Green function $G_l^{(+)C}(r,r')$ and making use of a differential equation method (Talukdar *et al* 1985) for evaluating $\bar{G}_l^{(+)C}(\alpha,\beta)$. In Appendix 1 we describe a slightly different method for obtaining the expression for $\bar{G}_l^{(R)C}(\alpha,\beta)$. As noted earlier we shall use (29) in (8) to obtain $f_l^{CS}(k,q)$ in closed form. Meanwhile, we would like to present a simple proof for the reality of $\varphi_l^{CS}(k,r)$. This is important because one knows that the regular solution of the Schrödinger equation should be real, but it is not quite obvious from our results in (29) and (32)–(39).

It is well-known that the reality of $\varphi_l^C(k,r)$ is proved by taking its complex conjugate and then making use of Kummer's first theorem (Slater 1960). As for the second term in (29) we note the following.
Since

$$\left(\frac{\beta_{li}-ik}{\beta_{li}+ik}\right)^{i\eta} = \exp(2\eta)\arctan\left(\frac{k}{\beta_{li}}\right) \tag{45}$$

the expression in (32) and (33) show that $Y_{ij}^{CS}(k)s$ are real. From (34) and (35) we see that for $I_{li}^{CS}(\beta_{li},k,r)$ it is sufficient to prove that $I_{l1}^{CS}(\beta_{l1},k,r)$ is real. To that end we rewrite (34) as

$$I_{l1}^{CS}(\beta_{l1},k,r) = -\frac{2^{-l}(l!)^{-1}}{2ik}r^{l+1}\exp(ikr)\Lambda_{\rho_{l1},1}(l+1+i\eta, 2l+2; -2ikr) \tag{46}$$

with

$$\Lambda_{\rho_{l1},1}(l+1+i\eta, 2l+2; -2ikr) = \sum_{n=0}^{\infty} \frac{\rho_{l1}^n}{n!} \theta_{n+1}(l+1+i\eta, 2l+2; -2ikr). \tag{47}$$

Taking the complex conjugate of (46) we have

$$[I_{l1}^{CS}(\beta_{l1}, k, r)]^* = \frac{2^{-l}(l!)^{-1}}{2ik} r^{l+1} \exp(-ikr) \Lambda_{\beta_{l1}^*,1}(l+1-i\eta, 2l+2; 2ikr). \tag{48}$$

Since $\rho_{l1}^* = 1 - \rho_{l1}$ we get

$$[I_{l1}(\beta_{l1}, k, r)]^* = \frac{2^{-l}(l!)^{-1}}{2ik} r^{l+1} \exp(-ikr) \Lambda_{1-\rho_{l1},1}(l+1-i\eta, 2l+2; 2ikr). \tag{49}$$

Babister (1967) notes that

$$\Lambda_{1-\rho,\sigma}(a,c;Z) = \exp(Z - i\pi\sigma) \Lambda_{\rho,\sigma}(c-a, c; Z\exp(i\pi)). \tag{50}$$

Using (50) in (49) we obtain

$$[I_{l1}^{CS}(\beta_{l1}, k, r)]^* = I_{l1}^{CS}(\beta_{l1}, k, r). \tag{51}$$

Thus $I_{l1}^{CS}(\beta_{l1}, k, r)$ is real. Equations (36)–(39) show that det ${}_2A_l^{CS}(k)$ and $a_{lij}^{CS}(k)$ are real if $\bar{G}_l^{(R)C}(\alpha,\beta)$ is real. The complex conjugate of $\bar{G}_l^{(R)C}(\alpha,\beta)$ is given by

$$\begin{aligned}[\bar{G}_l^{(R)C}(\alpha,\beta)]^* = \frac{2^{-2l}(l!)^{-2}\Gamma(2l+2)}{(l+1-i\eta)} &\left[\frac{1}{(\beta-ik)(\alpha^2+k^2)^{l+1}}\left(\frac{\alpha-ik}{\alpha+ik}\right)^{i\eta}\right. \\ &\times {}_2F_1\left(1, -i\eta-l; l+2-i\eta; \frac{\beta-ik}{\beta+ik}\right) - \frac{(\alpha+\beta)^{-2l-1}}{(\beta+ik)(\alpha+ik)} \\ &\left.\times {}_2F_1\left(1, -i\eta-l; l+2-i\eta; \frac{(\beta-ik)(\alpha-ik)}{(\beta+ik)(\alpha+ik)}\right)\right].\end{aligned} \tag{52}$$

If the hypergeometric functions in (52) are transformed by the recurrence relation (Magnus and Oberhettinger 1949)

$$\begin{aligned}{}_2F_1(a,b;c;Z) = &\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-Z)^{-a} {}_2F_1\left(a, 1+a-c; 1+a-b; \frac{1}{Z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-Z)^{-b} {}_2F_1\left(b, 1+b-c; 1+b-a; \frac{1}{Z}\right)\end{aligned} \tag{53}$$

we get $[\bar{G}_l^{(R)C}(\alpha,\beta)]^* = \bar{G}_l^{(R)C}(\alpha,\beta)$. This analysis proves our desired reality.

From (8) and (29) $[i,j=1,2]$ we obtain

$$\begin{aligned}f_l^{CS}(k,q) = \frac{(k^2-q^2)q^l}{(2l+1)!!} \sum_{L=0}^{l} \frac{(i)^{L-l}(l+L)!}{(2q)^L L!(l-L)!} &\left[\int_0^\infty dr\, r^{-L} \exp(iqr) \varphi_l^C(k,r)\right. \\ &\left.+ \frac{1}{\det {}_2A_l^{CS}(k)} \sum_{i,j=1}^{2} \lambda_{li} a_{lij}^{CS}(k)\, Y_{lj}^{CS}(k) \int_0^\infty dr\, r^{-L} \exp(iqr) I_{li}^{CS}(\beta_{li}, k, r)\right].\end{aligned} \tag{54}$$

In writing (54) we have used the expansion

$$qrh_l^{(+)}(qr) = \sum_{L=0}^{l} \frac{(i)^{2L-l}(l+L)!}{(2iqr)^L L!(l-L)!} \exp(iqr). \tag{55}$$

The integral $\int_0^\infty \mathrm{d}r r^{-L} \exp(iqr)\varphi_l^{\mathrm{C}}(k,r)$ can be obtained in closed form by using (43), while evaluation of $\int_0^\infty \mathrm{d}r r^{-L} \exp(iqr) I_{li}^{\mathrm{CS}}(\beta_{li}, k, r)$ requires the standard result (Babister 1967)

$$\int_0^\infty \exp(-bZ)Z^\nu \theta_\sigma(a,c;pZ)\,\mathrm{d}Z = \frac{\Gamma(\nu+\sigma+1)}{\sigma(\sigma+c-1)b^{\nu+\sigma+1}}$$
$$\times\, {}_3F_2(1, \sigma+a, \nu+\sigma+1; \sigma+1, \sigma+c; p/b),$$
$$\mathrm{Re}\,\sigma > 0, \mathrm{Re}(\sigma+c) > 1, \mathrm{Re}\,\nu > -1, \mathrm{Re}\,b > \mathrm{Re}\,p. \tag{56}$$

This enables us to write $f_l^{\mathrm{CS}}(k,q)$ as an infinite sum over n. To remove the aesthetically unpleasing n sum we write $f_l^{\mathrm{CS}}(k,q)$ as

$$f_l^{\mathrm{CS}}(k,q) = f_l^{\mathrm{C}}(k,q) + \frac{(k^2-q^2)q^l}{(2l+1)!!} \sum_{L=0}^{l} \frac{(i)^{L-l}(l+L)!}{(2q)^L L!(l-L)!} \frac{1}{\det {}_2A_l^{\mathrm{CS}}(k)}$$
$$\times \sum_{i,j=1}^{2} \lambda_{li} a_{lij}^{\mathrm{CS}}(k)\, Y_{lj}^{\mathrm{CS}}(k) F_{li}^{\mathrm{CS}}(\beta_{li}, k, q), \tag{57}$$

where the off-shell Coulomb Jost function (Talukdar *et al* 1983)

$$f_l^{\mathrm{C}}(k,q) = \frac{1}{(2l+1)!!}\left(\frac{q}{k+q}\right)^l \left(\frac{q+k}{q-k}\right)^{i\eta} \sum_{L=0}^{l} \frac{(l+1-L)(l+L)!}{L!}$$
$$\times \left(\frac{q-k}{2q}\right)^L {}_2F_1\left(l+1-i\eta, l+L; 2l+2; \frac{2q}{k+q}\right), \tag{58}$$

$$F_{l1}^{\mathrm{CS}}(\beta_{l1}, k, q) = \frac{2^{-l}(l!)^{-1}(i)^{L-l}}{2k} \frac{\partial^{l+1-L}}{\partial q^{l+1-L}} \sum_{n=0}^{\infty} \frac{\rho_{l1}^n}{n!} \lim_{\varepsilon\to 0} \int_0^\infty \mathrm{d}r$$
$$\times \exp(-[\varepsilon - i(k+q)]r)\theta_{n+1}(l+1+i\eta, 2l+2; -2ikr) \tag{59}$$

and

$$F_{l2}^{\mathrm{CS}}(\beta_{l2}, k, q) = [F_{l1}^{\mathrm{CS}}(\beta, k, q)]_{\beta=\beta_{l2}} + \frac{\beta_{l2}}{2(l+1)}\left[\frac{\partial}{\partial\beta} F_{l1}^{\mathrm{CS}}(\beta, k, q)\right]_{\beta=\beta_{l2}}. \tag{60}$$

From (56) and (59) we have

$$F_{l1}^{\mathrm{CS}}(\beta_{l1}, k, q) = -2^{-l}(l!)^{-1}(i)^{L-l-1} \frac{\partial^{l+1-L}}{\partial q^{l+1-L}} \frac{1}{(k+q)^2} \sum_{n=0}^{\infty} \left(\frac{k-i\beta_{l1}}{k+q}\right)^n$$
$$\times \frac{1}{(2l+2+n)}\, {}_2F_1\left(1, n+l+2+i\eta; n+2l+3; \frac{2k}{k+q}\right). \tag{61}$$

In writing (61) we have made some simplifications by substituting the value of ρ_{l1}. After certain algebraic manipulations, with the help of recurrence relations (Magnus and Oberhettinger 1949)

$$
{}_2F_1(a,b;c;Z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b-c+1;1-Z)
$$

$$
+(1-Z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}
$$

$$
\times {}_2F_1(c-a,c-b;c-a-b+1;1-Z), \tag{62}
$$

$$
{}_2F_1(a,b;c;Z)=(1-Z)^{-a}{}_2F_1\left(\mathrm{a},c-b;c;\frac{Z}{Z-1}\right), \tag{63}
$$

$$
{}_2F_1(a,b;c;Z)=(1-Z)^{c-a-b}{}_2F_1(c-a,c-b;c;Z) \tag{64}
$$

and the integral representation

$$
{}_2F_1(a,b;c;Z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \mathrm{d}t\,t^{b-1}(1-t)^{c-b-1}(1-tZ)^{-a} \tag{65}
$$

we get

$$
F_{l1}^{\mathrm{CS}}(\beta_{l1},k,q)=\frac{2^{-l}(l!)^{-1}(i)^{L-l}\Gamma(2l+2)}{(2k)^{2l+1}}\frac{\partial^{l+1-L}}{\partial q^{l+1-L}}\left\{\frac{1}{(\beta_{l1}-ik)}\right.
$$

$$
\times\left[\frac{\Gamma(i\eta-l)}{\Gamma(l+2+i\eta)}\left(\frac{q+k}{q-k}\right)^{i\eta}(q^2-k^2)^l{}_2F_1\left(1,i\eta-l;l+2+i\eta;\right.\right.
$$

$$
\left.\times\frac{\beta_{l1}+ik}{\beta_{l1}-ik}\right)-\frac{(q-k)^{2l}}{(i\eta-l)\Gamma(2l+2)}\left(\frac{2ik}{\beta_{l1}+ik}\right)^{2l+1}
$$

$$
\times\left\{{}_2F_1\left(1,i\eta-l;i\eta-l+1;\frac{(q-k)(\beta_{l1}+ik)}{(q+k)(\beta_{l1}-ik)}\right)-1\right\}
$$

$$
\left.\left.-(q+k)^{2l}X_{l1}^{\mathrm{CS}}(\beta_{l1},k,q)\right]\right\}, \tag{66}
$$

where

$$
X_{l1}^{\mathrm{CS}}(\beta_{l1},k,q)=\frac{1}{(i\eta-l)\Gamma(2l+2)}-\frac{(q-k)(\beta_{l1}-ik)}{(i\eta-l+1)(q+k)(\beta_{l1}+ik)}
$$

$$
\times\sum_{n=0}^{2l-1}\frac{(-1)^n}{\Gamma(n+3)\Gamma(2l-n)}\left(\frac{\beta_{l1}-ik}{\beta_{l1}+ik}\right)^n{}_2F_{1(n+1)}
$$

$$
\times\left(1,i\eta-l+1;i\eta-l+2;\frac{(q-k)(\beta_{l1}+ik)}{(q+k)(\beta_{l1}-ik)}\right). \tag{67}
$$

Here ${}_2F_{1(n+1)}$ stands for the first $(n+1)$ terms of the hypergeometric series with the given parameters. In deriving (66) we have also used the recurrence relation (Magnus and Oberhettinger 1949)

$$
C\,{}_2F_1(a,b;c;Z)-C\,{}_2F_1(a+1,b;c;Z)+bZ\,{}_2F_1(a+1,b+1;c+1;Z)=0 \tag{68}
$$

iteratively. The expression for $F^{CS}_{l2}(\beta_{l2},k,q)$ is given by

$$F^{CS}_{l2}(\beta_{l2},k,q)=\left[1-\frac{\beta_{l2}}{(2l+2)(\beta_{l2}-ik)}\right][F^{CS}_{l1}(\beta,k,q)]_{\beta=\beta_{l2}}$$
$$+\frac{2^{-l}(l!)^{-1}(i)^{L-l}\beta_{l2}\Gamma(2l+2)}{(2l+2)(2k)^{2l+1}(\beta_{l2}^2+k^2)}\frac{\partial^{l+1-L}}{\partial q^{l+1-L}}\left[\frac{\Gamma(i\eta-l)}{\Gamma(l+2+i\eta)}\right.$$
$$\times\left(\frac{q+k}{q-k}\right)^{i\eta}(q^2-k^2)^l\left\{(l+1+i\eta)-\left[1+\frac{2(l\beta_{l2}+k\eta)}{(\beta_{l2}-ik)}\right]\right.$$
$$\left.\times\,{}_2F_1\left(1,i\eta-l;l+2+i\eta;\frac{\beta_{l2}+ik}{\beta_{l2}-ik}\right)\right\}-\frac{(q+k)^{2l}}{(i\eta-l)\Gamma(2l+2)}$$
$$\times\left(\frac{2ik}{\beta_{l2}+ik}\right)^{2l+1}\left\{\left[(2l+1)-\frac{i(i\eta-l)(q+k)}{(\beta_{l2}-iq)}\right]\right.$$
$$-\left[(2l+1)-\frac{2ik(i\eta-l)}{(\beta_{l2}-ik)}\right]{}_2F_1\left(1,i\eta-l;i\eta-l+1;\right.$$
$$\left.\left.\left.\times\frac{(q-k)(\beta_{l2}+ik)}{(q+k)(\beta_{l2}-ik)}\right)\right\}-(\beta_{l2}+ik)(q+k)^{2l}X^{CS}_{l2}(\beta_{l2},k,q)\right], \tag{69}$$

where

$$X^{CS}_{l2}(\beta_{l2},k,q)=-\frac{2ik(q-k)}{(i\eta-l+1)(q+k)(\beta_{l2}^2+k^2)}\sum_{n=0}^{2l-1}\frac{(-1)^n}{\Gamma(n+3)\Gamma(2l-n)}$$
$$\times\left(\frac{\beta_{l2}-ik}{\beta_{l2}+ik}\right)^{n+1}\left[(n+1)\,{}_2F_{1(n+1)}\left(1,i\eta-l+1;i\eta-l+2;\right.\right.$$
$$\left.\times\frac{(q-k)(\beta_{l2}+ik)}{(q+k)(\beta_{l2}-ik)}\right)-\frac{(i\eta-l+1)(q-k)}{(i\eta-l+2)(q+k)}\left(\frac{\beta_{l2}+ik}{\beta_{l2}-ik}\right)$$
$$\left.\times\,{}_2F_{1(n)}\left(2,i\eta-l+2;i\eta-l+3;\frac{(q-k)(\beta_{l2}+ik)}{(q+k)(\beta_{l2}-ik)}\right)\right]. \tag{70}$$

Since the expressions in (66) and (69) involve a finite sum we can use them in (57) to wrlte $f^{CS}_l(k,q)$ in the "maximal reduced form". Some useful checks on the result for $f^{CS}_l(k,q)$ are now in order. In absence of the Graz potential $[\lambda_{li}=0, i=1,2]$ $f^{CS}_l(k,q)$ goes to $f^C_l(k,q)$. If the Coulomb interaction is turned off $[\eta\to 0]$ $f^{CS}_l(k,q)$ becomes identical with the Graz off-shell Jost function. van Haeringen (1983) has found a closed form expression for the off-shell Jost function for the Coulomb plus Yamaguchi potential. The Yamaguchi potential is the s-wave part of the Graz separable potential $[\lambda_{l2}=0]$. Thus, in the appropriate limit our result for $f^{CS}_l(k,q)$ should yield the result of van Haeringen which is shown below:

In the s-wave case (57) gives

$$f^{CS}_0(k,q)=f^C_0(k,q)+\frac{(k^2-q^2)}{\det{}_2A^{CS}_0(k)}\sum_{i,j=1}^{2}\lambda_{0i}a^{CS}_{0ij}(k)\,Y^{CS}_{0j}(k)F^{CS}_{0i}(\beta_{0i},k,q) \tag{71}$$

with the s-wave Coulomb off-shell Jost function $f_0^C(k,q)$ written as

$$f_0^C(k,q)=\left(\frac{q+k}{q-k}\right)^{i\eta}. \tag{72}$$

The quantities $Y_{0j}^{CS}(k)[j=1,2]$ and the elements of det ${}_2A_0^{CS}(k)$ can be obtained directly from (32), (33) and (36)–(39). From (66), (67), (69) and (70) $F_{0i}^{CS}(\beta_{0i},k,q)[i=1,2]$ can be expressed in the form

$$F_{01}^{CS}(\beta_{01},k,q)=\frac{f_0^C(k,q)}{(1+i\eta)(k^2-q^2)(\beta_{01}-ik)}\left[{}_2F_1\left(1,i\eta;2+i\eta;\frac{\beta_{01}+ik}{\beta_{01}-ik}\right)\right.$$
$$\left.+\frac{i(q-k)}{(\beta_{01}-iq)f_0^C(k,q)}\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{(q-k)(\beta_{01}+ik)}{(q+k)(\beta_{01}-ik)}\right)\right] \tag{73}$$

and

$$F_{02}^{CS}(\beta_{02},k,q)=\left[1-\frac{\beta_{02}}{2(\beta_{02}-ik)}\right][F_{01}^{CS}(\beta,k,q)]_{\beta=\beta_{02}}+\frac{\beta_{02}f_0^C(k,q)}{2(1+i\eta)(k^2-q^2)}$$
$$\times\frac{1}{(\beta_{02}^2+k^2)}\left[\left\{(1+i\eta)-\left(1+\frac{2k\eta}{\beta_{02}-ik}\right)\right.\right.$$
$$\left.\times\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{\beta_{02}+ik}{\beta_{02}-ik}\right)\right\}-\frac{1}{(\beta_{02}-iq)f_0^C(k,q)}$$
$$\times\left\{\frac{(1+i\eta)(k^2-q^2)}{(\beta_{02}-iq)}+i\left(1+\frac{2k\eta}{\beta_{02}-ik}\right)(q-k)\right.$$
$$\left.\left.\times\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{(q-k)(\beta_{02}+ik)}{(q+k)(\beta_{02}-ik)}\right)\right\}\right]. \tag{74}$$

For $\lambda_{02}=0$ the off-shell Jost function $f_0^{CS}(k,q)$ in (71) is identical with that of van Haeringen (1983). The s-wave on-shell Jost function derived from (1) and (71)–(74) is given by

$$f_0^{CS}(k)=\frac{\exp(\pi\eta/2)}{\Gamma(1+i\eta)}\left[1+\frac{1}{\det{}_2A_0^{CS}(k)}\sum_{i,j=1}^{2}\lambda_{0i}a_{0ij}^{CS}(k)\,Y_{0j}^{CS}(k)F_{0i}^{CS}(\beta_{0i},k)\right] \tag{75}$$

with

$$F_{01}^{CS}(\beta_{01},k)=\frac{1}{(1+i\eta)(\beta_{01}-ik)}\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{\beta_{01}+ik}{\beta_{01}-ik}\right) \tag{76}$$

and

$$F_{02}^{CS}(\beta_{02},k)=\left[1-\frac{\beta_{02}}{2(\beta_{02}-ik)}\right][F_{01}^{CS}(\beta,k)]_{\beta=\beta_{02}}+\frac{\beta_{02}}{2(1+i\eta)(\beta_{02}^2+k^2)}$$
$$\times\left[(1+i\eta)-\left(1+\frac{2k\eta}{\beta_{02}-ik}\right){}_2F_1\left(1,i\eta;2+i\eta;\frac{\beta_{02}+ik}{\beta_{02}-ik}\right)\right]. \tag{77}$$

In the next section we shall study the proton–proton half-off-shell T matrix in the ${}^1S_0$ channel by using (71) and (75) in the s-wave version of (20).

4. Half-shell $^1S_0$ p–p T matrix

In the expression for $f_0^{\mathrm{C}}(k,q)$ k is real positive and q is complex with Im $q>0$. One obtains $f_0^{\mathrm{C}}(k,q)$ for real positive k and q by taking the limit Im $q\to 0^+$, which yields

$$f_0^{\mathrm{C}}(k,q)=\begin{cases}\exp(\pi\eta)\left|\dfrac{q+k}{q-k}\right|^{i\eta}, & 0<q<k\\ \left|\dfrac{q+k}{q-k}\right|^{i\eta}, & 0<k<q.\end{cases} \tag{78}$$

Taking notice of (78) we have computed $T_0^{\mathrm{CS}}(k,q,k^2)$ for $q<k$ and $q>k$. We have chosen to work with (Crepinsek *et al* 1975) $\lambda_{01}=-2{\cdot}395\,\mathrm{fm}^{-3}$, $\lambda_{02}=58{\cdot}052\,\mathrm{fm}^{-3}$, $\beta_{01}=1{\cdot}244\,\mathrm{fm}^{-1}$ and $\beta_{02}=2{\cdot}3601\,\mathrm{fm}^{-1}$. We took $(2\eta k)^{-1}=28{\cdot}80\,\mathrm{fm}$. This is the proton Bohr radius.

In table 1 we present our results both for $T_0^{\mathrm{S}}(k,q,k^2)$ and $T_0^{\mathrm{CS}}(k,q,k^2)$ as a function of q for $E_{\mathrm{lab}}=20$ MeV. Note that the values of the half-off-shell T matrix for the pure short range interaction have been obtained by turning off the Coulomb interaction from $T_0^{\mathrm{CS}}(k,q,k^2)$. Thus, the two sets of numbers, namely those for $T_0^{\mathrm{S}}(k,q,k^2)$ and $T_0^{\mathrm{CS}}(k,q,k^2)$ are expected to provide a basis for looking into the role of Coulomb interaction in the $p-p$ half-shell scattering. As expected $T_0^{\mathrm{S}}(k,q,k^2)$ is a continuous function of the off-shell momentum q, and Re $T_0^{\mathrm{S}}(k,q,k^2)$ and Im $T_0^{\mathrm{S}}(k,q,k^2)$ decrease smoothly as q becomes large. In contrast to this $T_0^{\mathrm{CS}}(k,q,k^2)$ exhibits a discontinuity at the on-shell point $k=0{\cdot}49$. Beyond the on-shell point, however, Re $T_0^{\mathrm{CS}}(k,q,k^2)$

Table 1. Half-shell T matrices $T_0^{\mathrm{S}}(k,q,k^2)$ and $T_0^{\mathrm{CS}}(k,q,k^2)$ as a function of the off-shell momentum q at $E_{\mathrm{lab}}=20$ MeV. The number in braces stand for the power of 10 with which entries in the table should be multiplied to get the value of T matrices.

Off-shell momentum q in fm^{-1}	Re $T_0^{\mathrm{S}}(k,q,k^2)$	Im $T_0^{\mathrm{S}}(k,q,k^2)$	Re $T_0^{\mathrm{CS}}(k,q,k^2)$	Im $T_0^{\mathrm{CS}}(k,q,k^2)$
0·01	−6·8867(−1)	−8.7074(−1)	−1·3439(0)	−1·6572(0)
0·02	−6·8849(−1)	−8·7052(−1)	−9·7201(−1)	−1·1986(0)
0.04	−6·8781(−1)	−8·6966(−1)	−7·8572(−1)	−9·6892(−1)
0·06	−6·8668(−1)	−8·6823(−1)	−7·2294(−1)	−8·9150(−1)
0.08	−6·8510(−1)	−8·6623(−1)	−6·9068(−1)	−8·5172(−1)
0·10	−6·8307(−1)	−8·6367(−1)	−6·7035(−1)	−8·2664(−1)
0.20	−6·6668(−1)	−8·4295(−1)	−6·1763(−1)	−7·6163(−1)
0·40	−6·0829(−1)	−7·6911(−1)	−5·3332(−1)	−6·5767(−1)
0·45	−5·8998(−1)	−7·4597(−1)	−5·0169(−1)	−6·1867(−1)
0·47	−5·8240(−1)	−7·3638(−1)	−2·8553(8)	−3·5155(8)
0·49	−5·7469(−1)	−7·2663(−1)	—	—
0·51	−5·6686(−1)	−7·1674(−1)	−3·4086(8)	−4·1994(8)
0·55	−5·5098(−1)	−6·9665(−1)	−4·3403(−1)	−5·3523(−1)
0·60	−5·3080(−1)	−6·7114(−1)	−4·3129(−1)	−5·3185(−1)
0·80	−4·5046(−1)	−5·6956(−1)	−3·8461(−1)	−4·7428(−1)
1·00	−3·7708(−1)	−4·7678(−1)	−3·2837(−1)	−4·0493(−1)
2·00	−1·5995(−1)	−2·0224(−1)	−1·4327(−1)	−1·7668(−1)
3·00	−8·1620(−2)	−1·0320(−1)	−7·3614(−2)	−9·0778(−2)

and Im $T_0^{CS}(k,q,k^2)$ decrease with almost equal gradients as Re $T_0^S(k,q,k^2)$ and Im $T_0^S(k,q,k^2)$. Looking closely into our numbers we see that the Coulomb interaction affects the low off-shell momentum data more significantly and the two sets of numbers for $T_0^{CS}(k,q,k^2)$ and $T_0^S(k,q,k^2)$ are not practically discernible for large q values. On a very general ground one knows that the phase of the half-off-shell T matrix is the scattering phase shift $\delta_l(k)$ (Fuda and Whiting 1973). Our data in table 1 predict $\delta_0^{CS}(k) = 50{\cdot}96°$. This result is in exact agreement with that of Sett *et al* (1988).

This half-shell T matrix in (20) written in terms of Jost function in (57) and its on-shell version does not come out in separable form so as to provide a useful application to three-body calculation. However, a class of nuclear reactions are believed to probe the off-shell two nucleon force directly. These include $(p,2p)$ reaction, (p,p) bremsstrahlung and pion production near threshold. Here the time for the two-nucleon scattering process is restricted on one side only (Redish *et al* 1970). Thus, the centre of mass energy propagation is equal to the final relative kinetic energy. The T matrix applicable in this case is the half-shell T matrix (Redish 1973). Sharma and Jain (1982) and subsequently Kok *et al* (1983) confirmed that off-shell effects are sizeable for $(\alpha,2\alpha)$ reaction. In view of our results in table 1 similar studies are in order for the $(p,2p)$ reaction in which a single proton is knocked out of the nucleus and the momentum transfer distribution measured. The momentum transfer distribution is closely correlated with the distribution of momenta which the nuclear proton had before it was knocked out (McCarthy 1968). Further, since the final momentum variable is measured rather than integrated out, it is a more sensitive test for the reaction mechanism than inelastic scattering.

Acknowledgement

This work was supported in part by the Department of Atomic Energy, Government of India.

Appendix 1. Expression for $\bar{G}_l^{(R)C}(\alpha,\beta)$

From (42) we have

$$\bar{G}_l^{(R)C}(\alpha,\beta) = 2^{-l}(l!)^{-1}\int_0^\infty \mathrm{d}r\, r^l \exp(-\alpha r) I_l(\beta,k,r) \tag{A1}$$

with

$$I_l(\beta,k,r) = 2^{-l}(l!)^{-1}\int_0^\infty \mathrm{d}r'\, r'^l \exp(-\beta r') G_l^{(R)C}(r,r'). \tag{A2}$$

Substituting the value of $G_l^{(R)C}(r,r')$ from (24) in (A2) and invoking the change of variables $Z' = -2ikr'$ and $Z = -2ikr$ we can rewrite $I_l(\beta,k,r)$ as

$$I_l(\beta,k,r) = 2^{-l}(l!)^{-1}(2ik)^{-l-2}(-Z)^{l+1}\exp(-Z/2)\frac{1}{(2l+1)}$$

$$\times\left[\bar{\Phi}(l+1+i\eta, 2l+2; Z)\int_0^Z \mathrm{d}Z' \exp(\rho Z') Z'^{2l+1}\exp(-Z')\right.$$

$$\times \Phi(l+1+i\eta, 2l+2; Z') - \Phi(l+1+i\eta, 2l+2; Z)$$
$$\times \left. \int_0^Z dZ' \exp(\rho Z') Z'^{2l+1} \exp(-Z') \bar{\Phi}(l+1+i\eta, 2l+2; Z') \right], \quad \text{(A3)}$$

where

$$\rho = \frac{\beta + ik}{2ik}. \quad \text{(A4)}$$

We now expand $\exp(\rho Z)$ in power series and make use of the integral in (44). This yields

$$I_l(\beta, k, r) = -2^{-l}(l!)^{-1}(2ik)^{-1} r^{l+1} \exp(ikr) \sum_{n=0}^{\infty} \frac{\rho^n}{n!}$$
$$\times \theta_{n+1}(l+1+i\eta, 2l+2; -2ikr). \quad \text{(A5)}$$

From (A1) and (A5) we have

$$\bar{G}_l^{(R)C}(\alpha, \beta) = 2^{-2l}(l!)^{-2}(\alpha - ik)^{-2l-3} \sum_{n=0}^{\infty} \rho^n \frac{\Gamma(2l+2+n)}{\Gamma(n+2)}$$
$$\times \left(-\frac{2ik}{\alpha - ik} \right)^n {}_2F_1\left(1, n+l+2+i\eta; n+2; -\frac{2ik}{\alpha - ik} \right). \quad \text{(A6)}$$

In deriving (A6) we have used (56) together with ${}_3F_2(a, b, f; f, c; Z) = {}_2F_1(a, b; c; Z)$. As in the text we have removed the infinite sum in (A6) by making use of (62), (64) and (65), and arrived at

$$\bar{G}_l^{(R)C}(\alpha, \beta) = \frac{2^{-2l}(l!)^{-2}\Gamma(2l+2)}{(l+1+i\eta)(\beta - ik)} \left[(\alpha^2 + k^2)^{-l-1} \left(\frac{\alpha - ik}{\alpha + ik} \right)^{i\eta} \right.$$
$$\times {}_2F_1\left(1, i\eta - l; l+2+i\eta; \frac{\beta + ik}{\beta - ik} \right) - (\alpha - ik)^{-1}(\alpha + \beta)^{-2l-1}$$
$$\times \left. {}_2F_1\left(1, i\eta - l; l+2+i\eta; \frac{(\beta + ik)(\alpha + ik)}{(\beta - ik)(\alpha - ik)} \right) \right]. \quad \text{(A7)}$$

It is important to note that (A7) could also be derived from the Sturmian function representation of the Coulomb Green function (Chen and Chen 1972; Laha *et al* 1989).

References

Babister A W 1967 *Transcendental functions satisfying nonhomogeneous linear differential equations* (New York: MacMillan)

Chen J C Y and Chen A C 1972 *Adv. At. Mol. Phys.* **8** 71

Crepinsek L, Lang C B, Oberhumer H, Pleassas W and Zingl H 1975 *Acta Phys. Austriaca* **42** 139

Ford W F 1964 *Phys. Rev.* **133** B1116

Fuda M G and Whiting J S 1973 *Phys. Rev.* **C8** 1255

Jost R 1947 *Helv. Phys. Acta* **20** 256

Kok L P, Holwerda J E and de Maag J W 1983 *Phys. Rev.* **C27** 2548

Laha U, Roy B J and Talukdar B 1989 *J. Phys.* **A22** 3597

Magnus W and Oberhettinger F 1949 *Formulas and Theorems for the Special Functions of Mathematical Physics* (New York: Chelsea)

McCarthy I E 1968 *Introduction to nuclear theory* (New York: John Wiley)
Messiah A 1961 1962 *Quantum mechanics* (New York: Wiley-Interscience) Vols 1 and 2
Newton R G 1982 *Scattering theory of waves and particles* (New York: Springer Verlag)
Redish E F, Stephenson G J Jr and Lerner G M 1970 *Phys. Rev.* **C2** 1665
Redish E F 1973 *Phys. Lett.* **31** 617
Sett G C, Laha U and Talukdar B 1988 *J. Phys.* **A21** 3643
Sharma N R and Jain B K 1982 *Nucl. Phys.* **A377** 201
Slater L J 1960 *Confluent hypergeometric functions* (New York: Cambridge Univ. Press.)
Talukdar B, Mallick N, Sarkar S B and Mondal L 1977 *Phys. Rev.* **C15** 1257
Talukdar B, Saha S and Sasakawa T 1983 *J. Math. Phys.* **24** 683
Talukdar B, Ghosh D K and Sasakawa T 1984 *J. Math. Phys.* **25** 323
Talukdar B, Laha U and Bhattaru S R 1985 *J. Phys.* **A18** L359
van Haeringen H 1979 *J. Math. Phys.* **20** 1109
van Haeringen H 1983 *J. Math. Phys.* **24** 2467
van Leeuwen J M J and Reiner A S 1961 *Physics (Utrecht)* **27** 99