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Classical limit of relativistic quantal system with attractive Coulomb interaction

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Abstract. Using the appropriate harmonic oscillator states and reasonable approximations, we construct coherent wavepackets corresponding to the solutions of the Klein-Gordon equation for the attractive potential $V(r) = -k/r$, $k > 0$, in two and three space dimensions. We deduce the corresponding classical limit in two dimension by requiring that the expectation value $\langle r \rangle$ of the radial variable is large. In the case of three dimensions, besides the condition of large $\langle r \rangle$, we make the uncertainty $\Delta r = [\langle r^2 \rangle - \langle r \rangle^2]^{1/2}$ a minimum with respect to certain parameter of the wavepacket. We then investigate the trajectory traversed by the wavepacket in the classical limit. We find that the classical limit of this relativistic quantal problem gives, in the leading order, the same expression for the rate of motion of the perihelion as that given by the solution of the corresponding special relativistic classical dynamical problem. We also briefly discuss some of the subtle aspects of the classical limit of the relativistic quantal system, in general.

Keywords. Klein-Gordon equation for relativistic Coulomb problem; coherent state for harmonic oscillator; perhelion precession; dispersion with time.

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1. Introduction

The problem of determining the classical limits of quantal systems has been the topic of considerable interest since the formulation of quantum mechanics. In particular, classical limits of two of the most standard systems of non-relativistic quantum mechanics, namely, the harmonic oscillator and the attractive Coulomb problem have been investigated in detail, in terms of suitable wave packets (Schrödinger 1926; Sudarshan 1963; Glauber 1963; Brown 1973; Snieder 1983; Gerry 1986; Bhaumik *et al* 1986; Nandi and Shastry 1989). Gorbaczewski and Prorok reveal a manifest connection between the classical and quantized version of the kepler problem using Stochastic mechanics (Garbaczewski 1986; Gorbaczewski and Prorok 1987a, b). Qian and Huang (1986, 1987) have studied the classical limit of a non-relativistic quantum mechanical system in general, and hydrogen atom (H atom) in particular, in terms of "homogeneous ensemble". In our recent work (Nandi and Shastry 1989), we have examined the nature of the classical limit of two-dimensional (2*D*) and three dimensional (3*D*) non-relativistic H atom using a coherent wavepacket having minimum uncertainty product of position and momentum. The question arises: What is the corresponding special relativistic analogue? More specifically, does the H atom analyzed in terms of the Klein-Gordon (KG) equation have a classical limit such that one gets a transition from the relativistic quantal description to the corresponding relativistic classical dynamical result? These questions are examined in this paper. It should be stressed that while referring to H atom whether through KG equation or through Schrodinger equation, we merely imply a quantal system governed by the potential $V(r) = -k/r$, $k > 0$.

To examine the transition from relativistic quantal description to the corresponding relativistic classical description, we are guided by the following consideration. We assume that the correspondence principle is valid in the relativistic domain and that the transition from the non-relativistic to relativistic domain is smooth as far as the physical observables are concerned. However, certain subtle aspects like the physical interpretation of the wavefunction obtained by solving the KG equation and the uniqueness of the relativistic quantal formulation etc. will be discussed later. In the KG equation, the Coulomb interaction term may be introduced in two different ways. It may be termed together with the rest mass term by treating it as a pure scalar. It may also be treated as the fourth component of an interaction four vector where the other three components are equated to zero. For describing relativistic H atom, the latter choice is generally made because the Coulomb potential corresponds to the fourth Component of the electromagnetic interaction four vectror. However, purely as a quantum-mechanical problem, it is of interest to see whether these two different formulations of KG equation for attractive Coulomb potential lead to significantly different classical limits or not. This is also examined in this paper.

In order to examine the classical limit of the relativistic quantal system for the potential $V(r) = -k/r$, $k > 0$, it is necessary to recapitulate the corresponding special relativistic classical dynamical results. In a suitably chosen inertial frame of reference, the special relativistic classical dynamical equation of motion under central potential $V(r) = -k/r$ is

$$\dot{\mathbf{p}} = -kr^{-3}\mathbf{r}. \tag{1}$$

Here, $\mathbf{r}$ denotes position vector, $\mathbf{p} = \mu_0\gamma_0\dot{\mathbf{r}}$ denotes momentum vector, μ_0 denotes rest mass, $\gamma_0 = [1 - |\dot{\mathbf{r}}|^2/c^2]^{-1/2}$ and c denotes the speed of light. When $k = G\mu_{1g}\mu_{2g}$, then the equation (1) represents the relativistic Kepler problem where μ_{1g} and μ_{2g} refer to the gravitational masses of the interacting bodies. If the gravitational mass is assumed to be the same as inertial mass, then, it is known that the special theory of relativity leads to bound orbits in which perihelion motion at the rate (to the leading order) $|\Omega| = 2\pi[(G\mu_{1g}\mu_{2g})/Lc]^2$ per revolution is present (Engelke and Chander 1970). Here, G is gravitation constant. The symbol $\mathbf{L}$ denotes the angular momentum. On the other hand, if μ_{1g} and μ_{2g} are treated as constants independent of velocity, but equal to the corresponding rest masses, the bound state orbit is found to have a perihelion motion at the rate (to the leading order)

$$|\Omega| = \pi\left(\frac{G\mu_{1g}\mu_{2g}}{Lc}\right)^2 = \pi\left(\frac{k}{Lc}\right)^2 \tag{2}$$

per revolution (see for example, Goldstein 1980, p. 338). In this paper, our analysis corresponds to the latter version of the special relativistic Kepler problem. That is, in the interaction $V(r) = -k/r$, k is assumed to have no velocity dependence. It may be noted that for the motion of mercury in the sun's gravitational field the above results for the rate of perihelion motion obtained from special theory of relativity are much smaller than the result obtained from general theory of relativity given by (to the leading order) $|\Omega| = 6\pi(G\mu_{1g}\mu_{2g}/Lc)^2$ per revolution. However, in our analysis of transition from quantal to classical description of a system with the potential $V(r) = -k/r$ ($k > 0$ and velocity independent), within the framework of special theory of relativity, we can expect to obtain the result given in (2) for the rate of perihelion motion.

We use the coherent state formulation with reasonable approximations to obtain the classical limit of relativistic $2D$ and $3D$ H atom governed by the corresponding KG equations. The necessity of analyzing the classical limit of both $2D$ and $3D$ quantal problem has been elaborated in our earlier work (Nandi and Shastry 1989). The mathematical techniques used here are similar to those adopted by Bhaumik *et al* (1986) and us (Nandi and Shastry 1989) while studying the classical limit of H atom in the nonrelativistic case.

In § 2, we obtain the classical limit of $2D$ and $3D$ H atom represented by KG equation. In § 3, we discuss some subtle features of the analysis of the classical limit of relativistic quantal system in general.

2. Classical limit of 2*D* and 3*D* relativistic quantal system for attractive Coulomb potential

By treating the potential $V(r) = -k/r$, $k > 0$ as the fourth component of an interaction four vector in a suitable frame of reference where the other three components are assumed to be zero, the KG equation for H atom is written as

$$[-\hbar^2 c^2 \nabla^2 + \mu_0^2 c^4]\psi(\mathbf{r}) = [E - V(r)]^2 \psi(\mathbf{r}). \tag{3}$$

This equation is related to the relativistic relation

$$c^2 p^2 + \mu_0^2 c^4 = (E - V)^2. \tag{4}$$

By differentiating (4) with respect to time t and noting that $E - V = \mu_0 \gamma_0 c^2$, one can obtain the relativistic classical dynamical equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu_0 \gamma_0 \mathbf{u}) = -\nabla V(\mathbf{r}) = -\frac{k}{r^3}\mathbf{r}. \tag{5}$$

The conservation of angular momentum $\mathbf{L}$ for the classical dynamical system leads to the relation

$$\dot{\phi} = \frac{L}{\mu_0 \gamma_0 r^2} \tag{6}$$

where ϕ denotes the azimuthal angle variable. The solution of (5) and (6) for $V(r) = -k/r$, where k is positive and constant leads to a bound orbit with a motion of perihelion at the rate given in (2), in the leading order. In the case of a transition to the non-relativistic classical situation the equation of the orbit coincides with that of the elliptical orbit of the nonrelativistic Kepler problem with no motion of perihelion.

The $2D$ and $3D$ radial equations corresponding to (3) for the potential $V(r) = -k/r$,

$k>0$ are respectively (see for example, Schiff 1980 p. 470 for $3D$ case)

$$\frac{\mathrm{d}^2R_{nm}}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}R_{nm}}{\mathrm{d}r}+\left(\frac{E^2-\mu_0^2c^4}{\hbar^2c^2}+\frac{2Ek}{\hbar^2c^2r}+\frac{k^2}{\hbar^2c^2r^2}-\frac{m^2}{r^2}\right)R_{nm}=0, \tag{7}$$

where,

$$\psi(\mathbf{r})=R_{nm}(r)\Phi_m(\phi)$$

and

$$n=1,2,3,\ldots,\ m=0,\ \pm 1,\ \pm 2,\ldots,\ \pm(n-1).$$

$$\frac{\mathrm{d}^2R_{nl}}{\mathrm{d}r^2}+\frac{z}{r}\frac{\mathrm{d}R_{nl}}{\mathrm{d}r}+\left(\frac{E^2-\mu_0^2c^4}{\hbar^2c^2}+\frac{2Ek}{\hbar^2c^2r}+\frac{k^2}{\hbar^2c^2r^2}-\frac{l(l+1)}{r^2}\right)R_{nl}=0, \tag{8}$$

where,

$$\psi(\mathbf{r})=R_{nl}(r)\,Y_{lm}(\theta,\phi)$$

and

$$n=1,\ 2,\ 3,\ldots;\quad l=0,\ 1,\ 2,\ldots,(n-1);\quad m=0,\pm 1,\ \pm 2,\ldots,\pm l.$$

In the $2D$ case, r,ϕ refer to the plane polar coordinates and in the $3D$ case, r,θ,ϕ refer to the spherical polar co-ordinates. The radial equations (7) and (8) are substantially similar in mathematical structure, to the radial Schrodinger equation with an effective interaction term

$$V_{\text{eff}}(r)=-\frac{E}{\mu_0c^2}\frac{k}{r}-\frac{1}{2\mu_0c^2}\frac{k^2}{r^2} \tag{9}$$

and hence, are solved for eigenfunction and eigenenergy using the standard procedure.

Considering the potential $V(r)=-k/r$, $k>0$, as a pure scalar, one can also study the following form of KG equation using a similar procedure

$$[-\hbar^2c^2\nabla^2+(\mu_0c^2+V)^2]\psi(\mathbf{r})=E^2\psi(\mathbf{r}). \tag{10}$$

In table 1, we list the expressions for energy eigenvalues for non-relativistic and relativistic cases both in $2D$ and $3D$.

We consider (3) for detailed analysis in $2D$ and $3D$ cases. Using a similar procedure, (10) can also be analyzed. The results corresponding to this equation will be only summarized.

Using the notations,

$$\eta^4=\frac{4(\mu_0^2c^4-E^2)}{\hbar^2c^2},$$

$\nu=k/\hbar c$, $\lambda=8E\nu/\hbar c$, eq. (3) can be expressed as

$$\left[4\nabla^2+\frac{\lambda}{r}-\eta^4+\frac{4\nu^2}{r^2}\right]\psi(\mathbf{r})=0. \tag{11}$$

This can be compared with the corresponding non-relativistic equation (Nandi *et al* 1989)

$$\left[4\nabla^2+\frac{\lambda'}{r}-(\eta')^4\right]\psi(\mathbf{r})=0 \tag{12}$$

Table 1. Energy eigenvalues from the Schrödinger equation and KG equation in two and three dimensions for the potential $V(r) = -k/r$, $k > 0$.

Dimensions	Non-relativistic energy eigenvalue E_n $n = 1, 2, \ldots$	Relativistic energy eigenvalue from KG equation: When $V(r) = -k/r$, $k > 0$, is treated as fourth component of interaction four vector when the other three components are zero	Relativistic energy eigenvalue from KG equation: When $V(r) = -k/r$, $k > 0$ is treated as a four scalar
Two	$-\dfrac{\mu_0 k^2}{2\hbar^2(n-\frac{1}{2})^2}$	$E_{n,m} = \mu_0 c^2\left[1 - \dfrac{k^2}{2\hbar^2 c^2 (n-\frac{1}{2})^2} + \dfrac{3}{8} \times \dfrac{k^4}{\hbar^4 c^4 (n-\frac{1}{2})^4} - \dfrac{k^4}{2\lvert m\rvert \hbar^4 c^4 (n-\frac{1}{2})^3} + \cdots\right]$, $n = 1, 2, \ldots$ $m = 0, \pm 1, \pm 2, \ldots, \pm(n-1)$.	$E_{n,m} = \mu_0 c^2\left[1 - \dfrac{k^2}{2\hbar^2 c^2 (n-\frac{1}{2})^2} - \dfrac{1}{8} \times \dfrac{k^4}{\hbar^4 c^4 (n-\frac{1}{2})^4} + \dfrac{k^4}{2\lvert m\rvert \hbar^4 c^4 (n-\frac{1}{2})^3} + \cdots\right]$, $n = 1, 2, \ldots$ $m = 0, \pm 1, \pm 2, \ldots, \pm(n-1)$.
Three	$-\dfrac{\mu_0 k^2}{2\hbar^2 n^2}$	$E_{n,l} = \mu_0 c^2\left[1 - \dfrac{k^2}{2\hbar^2 c^2 n^2} + \dfrac{3}{8}\dfrac{k^4}{\hbar^4 c^4 n^4} - \dfrac{k^4}{2\hbar^4 c^4 (l+\frac{1}{2}) n^3} + \cdots\right]$ $n = 1, 2, \ldots$ $l = 0, 1, \ldots, (n-1)$	$E_{n,l} = \mu_0 c^2\left[1 - \dfrac{k^2}{2\hbar^2 c^2 n^2} - \dfrac{1}{8}\dfrac{k^4}{\hbar^4 c^4 n^4} + \dfrac{k^4}{2\hbar^4 c^4 (l+\frac{1}{2}) n^3} + \cdots\right]$ $n = 1, 2, \ldots$ $l = 0, 1, \ldots, (n-1)$

where,

$$\lambda' = \frac{\mu_0 k}{\hbar^2}, \quad (\eta')^4 = -\frac{8\mu_0 E}{\hbar^2}.$$

We see that, structurally, eqs (11) and (12) are same except for the term $(4v^2/r^2)\psi$ on the left side of the former. If, as a first step, one neglects $(4v^2/r^2)\psi$ term in (11) as a small perturbation, then the mathematical procedure based on construction of coherent state on harmonic oscillator basis can be used to construct the appropriate coherent wavepacket for the relativistic problem. For our purpose, this approximation is likely to be quite reasonable due to the fact that in the classical limit, we are really interested in stable orbits of the coherent wavepacket having large values of $\langle r \rangle$. Hence, using this approximation, we construct the coherent wavepacket for the relativistic problem at time $t = 0$. However, in order to study the subtle relativistic effects in analytical form, we include the effect of the full relativistic hamiltonian for investigating the time evolution of the wavepacket for $t > 0$. It is this crucial step which leads to the subtler result like the motion of perihelion in the classical limit.

2.1 *Classical limit of relativistic 2D H atom*

Following the procedure adopted earlier, and neglecting the term $(4v^2/r^2)\psi$, eq. (11) can be expressed in the form

$$[a_+^\dagger a_+ + a_-^\dagger + a_- + 1]\psi = \frac{\lambda}{2\eta^2}\psi \tag{13}$$

where, a_+, a_- and $a_+^\dagger$, $a_-^\dagger$ are annihilation and creation operators respectively and defined as

$$a_+ = \frac{1}{\sqrt{2\eta}}\left(\frac{\partial}{\partial \xi} + \eta^2 \xi^*\right), \quad a_- = \frac{1}{\sqrt{2\eta}}\left(\frac{\partial}{\partial \xi^*} + \eta^2 \xi\right)$$

$$a_+^\dagger = \frac{1}{\sqrt{2\eta}}\left(-\frac{\partial}{\partial \xi^*} + \eta^2 \xi\right), \quad a_-^\dagger = \frac{1}{\sqrt{2\eta}}\left(-\frac{\partial}{\partial \xi} + \eta^2 \xi^*\right). \tag{14}$$

The co-ordinates x, y are related to the complex co-ordinates ξ, ξ^* as follows:

$$x + iy = 2\xi^2, \quad x - iy = 2(\xi^*)^2. \tag{15}$$

Equation (13) appears identical to the Schrödinger equation for a $2D$ harmonic oscillator. The quantities $a_+^\dagger a_+$ and $a_-^\dagger a_-$ represent number operators having eigenvalues n_+ and n_- respectively, where, $n_\pm = 0, 1, 2, \ldots$

Hence,

$$n_+ + n_- + 1 = \frac{\lambda}{2\eta^2}. \tag{16}$$

It can be readily seen using (14) and (15) that

$$-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial n}\right)\psi = -\frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-)\psi. \tag{17}$$

Since, $\psi(\mathbf{r})$ is an eigenfunction of $-i\hbar(x(\partial/\partial y)-y(\partial/\partial x))$ having eigenvalue $m\hbar$, where, $m=0, \pm 1, \pm 2,\ldots$, we have

$$-n_+ + n_- = 2m. \tag{18}$$

Using (16) and (18) and on substituting the values of λ and η, we get

$$E=\mu_0 c^2\left[1-\frac{v^2}{2(n-\frac{1}{2})^2}+\frac{3}{8}\frac{\gamma^4}{(n-\frac{1}{2})^4}+\cdots\right] \tag{19}$$

where, $n=n_+ + m + 1 = 1, 2,\ldots$.

On the other hand, the solution of (7) leads to the following exact expression for energy

$$E_{\text{exact}}=\mu_0 c^2\left[1-\frac{v^2}{2(n-\frac{1}{2})^2}+\frac{3}{8}\frac{v^4}{(n-\frac{1}{2})^4}-\frac{v^4}{2|m|(n-\frac{1}{2})^3}+\cdots\right]. \tag{20}$$

This expansion is valid when $|m|>k/\hbar_c>0$ and we restrict to these values of m. Comparison of (19) and (20) indicates that omission of the term $(4v^2/r^2)\psi$ in (11) leads to the absence of the non-degenerate term $(-v^4/[2|m|(n-\frac{1}{2})^3])$ in the energy expression. The magnitude of this term is, however, sufficiently small in the large quantum number limit, as compared to that of the term $(-v^2/[2(n-\frac{1}{2})^2])$.

In view of the above observation, we approximate the wavepacket for the 2*D* relativistic H atom at $t=0$ by the coherent state constructed using the eigensolution of (13), and study its time development incorporating the role of the interaction term $-(4v^2/r^2)$ (or, equivalently, the term $-v^4/[2|m|(n-\frac{1}{2})^3]$) in the energy expression) also. Thus, we express the coherent state $|\alpha_+,\alpha_-\rangle$ at $t=0$ in terms of eigenfunctions $|n_+,n_-\rangle$ of the number operators $a_+^\dagger a_+$ and $a_-^\dagger a_-$ (Nandi and Shastry 1989):

$$|\alpha_+,\alpha_-\rangle=\sum_{n\pm=0}^{\infty}\exp[-\tfrac{1}{2}(|\alpha_+|^2+|\alpha_-|^2)]\frac{\alpha_+^{n_+}\alpha_-^{n_-}}{\sqrt{n_+!n_-!}}|n_+,n_-\rangle \tag{21}$$

where,

$$a_\pm|\alpha_+,\alpha_-\rangle=\alpha_\pm|\alpha_+,\alpha_-\rangle. \tag{22}$$

Following the method adopted earlier (Nandi and Shastry 1989), we write,

$$\alpha_\pm=|\alpha_\pm|\exp(\mp i\Delta) \tag{23}$$

along with the parametrization,

$$|\alpha_+|=\gamma\cos\chi,\quad |\alpha_-|=\gamma\sin\chi \tag{24}$$

such that $|\alpha_+|^2+|\alpha_-|^2=\gamma^2$ which is assumed to be large in the classical limit as γ^2 will be shown later to be proportional to $\langle r\rangle$.

Let us now introduce the time evolution of the coherent state for $t>0$. The coefficients of the coherent state are peaked around maximum of $(\alpha_+^{n_+}\alpha_-^{n_-})/\sqrt{n_+!n_-!}$. The following values of n_+ and n_- respectively are found to maximize the coefficients:

$$N_+=\gamma^2\cos^2\chi,\quad N_-=\gamma^2\sin^2\chi. \tag{25}$$

Expanding the energy E_{exact} as a power series around these values, we can write from (20),

$$E_{\text{exact}} = \mu_0 c^2 \left[1 - \frac{2v^2}{(n_+ + n_- + 1)^2} + \frac{6v^4}{(n_+ + n_- + 1)^4} - \frac{8v^4}{|n_+ - n_-|(n_+ + n_- + 1)^3} + \cdots \right]$$

$$\simeq E_{\text{peak}} + \hbar w_c(\delta_+ + \delta_-) + \hbar w_c'(\delta_+ - \delta_-) - \frac{3\hbar w_c}{2\gamma^2}(\delta_+ + \delta_-)^2 - \frac{3\hbar w_c'}{2\gamma^2}(\delta_+ - \delta_-)^2 + \cdots \tag{26}$$

where,

$$E_{\text{peak}} \simeq \mu_0 c^2 \left[1 - \frac{2v^2}{\gamma^4} + \frac{6v^4}{\gamma^8} - \frac{8v^4}{\gamma^8 |\cos 2\chi|} + \cdots \right] \tag{27}$$

$$w_c \simeq \frac{\mu_0 c^2}{\hbar} \left[\frac{4v^2}{\gamma^6} - \frac{24v^4}{\gamma^{10}} + \frac{24v^4}{\gamma^{10}|\cos 2\chi|} + \cdots \right] \tag{28}$$

$$w_c' \simeq \pm \left[\frac{\mu_0 c^2}{\hbar} \cdot \frac{8v^4}{\gamma^{10}\cos^2 2\chi} + \cdots \right] \tag{29}$$

and

$$\delta_\pm = n_\pm - N_\pm.$$

It may be noted that w_c' originates from the term $\{[-v^4/[2|m|(n-\frac{1}{2})^3]]\}$ in the expression for energy eigenvalue.

In (26), the energy E_{exact} is expressed in Taylor series around $N_\pm$ for which the coefficients of the coherent state $|\alpha_+, \alpha_-\rangle$ in (21) are peaked. The first three terms in (26) of E_{exact} give an expression for energy which is akin to the corresponding result for the harmonic oscillator. The fourth and the fifth terms in (26) for E_{exact} are smaller by a factor γ^2 (which, in turn, is of the order $N_\pm$ with respect to the second and the third terms and hence are small. Taking the first three terms the leading order terms one obtains, through the factor $\exp(-iE_{\text{exact}}t/\hbar)$, the following approximate time-dependence induced in α_+ and α_-, over and above some predominant phase factor $\exp(-iE_{\text{peak}}t/\hbar)$:

$$\alpha_\pm(t) = \alpha_\pm \exp(-i(w_c \pm w_c')t). \tag{30}$$

This follows from a procedure akin to that adopted in the case of deduction of the time-dependence of the harmonic oscillator coherent state. A similar procedure for obtaining the approximate time-dependence for non-relativistic H atom coherent state using the harmonic oscillator basis has been adopted in the earlier papers (Bhaumik *et al* 1986; Nandi and Shastry 1989).

The time dependent coherent state is given by

$$|\alpha_+, \alpha_-\rangle = |\gamma, \chi, \Delta, t\rangle = |\gamma\cos\chi\exp(-i(\Delta + w_c + w_c')t), \times\gamma\sin\chi\exp(i(\Delta - w_c + w_c')t)\rangle. \tag{31}$$

Using (14), (15) and (22) and properties of the above coherent state, the expectation values of the dynamical variables x, y, r and angular momentum L (where $L = xp_y - yp_x$) are found out. Choosing $\Delta = 0$, the results, up to the order of $w_c' t$ are as follows:

$$\langle x \rangle = \frac{\gamma^2}{\eta^2}[\cos 2w_c t + \sin 2\chi - \cos 2\chi \sin 2w_c t \sin 2w_c' t]$$

$$= \langle x_0 \rangle - \langle y_0 \rangle \sin 2w_c' t \tag{32a}$$

$$\langle y \rangle = \frac{\gamma^2}{\eta^2}[\cos 2\chi \sin 2w_c t + (\cos 2w_c t + \sin 2\chi)\sin 2w_c' t]$$

$$= \langle y_0 \rangle + \langle x_0 \rangle \sin 2w_c' t \tag{32b}$$

$$\langle r \rangle = \frac{\gamma^2}{\eta^2}[1 + \sin 2\chi \cos 2w_c t] = \langle r_0 \rangle \tag{32c}$$

$$\langle L \rangle = \tfrac{1}{2}\hbar\gamma^2 \cos 2\chi = \langle L_0 \rangle \tag{32d}$$

where, $\langle x_0 \rangle$, $\langle y_0 \rangle$, $\langle r_0 \rangle$ and $\langle L_0 \rangle$ are the corresponding expectation values for non-relativistic $2D$ H atom (Nandi and Shastry 1989):

$$\langle x_0 \rangle = \frac{\gamma^2}{\eta^2}[\cos 2w_c t + \sin 2\chi] \tag{33a}$$

$$\langle y_0 \rangle = \frac{\gamma^2}{\eta^2}\cos 2\chi \sin 2w_c t \tag{33b}$$

These values of $\langle x \rangle$ and $\langle y \rangle$ satisfy the following equation of orbit:

$$\frac{\left(\langle x \rangle - \frac{\gamma^2}{\eta^2}\sin 2\chi\right)^2}{(\gamma^2/\eta^2)^2} + \frac{(\langle y \rangle)^2}{\left(\frac{\gamma^2}{\eta^2}\cos 2\chi\right)^2}$$

$$= 1 + \frac{2\eta^2}{\gamma^2}\sin 2w_c' t\left[\frac{\langle x_0 \rangle}{\cos 2\chi}\sin 2w_c t - \langle y_0 \rangle \cos 2w_c t\right]. \tag{34}$$

If one neglects the second term on the right side of (34), then this represents an ellipse with the symmetry axes coinciding with the co-ordinate axes and one of the foci being the origin of the co-ordinate system:

$$\frac{(\langle x \rangle - \mathsf{a}\varepsilon)^2}{\mathsf{a}^2} + \frac{(\langle y \rangle)^2}{\mathsf{b}^2} = 1 \tag{35}$$

where, semimajor axis:

$$\mathsf{a} = \gamma^2/\eta^2, \tag{36a}$$

semi minor axis:

$$\mathsf{b} = (\gamma^2/\eta^2)\cos 2\chi, \tag{36b}$$

eccentricity:

$$\varepsilon = \sin 2\chi. \tag{36c}$$

The expectation value of energy other than the rest energy term is (to the leading order):

$$\langle E - \mu_0 c^2 \rangle \simeq -\frac{2\mu_0 k^2}{\hbar^2 \gamma^4}. \tag{36d}$$

The time period of revolution is

$$T = \frac{2\pi}{2w_c} \simeq \frac{\pi \hbar^3 \gamma^6}{4\mu_0 k^2}\left[1 + \frac{6v^2}{\gamma^4} - \frac{6v^2}{\gamma^4 |\cos 2\chi|} + \cdots\right]. \tag{36e}$$

Therefore, to the leading order, the elliptical orbit given by (35) along with the period of revolution and the energy expectation value (other than the rest energy), is identical with that for the $2D$ non-relativistic H atom, which, in turn, is same as the elliptical orbit of the Kepler problem in non-relativistic classical mechanics (Nandi and Shastry 1989). Clearly, the deviation from the elliptical orbit, like the motion of perihelion, will be generated by the second term on the right side of (34).

Considering w_c' and eccentricity ε to be small (χ small), some algebric calculation leads to the following polar form of the equation given by (34):

$$r = \frac{a(1-\varepsilon^2)}{1 - \varepsilon \cos(1+\alpha)\phi} \tag{37}$$

where,

$$\alpha = \frac{w_c'}{w_c}. \tag{38}$$

Here, r, ϕ represent the polar co-ordinate with one of the foci of the ellipse (35) as the pole and the major axis as the reference axis. The quantity $(1+\alpha)\phi$ must increase by an integral multiple of 2π, if r is to return to some initial value. Obviously, this will not occur for an increase of ϕ by 2π but occurs for an increase of ϕ by say, $(2\pi + \Omega)$. Then Ω is the rate of perihelion motion and the magnitude of Ω is given by

$$|\Omega| = 2\pi|\alpha| = 2\pi\left|\frac{w_c'}{w_c}\right| \simeq \pi\left(\frac{k}{Lc}\right)^2\left[1 - \frac{3}{2}\left(\frac{k}{Lc}\right)^2 |\cos 2\chi|(1 - |\cos 2\chi|) + \cdots\right] \tag{39}$$

per revolution. In the leading order, this is same as the corresponding special relativistic classical mechanical value of $|\Omega|$ given in (2).

Thus, we see that if the approximate coherent state constructed at $t = 0$ is allowed to evolve by including the effect of the term $4v^2/r^2$ in the hamiltonian, one generates a wavepacket which reproduces, to the leading order, the motion of perihelion in the same rate as obtained in the special relativistic classical dynamics. It should be stressed that (2) is also a leading order expression and there are much smaller higher order corrections. This indicates that our present formulation successfully incorporates the leading order special relativistic effects in the classical limit.

2.2 Classical limit of relativistic 3D H atom

Following a similar treatment as done in the previous subsection and those adopted in our earlier paper (Nandi and Shastry 1989), the classical limit of relativistic $3D$ H atom can be studied. The energy expression for relativistic $3D$ H atom obtained on neglecting the term $(4\nu^2/r^2)\psi$ in (11) and treating it on the four dimensional or two $2D$ harmonic oscillator basis with a constraint (Bhaumik *et al* 1986), is given by

$$E = \mu_0 c^2\left[1 - \frac{\nu^2}{2n^2} + \frac{3}{8}\frac{\nu^4}{n^4} + \cdots\right] \tag{40}$$

where,

$$n = 1, 2, \ldots.$$

From the expression for the corresponding exact energy obtained by solving the radial eq. (8) and listed in table 1, it is obvious that the expressions for E and E_{exact} agree up to the third term.

Hence, following the same idea as that mentioned in the last subsection, the approximate coherent state expression at $t = 0$ is structurally same as that in the corresponding non-relativistic case. Let $|\alpha_+, \alpha_-, \beta_+, \beta_-\rangle$ represent the coherent state constructed using the corresponding four dimensional or two $2D$ harmonic oscillator representation or the KG eq. (11) on neglecting the term $(4\nu^2/r^2)\psi$. Detail of the mathematical steps involved are similar to those described in the earlier works (Bhaumik *et al* 1986; Nandi and Shastry 1989). Writing,

$$\alpha_\pm = |\alpha_\pm|\exp(\mp i\Delta_1)$$

and

$$\beta_\pm = |\beta_\pm|\exp(\pm i\Delta_2)$$

and using the following parametrization

$$|\alpha_+| = \gamma\cos\chi\cos\delta \quad |\alpha_-| = \gamma\sin\chi\cos\delta$$

$$|\beta_+| = \gamma\sin\chi\sin\delta \quad |\beta_-| = \gamma\cos\chi\sin\delta$$

such that $|\alpha_+|^2 + |\alpha_-|^2 + |\beta_+|^2 + |\beta_-|^2 = \gamma^2 \gg 1$ in the classical limit where $\langle r\rangle$ is large, and adopting the similar method of constructing the time dependence of the wavepacket for $t > 0$, as done in the last subsection, we get,

$$\begin{aligned}\langle x\rangle &= \frac{\gamma^2}{\eta^2}\sin 2\delta[\cos(\Delta_1 + \Delta_2 + 2w_c' t)\{\cos 2w_c t + \tfrac{1}{2}\sin 2\chi \\ &\quad \times(\cos^2\chi\cos^2\delta + \sin^2\chi\sin^2\delta)^{-1/2}(\cos^2\chi\sin^2\delta + \sin^2\chi\cos^2\delta)^{-1/2}\} \\ &\quad - \sin(\Delta_1 + \Delta_2 + 2w_c' t)\cos 2\chi\sin 2w_c t] \\ &= \cos(\Delta_1 + \Delta_2 + 2w_c' t)\langle x_0\rangle - \sin(\Delta_1 + \Delta_2 + 2w_c' t)\langle y_0\rangle\end{aligned} \tag{41a}$$

$$\begin{aligned}\langle y\rangle &= \frac{\gamma^2}{\eta^2}\sin 2\delta[\sin(\Delta_1 + \Delta_2 + 2w_c' t)\{\cos 2w_c t + \tfrac{1}{2}\sin 2\chi \\ &\quad \times(\cos^2\chi\cos^2\delta + \sin^2\chi\sin^2\delta)^{-1/2}(\cos^2\chi\sin^2\delta + \sin^2\chi\cos^2\delta)^{-1/2}\}\end{aligned}$$

$$+\cos(\Delta_1+\Delta_2+2w_c't)\cos 2\chi \sin 2w_c t]$$

$$=\sin(\Delta_1+\Delta_2+2w_c't)\langle x_0\rangle+\cos(\Delta_1+\Delta_2+2w_c't)\langle y_0\rangle \tag{41b}$$

$$\langle z\rangle=\frac{\gamma^2}{\eta^2}\sin 2\chi\cos 2\delta[\cos 2w_c t+\tfrac{1}{2}\sin 2\chi$$

$$(\cos^2\chi\cos^2\delta+\sin^2\chi\sin^2\delta)^{-1/2}(\cos^2\chi\sin^2\delta+\sin^2\chi\cos^2\delta)^{-1/2}]$$

$$=\langle z_0\rangle \tag{41c}$$

$$\langle r\rangle=\frac{\gamma^2}{\eta^2}[2(\cos^2\chi\cos^2\delta+\sin^2\chi\sin^2\delta)^{1/2}(\cos^2\chi\sin^2\delta+\sin^2\chi\cos^2\delta)^{1/2}$$

$$+\sin 2\chi\cos 2w_c t]$$

$$=\langle r_0\rangle \tag{41d}$$

$$\langle L_z\rangle=\frac{\gamma^2}{2}\hbar\cos 2\chi=\langle L_{z_0}\rangle \tag{41e}$$

where,

$$w_c=\frac{\mu_0 c^2}{\hbar}\left[\frac{4v^2}{\gamma^6}-\frac{24v^4}{\gamma^{10}}+\frac{24v^4}{\gamma^{10}|\cos 2\chi|}+\cdots\right] \tag{42a}$$

and

$$w_c'=\pm\frac{\mu_0 c^2}{\hbar}\left[\frac{8v^4}{\gamma^{10}\cos^2 2\chi}+\cdots\right] \tag{42b}$$

Here, $\langle x_0\rangle$, $\langle y_0\rangle$, $\langle z_0\rangle$, $\langle r_0\rangle$ and $\langle L_{z_0}\rangle$ are the expectation values of the corresponding variables in the 3D non-relativistic case.

Equations (41a, c) show that the projection of the actual orbit on the $z-x$ plane is given by

$$\langle z\rangle=\frac{\sin 2\chi}{\tan 2\delta\cos(\Delta_1+\Delta_2+2w_c't)}\langle x\rangle+\frac{\gamma^2}{2\eta^2}\sin 4\chi\cos 2\delta\sin 2w_c t$$

$$\times\tan(\Delta_1+\Delta_2+2w_c't). \tag{43}$$

This equation indicates a significant difference with the corresponding result for 3D non-relativistic H atom. In the non-relativistic case, for $(\Delta_1+\Delta_2)=0$, the projection of the actual orbit on the $z-x$ plane is (Nandi and Shastry 1989)

$$\langle z_0\rangle=\frac{\sin 2\chi}{\tan 2\delta}\langle x_0\rangle \tag{44}$$

which represents a fixed straight line making an angle

$$\theta_0=\tan^{-1}\left(\frac{\sin 2\chi}{\tan 2\delta}\right)$$

with the x-axis. But, in the corresponding relativistic case, even on putting $(\Delta_1+\Delta_2)=0$ in (43), the resulting equation does not represent a fixed straight line. This means that the actual orbit is not strictly planar, though the mean (time average) orbit is confined in the same plane as in the corresponding non-relativistic case, when considered up to terms of the order of $w_c' t$, since, the time average of $\sin 2w_c t \sin 2w_c' t$ is zero.

However, the requirement that $\Delta r = [\langle r^2\rangle - \langle r\rangle^2]^{1/2}$ a minimum, which is appropriate for the classical limit, implies that $\delta = (\pi/4)$. In this case, for $(\Delta_1 + \Delta_2) = 0$, the projection of the actual orbit on the $z - x$ plane becomes identical with that in the corresponding non-relativistic case. In this case, $\langle z\rangle = 0$ and the expressions for $\langle x\rangle$, $\langle y\rangle$, $\langle r\rangle$ and $\langle L_z\rangle = \langle L\rangle$ become identical with those given in eqs (32a–d) in the last subsection. Therefore, the energy expectation value, the equation of the orbit and its characteristic parameters, the period of revolution along with the rate of perihelion motion for the $3D$ relativistic H atom in the limit of large $\langle r\rangle$ and minimum Δr, are identical with the corresponding results for $2D$ relativistic H atom in the limit of large $\langle r\rangle$. These results are listed in the appendix.

The wave eq. (10) can also be analyzed and the classical limit can be obtained both in $2D$ and $3D$ cases by methods parallel to those described in this section. These results are also summarized in the appendix. The results indicate that, to the leading order, both versions of the KG equation, namely (3) and (10) lead to similar relativistic classical limits.

The dispersion with time of the coherent state of the $2D$ and $3D$ relativistic H atom in the classical limit can be analyzed in the same way as done in case of $2D$ and $3D$ non-relativistic H atom (Nandi and Shastry 1989). For the special case of the wavepacket moving in a circular orbit ($\chi = 0$), considering up to the fifth term in the energy expression (26), we get

$$\langle r\rangle = \frac{\gamma^2}{\eta^2} \tag{45a}$$

$$\langle x\rangle = \frac{1}{\eta^2}\exp\left(-\frac{18(w_c \pm w_c')^2 t^2}{\gamma^2} - \frac{1}{2\gamma^2}\right)(\gamma^2 \cos(2w_c t \pm 2w_c' t) + 6(w_c \pm w_c')t \sin(2w_c t \pm 2w_c' t)) \tag{45b}$$

$$\langle y\rangle = \frac{1}{\eta^2}\exp\left(-\frac{18(w_c \pm w_c')^2 t^2}{\gamma^2} - \frac{1}{2\gamma^2}\right)(\gamma^2 \sin(2w_c t \pm 2w_c' t) - 6(w_c \pm w_c')t \cos(2w_c t \pm 2w_c' t)). \tag{45c}$$

The above expressions show that as $t \to \infty$, both $\langle x\rangle$ and $\langle y\rangle$ go to zero like

$$\exp\left(-\frac{18(w_c \pm w_c')^2 t^2}{\gamma^2} - \frac{1}{2\gamma^2}\right)$$

keeping $\langle r\rangle$ constant. That is, the wavepacket spreads uniformly over a circular ring of large radius in the dissipation time

$$T_s \simeq \frac{\gamma}{\sqrt{18}(w_c \pm w_c')}. \tag{46}$$

This time T_s is practically same as the dissipation time in the corresponding non-relativistic situation, namely, $\gamma/\sqrt{18}\, w_c$, since, $|w_c'| \ll |w_c|$ because $|w_c'/w_c| \simeq \pi(k/Lc)^2 \ll 1$. It is found that T_s is of the order of 10^{36} years for planet like the earth.

3. Discussion

The results of §2 indicate that in the classical limit, the motion of the coherent wavepacket predicts, to the leading order, the motion of perihelion as that deduced from the corresponding special relativistic classical dynamical equation for the potential $V(r) = -k/r$, $k > 0$. The KG equation for this potential differs in its mathematical structure, from the corresponding Schrodinger equation by the presence of the term involving r^{-2}, in addition to the usual interaction $\propto r^{-1}$. Therefore, the analysis of the KG equation in terms of the harmonic oscillator type coherent state requires some appropriate approximations which in the classical limit do not overshadow the subtle relativistic features.

Some features of the analysis for classical limit of 2*D* and 3*D* H atom by using coherent wavepacket on harmonic oscillator basis and the particular type of parametrization and time evolution of the state have already been discussed in our earlier paper (Nandi and Shastry 1989). These are not repeated here. It may, of course, be pointed out that the results obtained in the last section are valid only for small values of eccentricity of the orbit of motion.

Now, we briefly discuss some other important aspects associated with the present analysis of the classical limit of the relativistic H atom. It is well-known that even though KG equation provides a relativistic wave equation, its solutions do not confirm the same interpretation as that of the solutions of Schrödinger equation (see, for example, Schiff 1968, p. 468). In spite of this, we believe that the classical limit of coherent wavepacket from the solution of the KG equation for the potential $V(r) = -(k/r)$, $k > 0$ is important. This is because the solution of the KG equation and the corresponding results for eigenvalues and eigenfunctions are consistent with those in the corresponding non-relativistic case. There is also an additional correspondence between the KG equation and relativistic classical mechanics: The study of the classical limit of non-relativistic quantal system through Hamilton-Jacobi theory and the classical Lagrangian is well-known (see, for example, Goldstein 1980; p. 490). Recently, KG equation has been deduced for a free particle, starting with the relativistic Lagrangian within the framework of Hamilton-Jacobi formalism (Kwong 1987). This supports the idea that the present study is relevant in understanding the classical limit of relativistic quantal system for potential $V(r) = -(k/r)$, $k > 0$.

In the above context, we point out that recently, another form of the KG equation has been derived and this form leads to a continuity equation that has positive definite probability density (Kostin 1987). The form of this new KG equation is

$$[E - V(r)]^2\psi = -c^2\hbar^2\nabla^2\psi + \mu_0^2c^4\psi - \frac{\hbar^2c^2}{E - V + \mu_0c^2}\nabla V\cdot\nabla\psi \tag{47}$$

This equation differs from the original KG equation for a scalar potential in the third term on the right side of (47). However, in the limit of large mass, this additional term may be considered very small compared to the other interaction terms in (47). Further, it should be noted that presence of $\hbar^2$ in the interaction term $((-\hbar^2c^2/E - V + \mu_0c^2)\nabla V\cdot\nabla)$ in (47) will make the effect of this term negligible in the classical limit interpreted as $\hbar \to 0$. However, a quantitative conclusion about the classical limit obtained from (47) requires a detailed mathematical analysis.

The deduction of classical limit, in general, of relativistic quantal system governed by KG equation poses additional problems. For example, in a suitable reference frame, the radial KG equation for 3*D* relativistic harmonic oscillator has the form (Lipas 1970):

$$-\frac{\hbar^2}{2\mu_0}\left[\nabla_r^2 - \frac{l(l+1)}{r^2}\right]R + VR = E'R + \frac{E' - V}{2\mu_0 c^2}R \tag{48}$$

where,

$$V = \tfrac{1}{2}kr^2, \quad E' = E - \mu_0 c^2.$$

This can be rearranged into the mathematical form of a radial Schrodinger equation with the effective potential

$$\left[\frac{1}{2}\frac{kE}{\mu_0 c^2}r^2 - \frac{k^2}{8\mu_0 c^2}r^4\right].$$

It is clear that such a potential cannot generate genuine bound states, unlike the case of a non-relativistic harmonic oscillator, because of the tunnelling through the potential barrier present in the above stated effective potential. Even if one treats $V(r) = \frac{1}{2}kr^2$ as a Lorentz scalar, this problem remains. Similar situation can be expected in the case of other confining potentials. We have also examined the special relativistic dynamical equation for 2*D* harmonic oscillator and found that classical analysis leads to bounded orbits. Hence, in general, the correlation between the relativistic quantal problem and the corresponding classical special relativistic result needs further investigation.

The problem of classical limit of quantal system for the attractive Coulomb potential, based on the wavepackets constructed from the solution of Dirac equation is also of interest, since, unlike the case of KG equation, the solution ψ of the Dirac equation has $\psi^\dagger\psi > 0$. It is well-known that the electron spin which gets built in naturally in the Dirac approach, is a special relativistic quantal feature. In addition to Dirac equation, recently there has been some study of the relativistic quantal problem for Coulomb system using the quasipotential approach (Nagiyev 1988). One of the main features of this approach is that it depends on one time argument and allows for probability interpretation. Hence, the investigation of classical limits based on Dirac equation and relativistic quasipotential equation are of some additional interest. It is also desirable to investigate the classical limit by incorporating the relativistic convariance in a general way. Further investigation of the classical limit of relativistic quantal system taking into account the various aspects discussed above is being planned.

Appendix

Main results pertaining to the classical limit of non-relativistic and relativistic quantal system governed by the potential $V(r) = -k/r$, $k > 0$ in two dimensions are given below. These results are valid for three dimensional case corresponding to a properly chosen wavepacket. See the text for details.

(i) *Energy expectation value*

Non-relativistic case: $\langle E\rangle = -\dfrac{2\mu_0 k^2}{\hbar^2\gamma^4}$

Relativistic case corresponding to

(a) equation (3): $\langle E-\mu_0 c^2\rangle = -\dfrac{2\mu_0 k^2}{\hbar^2\gamma^4} + \dfrac{6\mu_0 k^4}{\hbar^4 c^2\gamma^8} - \dfrac{8\mu_0 k^4}{\hbar^4 c^2\gamma^8|\cos 2\chi|} + \cdots$

(b) equation (10): $\langle E-\mu_0 c^2\rangle = -\dfrac{2\mu_0 k^2}{\hbar^2\gamma^4} - \dfrac{2\mu_0 k^4}{\hbar^4 c^2\gamma^8} + \dfrac{8\mu_0 k^4}{\hbar^4 c^2\gamma^8|\cos 2\chi|} + \cdots$

(ii) *Frequency of periodic motion*

Non-relativistic case: $2w_c = \dfrac{8\mu_0 k^2}{\hbar^3\gamma^6}$

Relativistic case corresponding to

(a) equation (3): $2w_c = \dfrac{2\mu_0 c^2}{\hbar}\left[\dfrac{4k^2}{\hbar^2 c^2\gamma^6} - \dfrac{24k^4}{\hbar^4 c^4\gamma^{10}} + \dfrac{24k^4}{\hbar^4 c^4\gamma^{10}|\cos 2\chi|} + \cdots\right]$

(b) equation (10): $2w_c = \dfrac{2\mu_0 c^2}{\hbar}\left[\dfrac{4k^2}{\hbar^2 c^2\gamma^6} + \dfrac{8k^4}{\hbar^4 c^4\gamma^{10}} - \dfrac{24k^4}{\hbar^4 c^4\gamma^{10}|\cos 2\chi|} + \cdots\right]$

(iii) *Eccentricity*

Non-relativistic case: $\varepsilon = \sin 2\chi$

Relativistic case (ignoring motion of perihelion) corresponding to

(a) equation (3): $\varepsilon = \sin 2\chi$

(b) equation (10): $\varepsilon = \sin 2\chi$

(iv) *Semimajor axis* (*a*)

Non-relativistic case: $\mathrm{a} = \dfrac{\gamma^2}{\eta^2}$

Relativistic case (ignoring motion of perihelion corresponding to

(a) equation (3): $\mathrm{a} = \dfrac{\gamma^2}{\eta^2}$

(b) equation (10): $\mathrm{a} = \dfrac{\gamma^2}{\eta^2}$

(v) *Equation of the orbit*

Non-relativistic case: $\dfrac{\left(\langle x\rangle - \dfrac{\gamma^2}{\eta^2}\sin 2\chi\right)^2}{(\gamma^2/\eta^2)^2} + \dfrac{(\langle y\rangle)^2}{\left(\dfrac{\gamma^2}{\eta^2}\cos 2\chi\right)^2} = 1$

Relativistic case corresponding to

(a) equation (3): $$\frac{\left(\langle x\rangle - \frac{\gamma^2}{\eta^2}\sin 2\chi\right)^2}{(\gamma^2/\eta^2)^2} + \frac{(\langle y\rangle)^2}{\left(\frac{\gamma^2}{\eta^2}\cos 2\chi\right)^2} = 1$$

$$+\frac{2\eta^2}{\gamma^2}\sin 2w_c't\left[\frac{\langle x_0\rangle}{\cos 2\chi}\sin 2w_ct - \langle y_0\rangle\cos 2w_ct\right]$$

(b) equation (10): $$\frac{\left(\langle x\rangle - \frac{\gamma^2}{\eta^2}\sin 2\chi\right)^2}{(\gamma^2/\eta^2)^2} + \frac{(\langle y\rangle)^2}{\left(\frac{\gamma^2}{\eta^2}\cos 2\chi\right)^2} = 1$$

$$+\frac{2\eta^2}{\gamma^2}\sin 2w_c't\left[\frac{\langle x_0\rangle}{\cos 2\chi}\sin 2w_ct - \langle y_0\rangle\cos 2w_ct\right]$$

where $w_c' = \pm\left[\frac{\mu_0}{c^2\hbar^5}\cdot\frac{8k^4}{\gamma^{10}\cos^2 2\chi} + \cdots\right]$

(vi) *Rate of perihelion motion*

Non-relativistic case: $|\Omega| = 0$ per revolution
Relativistic case corresponding to

(a) equation (3): $$|\Omega| = \pi\left(\frac{k}{Lc}\right)^2\left[1 - \frac{3}{2}\left(\frac{k}{Lc}\right)^2|\cos 2\chi|(1 - |\cos 2\chi|) + \cdots\right]$$

per revolution

(b) equation (10): $$|\Omega| = \pi\left(\frac{k}{Lc}\right)^2\left[1 - \frac{3}{2}\left(\frac{k}{Lc}\right)^2|\cos 2\chi| - \frac{1}{2}\left(\frac{k}{Lc}\right)^2\cos^2 2\chi + \cdots\right]$$

per revolution

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