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The dimensional reduction of eleven dimensional supergravity into a product of Robertson–Walker spacetime and the seven sphere

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Abstract. The dimensionai reduction of eleven dimensional supergravity is discussed. It is shown that there is no dimensional reduction onto Robertson–Walker space with the asymmetric tensor F giving a realistic fluid. Furthermore it is shown that the ansatz's for the scale factor R:$R = at^n$, $R = a\exp(bt^n)$, and $R = aZ^n$, there is no dimensional reduction except the known example of the Freund-Rubin-Englert solution.

Keywords. Supergravity; cosmology.

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1. Introduction

The existing dimensional reductions of Cremmer *et al* (1978), Cremmer and Julia (1979) eleven dimensional supergravity, namely Freund and Rubin's (1980) and Englert (1982), have the external dimensions given by anti-deSitter spacetime with a very large cosmological constant. A dimensional reduction scheme where the extra dimensions are compact is referred to as a compactification and the above cases are examples of this. The cosmological constant in anti-deSitter spacetime is simply an unphysical perfect fluid (see for example p. 124 of Hawking and Ellis (1973)). The fluid has the density and pressure of the same magnitude, but with the density negative. Thus it is of interest to investigate whether there are perfect fluid compactifications. The extreme conditions of the early universe may provide a testing ground for supergravity, and therefore we consider dimensional reduction into Robertson–Walker spacetimes. Cosmological solutions to supergravity have been considered before by Emelyanov *et al* (1986). We show that compactifications onto Robertson–Walker spacetime necessarily lead to a cosmological constant and cannot give a physical fluid. Furthermore it is shown that subject to several ansatz's for the scale factor R there is no new solution.

We use the following conventions and notations: range of indices, $M, N, \ldots\ 1, \ldots, 11$; $\alpha, \beta, \ldots\ 1, \ldots, 4$; $\mu, \nu, \ldots\ 1, 2, 3$; $i, j, \ldots\ 5, \ldots, 11$; signature $+++-$, $+++++++$; the modulus is understood in roots, *i.e.* $\sqrt{g_4} = \sqrt{|g_4|}$; $F(e, i)$ means any component of the antisymmetric tensor F^{MNPQ} with e external and i internal indices; $F^2(e, i)$ means the scalar $F_{MNPQ}F^{MNPQ}$ made from the antisymmetric tensor F^{MNPQ} with e external and i internal indices; $g = \det\ g_{MN}$; $g^4 = \det\ g_{\alpha\beta}$; $F^2 = F_{MNPQ}F^{MNPQ}$, all other conventions are those of Weinberg (1972).

2. The field equations

The bosonic part of the Cremmer-Julia-Scherk equations for eleven dimensional supergravity are

$$R_{MN} = \kappa_1 \left(F_{MPQR} F_N^{PQR} - \frac{1}{12} g_{MN} F^2 \right). \tag{1}$$

and

$$\partial_M(\sqrt{g^{(11)}} F^{MN_1N_2N_3}) = \frac{\kappa_2}{\sqrt{g^{(11)}}} \varepsilon^{P_1P_2P_3P_4Q_1Q_2Q_3Q_4N_1N_2N_3} F_{P_1P_2P_3P_4} F_{Q_1Q_2Q_3Q_4}, \tag{2}$$

where

$$\kappa_1 = -\frac{\pi G}{6}, \quad \kappa_2 = -\frac{\sqrt{2}}{2(4!)^2}.$$

We consider dimensional reduction into

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\mathrm{d}\Sigma_3^2 + Z^2(x^\alpha)\mathrm{d}\Sigma_7^2 \tag{3}$$

when $Z = 0$ this is Robertson–Walker spacetime, a particular choice of the scale factor R gives anti-deSitter spacetime. $\mathrm{d}\Sigma_3^2$ and $\mathrm{d}\Sigma_7^2$ are maximally symmetric spaces with external curvature k_e and internal curvature k_i. The Ricci tensor is (generalizing p. 471 of Weinberg 1972.)

$$R_{tt} = \frac{7Z''}{Z} + \frac{3R''}{R} \tag{4}$$

$$R_{\mu\nu} = -\frac{g_{\mu\nu}}{R^2}\left(2R'^2 + RR'' + \frac{7RR'Z'}{Z} + 2k_e \right) \tag{5}$$

$$R_{ij} = -\frac{g_{ij}}{Z^2}\left(6Z'^2 + ZZ'' + \frac{3ZZ'R'}{R} + 2k_i \right) \tag{6}$$

where $Z' = \partial_t Z$. Maximal symmetry implies that there are no cross terms $F^2(1,3)$, $F^2(2,2)$, and $F^2(3,1)$, thus there are only two types of terms $F^2(0,4)$ and $F^2(4,0)$. The indices on $F^{\alpha\beta\gamma\delta}$ are asymmetric therefore $F^{t\beta\gamma\delta}F_{t\beta\gamma\delta}$ must occur one third as often as $F^{\mu\beta\gamma\delta}F_{\mu\beta\gamma\delta}$ (μ summed), so that

$$F^2(0,4) \equiv F_{ijkl}F^{ijkl}, \quad F^2(4,0) = F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta},$$
$$F_{t\beta\gamma\delta}F^{t\beta\gamma\delta} = \tfrac{1}{4}F^2(4,0), \quad F_{\mu\beta\gamma\delta}F^{\mu\beta\gamma\delta} = \tfrac{3}{4}F^2(4,0) \tag{7}$$

giving

$$\frac{12}{\kappa_1}R_t^t = 2F^2(4,0) - F^2(0,4)$$

$$\frac{12}{\kappa_1}R_\mu^\mu = 6F^2(4,0) - 3F^2(0,4)$$

$$\frac{12}{\kappa_1}R_i^i = -7F^2(4,0) + 5F^2(0,4). \tag{8}$$

The perfect fluid stress is

$$T_{\alpha\beta} = pg_{\alpha\beta} + (p+\rho)U_\alpha U_\beta \tag{9}$$

where p is the pressure and ρ is the density, and U_α is a time-like vector. Taking $U_t = 1$ and $U_\mu = 0$ and the line element (3) gives

$$\begin{aligned} -6\kappa_3 p &= 3R^t_t + R^\mu_\mu = \kappa_1(F^2(4,0) - \tfrac{1}{2}F^2(0,4)), \\ 2\kappa_3\rho &= R^\mu_\mu - R^t_t = \frac{\kappa_1}{3}(F^2(4,0) - \tfrac{1}{2}F^2(0,4)), \end{aligned} \tag{10}$$

therefore

$$\rho = -p = \frac{\kappa_1}{6\kappa_3}(F^2(4,0) - \tfrac{1}{2}F^2(0,4)) \tag{11}$$

so that the stress must have either the pressure or the density of the unphysical sign.

For the line element (3) the wave equation (2) becomes

$$\partial_\alpha(\sqrt{g^{(11)}}\,F^{\alpha\beta_1\beta_2\beta_3}) = \kappa_2\,\varepsilon^{\gamma_1\gamma_2\gamma_3\gamma_4\delta_1\delta_2\delta_3\delta_4\beta_1\beta_2\beta_3}F_{\gamma_1\gamma_2\gamma_3\gamma_4}F_{\delta_1\delta_2\delta_3\delta_4}. \tag{12}$$

there must be a repeated index, and therefore the right hand side vanishes identically, thus (12) has the solution

$$F^{\alpha\beta\gamma\delta} = \frac{f(\neq x^\alpha)\varepsilon^{\alpha\beta\gamma\delta}}{\sqrt{g^{(11)}}} \tag{13}$$

absorbing $g^{(7)}$ into $f(\neq x^\alpha)$ gives

$$\begin{aligned} F^2(4,0) &= F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta} = \frac{f^2(\neq x^\alpha)\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta}}{|g^{(11)}|} \\ &= \frac{g^{(4)}}{|g^{(4)}|}\frac{f^2(\neq x^\alpha)}{g^{(7)}} \equiv -f^2. \end{aligned} \tag{14}$$

For $F^2(0,4)$ comparison with Englert gives

$$\begin{aligned} F^2(0,4) &= F_{mnop}F^{mnop} = \frac{2}{3}\lambda^2 k_i^2 S^m_m \\ &= \frac{14}{3}\lambda^2 k_i^2 \end{aligned} \tag{15}$$

where λ is a constant. Thus

$$\begin{aligned} \frac{12}{\kappa_1}R^t_t &= 2F^2(4,0) - F^2(0,4) = -2f^2 - \frac{2\cdot 7}{3}k_i^2\lambda^2 \equiv \alpha^2\cdot\frac{12}{\kappa_1} \\ \frac{12}{\kappa_1}R^\mu_\mu &= 6F^2(4,0) - 3F^2(0,4) = -6f^2 - \frac{6\cdot 7}{3}k_i^2\lambda^2 \equiv 3\alpha^2\cdot\frac{12}{\kappa_1} \\ \frac{12}{\kappa_1}R^i_i &= -7F^2(4,0) + 5F^2(0,4) = 7f^2 + \frac{2\cdot 5\cdot 7}{3}k_i^2\lambda^2 \equiv -7\beta^2\cdot\frac{12}{\kappa_1}. \end{aligned} \tag{16}$$

From (4) and (14) the field equations become

$$\alpha^2 = -\frac{7Z''}{Z} - \frac{3R''}{R} \tag{17}$$

$$\alpha^2 = -\left[\frac{2R'^2}{R^2} + \frac{R''}{R} + \frac{7R'Z'}{RZ} + \frac{2k_e}{R^2}\right] \tag{18}$$

$$\beta^2 = +\frac{6Z'^2}{Z^2} + \frac{Z''}{Z} + \frac{3R'Z'}{RZ} + \frac{2k_i}{Z^2}. \tag{19}$$

Note that as $F^2(4,0)$ is negative and constant, and $F^2(0,4)$ is positive and constant, the pressure and density (11) must be equivalent to a negative cosmological constant.

3. The Freund-Rubin-Englert solution

To recover the Freund-Rubin-Englert solution choose

$$k_e = -1, \quad k_i = +1, \quad R = b\cos at, \quad Z' = 0. \tag{20}$$

Then the field equations (17), (18) and (19) become

$$\alpha^2 = 3a^2 \tag{21}$$

$$\alpha^2 = 2\left(-a^2 \operatorname{tn}^2 at - \frac{1}{b^2}\sec^2 at\right) - a^2 \tag{22}$$

$$\beta^2 = \frac{2k_I}{Z^2} \tag{23}$$

respectively, now choosing

$$b^2 = \frac{1}{a^2} \tag{24}$$

and

$$Z^2 = \frac{24}{-\kappa_1\left(f^2 + \frac{10\lambda^2}{3}\right)} \tag{25}$$

gives the Freund-Rubin-Englert solution.

4. The ansatz $R = at^n$

First consider the case

$$R = a \tag{26}$$

then (17) and (18) give

$$-\alpha^2 = \frac{7Z''}{Z} = \frac{2k_e}{a^2} \tag{27}$$

the differential equation for Z has solution

$$Z = c\cos bt, \quad b = \sqrt{\frac{\alpha^2}{7}}\,. \tag{28}$$

Now (19) gives

$$-\beta^2 - \frac{\alpha^2}{7} + \frac{2k_i}{c^2} = \operatorname{tn}^2 bt\left(\frac{2k_i}{c^2} + 6b^2\right) \tag{29}$$

both sides must vanish independently. The right hand side gives

$$\frac{k_i}{c^2} = -\frac{3\alpha^2}{7} \tag{30}$$

therefore

$$k_i = -1, \quad c^2 = \frac{7}{3\alpha^2}. \tag{31}$$

Now the left hand side gives

$$-\beta^2 - \frac{\alpha^2}{7} + \frac{2k_i}{c^2} = -\beta^2 - \alpha^2 = 0 \tag{32}$$

therefore there is no new solution.

Secondly consider the case

$$R = at \tag{33}$$

then (17) and (18) give

$$-\alpha^2 = \frac{7Z''}{Z} = \frac{2}{t} + \frac{7Z'}{tZ} + \frac{2k_e}{a^2t^2} \tag{34}$$

the first equality has (28) as a solution again. Substituting this into the second equality gives

$$-\alpha^2 = \frac{2}{7} - \frac{7b}{t}\operatorname{tn} bt + \frac{2k_e}{a^2t^2} \tag{35}$$

from which it is clear that there is no new solution.

For the general case

$$R = at^n \tag{36}$$

equations (17), (18), and (19) become

$$-\alpha^2 = \frac{7Z''}{Z} + \frac{3n(n-1)}{t^2} \tag{37}$$

$$-\alpha^2 = \frac{n(3n-1)}{t^2} + \frac{7n}{t}\frac{Z'}{Z} - \frac{2k_e t^{-2n}}{a^2} \tag{38}$$

$$\beta^2 = \frac{6Z'^2}{Z^2} + \frac{Z''}{Z} - \frac{3n}{t}\frac{Z'}{Z} - \frac{2k_i}{Z^2} \tag{39}$$

respectively. Equation (38) gives

$$\frac{Z'}{Z}=\left(\frac{\alpha t}{7n}-\frac{(3n-1)}{7t}-\frac{2k_e t^{-2n}}{a^2}\right) \tag{40}$$

Differentiating (39) and using (40) gives

$$\begin{aligned}\frac{Z''}{Z}&=\left(\frac{\alpha t}{7n}-\frac{(3n-1)}{7t}-\frac{2k_e t^{-2n}}{a^2}\right)^2+\frac{\alpha}{7n}+\frac{(3n-1)}{7t^2}+\frac{(2n-1)2k_i t^{2n}}{7na^2}\\ &=\frac{\alpha}{7}-\frac{3n(n-1)}{t^2}.\end{aligned} \tag{41}$$

Therefore

$$\begin{aligned}0&=t^2\cdot\frac{\alpha^2}{7^2n^2}+t'\cdot\frac{\alpha}{7}\left(-1+\frac{1}{n}-\frac{2(3n-1)}{7n}\right)\\ &+t^{-2}\cdot\left(3n(n-1)+\frac{(3n-1)}{7}+\frac{(3n-1)^2}{7^2}\right)\\ &+t^{-2n}\cdot\frac{2k_e}{7^2na^2}(20n-9)+t^{2n-2}\cdot\frac{-2\alpha^2k_e}{7^2n^2a^2}\\ &+t^{2-4n}\cdot\frac{2^2k_e^2}{7^2n^2a^2}.\end{aligned} \tag{42}$$

The t^2 term must cancel. The $n=0$ case has been considered. The only choice of n which allows for the cancellation of the t^2 term is $n=-1$, when the $-2n$ term may cancel with it, however then there is a t^6 term which does not cancel. Therefore there is no new solution.

5. The ansatz $R=ae^{bt^n}$

Let

$$R=ae^{br} \tag{43}$$

and

$$Z=ce^{dz} \tag{44}$$

where r and z are functions of t. Then the field equations (17), (18), and (19) become

$$7d^2z'^2=-\alpha^2-7dz''-3b^2r'^2-3br'' \tag{45}$$

$$7bdr'z'=-\alpha^2-3b^2r'^2-br''-\frac{2k_e e^{-2b}}{a^2} \tag{46}$$

$$7d^2z'^2=\beta^2-dz''-3bdr'z'-\frac{2k_i e^{-2dz}}{c^2} \tag{47}$$

respectively. Equations (46) and (47)

$$0=-\alpha^2-6dz''-3b^2r'^2-3br''-\beta^2+3bdr'z'+\frac{2k_i}{c^2}e^{-2dz}. \tag{48}$$

To proceed choose r to be of the form

$$r = t^n. \tag{49}$$

The case $n = 0$ gives $r = a$ constant and has been dealt with in §4. Now (46) gives

$$z' = \frac{1}{7nb\mathrm{d}}\left(-\alpha^2 t^{1-n} - 3b^2 n^2 t^{n-1} - \frac{bn(n-1)}{t} - \frac{t^{1-n} 2k_e e^{-2br}}{a^2} \right), \tag{50}$$

z'' can be produced by differentiating (50), substituting into equation (48) gives

$$\begin{aligned} 0 = & -\alpha^2 - \beta^2 - 3b^2 n^2 t^{n-1} - 3bn(n-1)t^{n-2} + \frac{2k_i e^{-2\mathrm{d}z}}{c^2} \\ & - \frac{6}{7nb}\left(-\alpha^2(1-n)t^{-n} - 3b^2 n^2 (n-1) t^{n-2} + bn(n-1)t^{-2} \right. \\ & \left. + \frac{(n-1)t^{-2} 2k_e e^{-2br}}{a^2} \right) + \frac{3}{7}\left(-\alpha^2 - 3b^2 n^2 t^{2n-2} - bn(n-1)t^{n-2} \right. \\ & \left. - \frac{2k_e e^{-2br}}{a^2} \right). \end{aligned} \tag{51}$$

For the $n = 1$ case (51) reduces to

$$0 = -\frac{10}{7}\alpha^2 - \beta^2 - 3b^2 \tag{52}$$

as all the terms are negative there is no new solution. For $n \neq 1$ equating the exponential terms gives

$$-\frac{6(n-1)2k_e t^{-n} e^{-2br}}{7nba^2} - \frac{3t^{n-1} 2k_e t^{1-n} e^{-2br}}{7a^2} + \frac{2k_i e^{-2\mathrm{d}z}}{c^2} = 0 \tag{53}$$

from which it can be seen that there is no solution unless $k_e = k_i = 0$, for which it can be shown that there are powers of t which do not cancel.

6. The ansatz $R = cZ^n$

For

$$R = cZ^n, \tag{54}$$

equations (17), (18), and (19) become

$$-\alpha^2 = (7 + 3n)\frac{Z''}{Z} + 3n(n-1)\frac{Z'^2}{Z^2} \tag{55}$$

$$-\alpha^2 = n\frac{Z''}{Z} + 3n(n+2)\frac{Z'^2}{Z^2} + \frac{2k_e Z^{-2n}}{c^2} \tag{56}$$

$$\beta^2 = \frac{Z''}{Z} + 3(n+2)\frac{Z'^2}{Z^2} + 2k_i Z^{-2} \tag{57}$$

respectively. Now (56) and (57) give

$$\alpha^2 + n\beta^2 = 2nk_i Z^{-2} - \frac{2k_e}{c^2} Z^{-2n} \tag{58}$$

and (55) has solution

$$Z = a \cos^{\mathrm{d}} et \tag{59}$$

provided

$$\mathrm{d} = \frac{7 + 3n}{7 + 3n^2} \tag{60}$$

and

$$\alpha^2 = e^2 \mathrm{d}(7 + 3n). \tag{61}$$

From equations (58) and (59) it is apparent that there is no solution subject to (54).

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