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Spherically symmetric static inhomogeneous cosmological models

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Abstract. Spherically symmetric static cosmological models filled with black-body radiation are considered. The models are isotropic about a central observer but inhomogeneous. It is suggested that the energy density of the free gravitational field, which is coupled to the isotropic radiation energy density, might play an important role in generating sufficient field (vacuum) energy (when converted into thermal energy) and initiate processes like inflation. On the central world line the energy density of the free gravitational field vanishes whereas the proper pressure and density of the isotropic black-body radiation are constants. Further, it is shown that the cosmological constant is no more arbitrary but given in terms of the central pressure and density. Also, at its maximum value the energy density of the free gravitational field is proved to be equal to one third of the combined value of radiation pressure and density.

Keywords. Cosmology; energy density of the free gravitational field; black-body radiation, inflation.

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1. Introduction

Historically, relativistic cosmology was born with Einstein's spherically symmetric static model. In order to make this model physically meaningful the cosmological term was added to the original field equations. However, Hubble's discovery that the galaxies are receding from each other could not be explained within the framework of static models. Thus, the Friedmann–Robertson–Walker (FRW) models were widely accepted for the description of the present epoch of the universe with their spatially homogeneous and isotropic character. But there are several objections to the FRW models which include their singular origin, existence of horizons, a high degree of observed inhomogeneities in nebular distribution within short ranges, say far enough to include the virgo cluster, and evidence for considerable anisotropy in expansion velocities among galaxies. To explain some of the difficulties faced by the FRW models Tolman (1934, 1949), Omer (1949), Bondi (1947), Bonnor (1972, 1974) and others have considered inhomogeneous and anisotropic models without disturbing spherical symmetry. However, these models are quite complicated and fitting observational data with a certain amount of reliability is doubtful.

In this background interest in spherically symmetric static cosmological models has been resuscitated by Ellis (1978), Ellis *et al* (1978) and Collins (1983). In the static case, the spatial isotropy assumption of FRW models has been modified by "that the universe is precisely isotropic about some particular observer". Similarly the spatial homogeneity condition is replaced by "that the universe is static". Thus, the space-time metric was chosen as

$$ds^2 = -dr^2 - f^2(r)(d\theta^2 + \sin^2\theta d\phi^2) + g^2(r)dt^2. \quad (1)$$

Here, the parameter r increases monotonically as we move outwards from the centre of the system. However, $f(r)$ in general need not be an increasing function of r. The smoothness conditions on f and g are given by

$$\left.\begin{array}{ll}\text{(i)} & f(r)\to 0,\ \ f'(r)\to 1,\ \ f''(r)/f(r)\to \text{finite limit}\\ \text{(ii)} & g(r)\to \text{finite non-zero limit},\ g'(r)\to 0\end{array}\right\}\text{as } r\to 0$$

where a prime for f and g denotes a differentiation with respect to r. Further $f(r)$ and $g(r)$ are assumed to be regular non-zero functions of r on the interval $0 < r < \mathbf{a}$ where $\mathbf{a}$ may be finite or infinite. Finally the function $g(r)$ is normalized by the condition that $g(0) = 1$. Thus, the space-time (1) is spherically symmetric about the central world-line at $r = 0$ which is denoted by C.

The static cosmological models considered by Ellis *et al* (1978) have two centres, our galaxy being near one of them and the initial singularity representing the second. In contrast to the interpretation of redshifts in the FRW models as being cosmological Doppler shifts, in the static models they are being interpreted as cosmological gravitational redshifts. The 3°K microwave background radiation originates from the singular centre of the static models.

In the present paper we obtain an analytic solution representing spherically symmetric static inhomogeneous cosmological models filled with black-body radiation. Also, following the method adopted in the study of non-static cosmological models (Krishna Rao 1990) the energy density of the free gravitational field represented by the eigen value of the Weyl conformal tensor has been introduced as a new parameter in relativistic cosmology. It was argued in the above cited paper that the energy density of the free gravitational field contributes for the expansion of the universe and also increases the deceleration rate effecting the present Hubble velocity in a shorter time. This has relevance in the context of inflationary models needing more vacuum (field) energy to undergo an exponential expansion. It is hoped that the energy density of the free gravitational field, rather than the cosmological constant Λ, plays a decisive role to explain inflation as well as grand unification processes by generating sufficient field energy which can be transformed into thermal energy.

In §2 the field equations are derived for spherically symmetric static space-times filled with a perfect fluid by introducing the parameters $w = \{(4\pi/3)(\rho + \varepsilon) + (\Lambda/6)\}f^2$ and $v = (4\pi p - (\Lambda/2))f^2$. Isothermal solutions with an equation of state $(8\pi p - \Lambda) = k\{8\pi(\rho + \varepsilon) + \Lambda\}$ are discussed in §3 and an analytical solution corresponding to the case $k = 1/3$ has been derived in §4. The paper ends with concluding remarks in §5.

2. Field equations

We assume that the cosmological models are filled by a perfect fluid of proper density ρ and pressure p. Thus, using Einstein's equations with a cosmological constant,

$$-8\pi T_a^b = -8\pi[(\rho + p)u_a u^b - p g_a^b] = R_a^b - \tfrac{1}{2}g_a^b R + \Lambda g_a^b, \quad (2)$$

where $u^a = (0, 0, 0, g^{-1}(r))$ and adopting our earlier procedure (Krishna Rao and Annapurna 1985), we get the expressions for pressure and density as below:

$$8\pi p - \Lambda = (2f'g'/fg) + (f'/f)^2 - (1/f^2) \tag{3}$$

$$8\pi(p + \varepsilon) - \Lambda = (2f'g'/fg) + (2f''/f) - (f'/f)^2 + (1/f^2), \tag{4}$$

$$8\pi\rho + \Lambda = -(2f''/f) - (f'/f)^2 + (1/f^2), \tag{5}$$

where

$$8\pi\varepsilon = (f''/f) - (f'/f)^2 - (g''/g) + (f'g'/fg) + (1/f^2), \tag{6}$$

is the eigen value of the Weyl conformal tensor (Krishna Rao 1966).

Now, the combination of equations $\{(4) + (5) - (3)\}$ gives

$$f'^2 = 1 - 2w, \tag{7}$$

where

$$w = \left\{\frac{4\pi}{3}(\rho + \varepsilon) + \frac{\Lambda}{6}\right\} f^2. \tag{8}$$

The coupling of ε to the material energy density ρ immediately suggests that the former may be interpreted as the energy density of the free gravitational field (Krishna Rao and Annapurna 1986).

Again doubling (3) and adding to (4) and (5), we get

$$\{4\pi(\rho + \varepsilon + 3p) - \Lambda\}(f/f') = 3g'/g. \tag{9}$$

Also, the equation for radial pressure gradient is given by

$$p'/(\rho + p) = -g'/g. \tag{10}$$

Thus, eliminating g'/g from (9) and (10), and making use of (7) and (8), we get

$$3\mathrm{d}p/\mathrm{d}f = -(\rho + p)\{4\pi(\rho + \varepsilon + 3p) - \Lambda\} f(1 - 2w)^{-1}. \tag{11}$$

Once again making the combination $\{2 \times (5) + (3) - (4)\}$ we get

$$4\pi(2\rho - \varepsilon) + \Lambda = -(3f''/f). \tag{12}$$

From (7) and (12), it follows that

$$4\pi(2\rho - \varepsilon) + \Lambda = 3f^{-1}(\mathrm{d}w/\mathrm{d}f). \tag{13}$$

We now define a new parameter $v(f)$ as

$$v(f) = \left(4\pi p - \frac{\Lambda}{2}\right) f^2, \tag{14}$$

and obtain from (3) and (7)

$$g'/g = (w + v)/f(1 - 2w)^{1/2}. \tag{15}$$

Now, from (10), (11) and (15) we get

$$f^{-1}\left(\frac{\mathrm{d}f}{\mathrm{d}w}\right) = \left\{(1 - 2w)\left(\frac{\mathrm{d}v}{\mathrm{d}w}\right) + (w + v)\right\}\{2v - (w^2 + 6vw + v^2)\}^{-1}. \tag{16}$$

Another important relation which can be deduced from (5), (7) and (16) is

$$\left(4\pi\rho+\frac{\Lambda}{2}\right)f^2=w\left\{\left(\frac{\mathrm{d}v}{\mathrm{d}w}\right)-\beta\right\}\left\{\left(\frac{\mathrm{d}v}{\mathrm{d}w}\right)-\alpha\right\}^{-1}, \tag{17}$$

where

$$\alpha=-(w+v)/(1-2w),\quad \beta=-v(2-5w-v)/w(1-2w). \tag{18}$$

From (16), it is worth noting that if v is given as a function of w, the equation is readily integrable. Similarly, ρ can be evaluated from (17).

The significance of the families of hyperbolas $H=0=2v-(w^2+6wv+v^2)$ and parabolas $\mathrm{d}v/\mathrm{d}w=\alpha$ as well as the integral curves of $\mathrm{d}v/\mathrm{d}w=\beta$ was discussed by Bondi (1964).

3. Isothermal solutions

Assuming that the non-luminious, non-baryonic material forms the bulk of the mass of the universe and clusters on galactic scales, the measurements of rotation curves around galaxies give the linear mass-radius relation, $M(r)\propto r$ resulting in $\rho\propto 1/r^2$. This result is characteristic of isothermal gas spheres bound by gravitation (Wilczek 1983). So, in this section we consider cosmological situations obeying an isothermal pressure-density relation.

We have seen earlier that the field energy density ε has been coupled to the material energy density ρ. Thus, the classical pressure-density relation $p=k\rho$ for isothermal solutions needs modification in the context of curved geometry as well as the presence of the cosmological constant. Therefore, choosing the equation of state as

$$p-\frac{\Lambda}{8\pi}=k\left(\rho+\varepsilon+\frac{\Lambda}{8\pi}\right), \tag{19}$$

and making use of (8), (14) and (19), we get $v=3kw$, which on substitution in (16) gives

$$f^{-1}\left(\frac{\mathrm{d}f}{\mathrm{d}w}\right)=\{3k(1-2w)+(3k+1)w\}w^{-1}(A-Bw)^{-1}, \tag{20}$$

where $A=6k$ and $B=9k^2+18k+1$. We now integrate (20) to obtain

$$\frac{f^2}{2R^2}=w(A-Bw)^{((A-2)/B)-1}, \tag{21}$$

R being a constant. Defining a new function $l(f)$ such that

$$l(f)=(A-Bw)^{(A-2)/B}, \tag{22}$$

the expression for w is given by

$$w=Af^2\{2R^2l(f)+Bf^2\}^{-1}. \tag{23}$$

Again from (15) and (19) we obtain the integral for g as

$$\log g=\int\frac{(3k+1)Af\,\mathrm{d}f}{\{2R^2l(f)+(B-2A)f^2\}}+\text{constant} \tag{24}$$

4. A cosmological model filled with isotropic black-body radiation

The existence of 3°K isotropic microwave background radiation suggests a hot phase in an earlier epoch of the universe. In fact such a scenario points out that at temperatures of the order of 3×10^3°K matter and radiation have got decoupled. For a universe filled with a black-body radiation we put $k = 1/3$ in (19) such that $v = w$, $A = 2$, $B = 8$ and consequently (22) gives $l(f) = 1$. Accordingly from (23) and (24) we get respectively

$$v = w = f^2/(R^2 + 4f^2), \tag{25}$$

$$g^2 = 2\gamma(R^2 + 2f^2), \tag{26}$$

γ being a constant. From the above expressions we note that the metric (1) is regular at $f = 0$. Thus, denoting the initial pressure on the central world-line as p_c we get

$$2/R^2 = 8\pi p_c - \Lambda. \tag{27}$$

We thus eliminate R^2 from (25) and (26) making use of (27). Therefore,

$$w = v = \left(4\pi p_c - \frac{\Lambda}{2}\right) f^2/\{1 + 2(8\pi p_c - \Lambda)f^2\}, \tag{28}$$

and

$$g^2 = 4\gamma(8\pi p_c - \Lambda)^{-1}\{1 + (8\pi p_c - \Lambda)f^2\}. \tag{29}$$

In the case of bounded distributions it is possible to evaluate γ making use of the boundary pressure p_b.

From (8), (19) and (28), we get (after putting $k = 1/3$)

$$8\pi p - \Lambda = (8\pi p_c - \Lambda)\{1 + 2(8\pi p_c - \Lambda)f^2\}^{-1}. \tag{30}$$

Also, (17) simplifies to

$$8\pi\rho + \Lambda = (8\pi p_c - \Lambda)\{3 + 2(8\pi p_c - \Lambda)f^2\}\{1 + 2(8\pi p_c - \Lambda)f^2\}^{-2}. \tag{31}$$

Finally substituting from (30) and (31) in (19), with $k = 1/3$, we obtain

$$8\pi\varepsilon = 4(8\pi p_c - \Lambda)^2 f^2\{1 + 2(8\pi p_c - \Lambda)f^2\}^{-2}. \tag{32}$$

From (32) we note that on the central world line $\varepsilon = 0$ and $8\pi\rho_c + \Lambda = 3(8\pi p_c - \Lambda)$ giving the value for Λ as

$$\Lambda = 2\pi(3p_c - \rho_c). \tag{33}$$

Thus, our modified equation of state given by (19) enables us to determine the cosmological constant in terms of initial pressure and density on the central world-line. Also in the limiting case when $R \to \infty$, we get $\rho + p = 0$, $\varepsilon = 0$ and in order to make ρ positive, the cosmological constant has to be chosen as negative. The resulting space-time is isomorphic to that of anti-de Sitter cosmological model.

We have mentioned in § 1 that the energy density of the free gravitational field ε contributes for the expansion of the universe in the non-static inhomogeneous and anisotropic cosmological models. In inflationary cosmologies the initial rapid expansion of the universe was attributed to negative pressures. The formation of the galaxies is to be explained before inflation would smooth out inhomogeneities heralding the present FRW epoch. So, even though we are working here with a static model the maximum and minimum values of ε may provide some information regarding the inflationary phase. It may be verified easily that the minimum and maximum values of ε correspond to $f = 0$ and $f = R/2$ respectively. Also at $f = R/2$ the expressions for ρ, p and ε are given by

$$4\pi\rho + \frac{\Lambda}{2} = 8\pi p - \Lambda = 8\pi\varepsilon = \pi(p_c + \rho_c) = 1/R^2 \tag{34}$$

which indicate that at its maximum value the contribution of ε is equal to one third of the combined value of ρ and p.

5. Concluding remarks

The cosmological model presented here is static and inhomogeneous but singularity free. It may be considered as a forerunner for more general non-static and inhomogeneous models filled with isotropic black-body radiation. Also, the energy density of the free gravitational field, which is coupled to the radiation energy density, enters relativistic cosmology as a new parameter. We may note that, for simplicity, we have chosen the coupling constant as unity. However, the value of ε in relation to ρ and p can be raised substantially by an appropriate choice of the coupling constant. It is well-known (see Abbott and Pi 1986; Borner 1988) that in a radiation dominated universe $\rho \propto T^4$ and therefore $\varepsilon \propto T^4$ where T is the temperature. Also, the inflation which preceeds the radiation dominated phase may be interpreted as due to conversion of the energy of the free gravitational field (represented by ε) into thermal energy. During the inflationary phase it might be possible that ε has values higher than fourth power in T.

In contrast to the traditional notion of mining the vacuum (anti-de Sitter space-time with $8\pi\rho = -8\pi p = \Lambda$) wherein the negative pressure supplies the necessary energy for inflation, the above scenario fits more naturally into the scheme since ε is the invariant of the conformal Weyl tensor and inflation can be explained even when the controversial cosmological term is dropped from the field equations. The maximum value for ε occurs at $f = R/2$ but however ε dominates over ρ only after $f = \sqrt{3}R/2$. Finally, we point out that the presence of initial inhomogeneities, through a non-vanishing ε, are conductive for galaxy formation.

The values of $w(=v)$, $4\pi\varepsilon R^2$, $(4\pi p - (\Lambda/2))R^2$ and $(4\pi\rho + (\Lambda/2))R^2$ are given as functions of $x(=2f/R)$ in the accompanying table 1 and the graphs describing their relative behaviour are also drawn in figure 1.

Table 1. Numerical values of w, $4\pi\varepsilon R^2$, $(4\pi p-(\Lambda/2))R^2$ and $(4\pi\rho+(\Lambda/2))R^2$.

$x=\dfrac{2f}{R}$ (1)	$w=\dfrac{x^2}{4(1+x^2)}$ (2)	$4\pi\varepsilon R^2=\dfrac{2x^2}{(1+x^2)^2}$ (3)	$\left(4\pi p-\dfrac{\Lambda}{2}\right)R^2=\dfrac{1}{(1+x^2)}$ (4)	$\left(4\pi\rho+\dfrac{\Lambda}{2}\right)R^2=\dfrac{3+x^2}{(1+x^2)^2}$ (5)
0·1	0·0025	0·0196	0·9901	2·9507
0·2	0·0096	0·0740	0·9615	2·8107
0·3	0·0206	0·1515	0·9174	2·6007
0·4	0·0345	0·2378	0·8621	2·3484
0·5	0·0500	0·3200	0·8000	2·0800
0·6	0·0662	0·3893	0·7353	1·8166
0·7	0·0822	0·4414	0·6711	1·5720
0·8	0·0976	0·4759	0·6098	1·3534
0·9	0·1119	0·4945	0·5525	1·1630
1·0	0·1250	0·5000	0·5000	1·0000
1·1	0·1369	0·4955	0·4525	0·8620
1·2	0·1475	0·4837	0·4098	0·7458
1·3	0·1570	0·4671	0·3717	0·6481
1·4	0·1655	0·4474	0·3378	0·5661
1·5	0·1731	0·4260	0·3077	0·4970
1·6	0·1798	0·4040	0·2809	0·4387
1·7	0·1858	0·3820	0·2571	0·3892
1·8	0·1910	0·3604	0·2358	0·3471
1·9	0·1958	0·3397	0·2169	0·3110
2·0	0·2000	0·3200	0·2000	0·2800
2·1	0·2037	0·3014	0·1848	0·2532
2·2	0·2072	0·2838	0·1712	0·2299
2·3	0·2103	0·2674	0·1590	0·2095
2·4	0·2130	0·2521	0·1479	0·1917
2·5	0·2155	0·2378	0·1379	0·1760
2·6	0·2178	0·2245	0·1289	0·1621
2·7	0·2198	0·2122	0·1206	0·1497
2·8	0·2217	0·2007	0·1131	0·1387
2·9	0·2235	0·1900	0·1063	0·1289
3·0	0·2250	0·1800	0·1000	0·1200
3·5	0·2311	0·1396	0·0755	0·0869
4·0	0·2353	0·1107	0·0588	0·0657
4·5	0·2382	0·0897	0·0471	0·0515
5·0	0·2416	0·0740	0·0385	0·0414
5·5	0·2420	0·0620	0·0320	0·0340
6·0	0·2433	0·0526	0·0270	0·0285
6·5	0·2443	0·0452	0·0231	0·0242
7·0	0·2450	0·0392	0·0200	0·0208
7·5	0·2457	0·0343	0·0175	0·0181
8·0	0·2461	0·0303	0·0154	0·0159
8·5	0·2466	0·0270	0·0137	0·0140
9·0	0·2471	0·0241	0·0122	0·0125
9·5	0·2473	0·0217	0·0110	0·0112
10·0	0·2475	0·0196	0·0099	0·0101

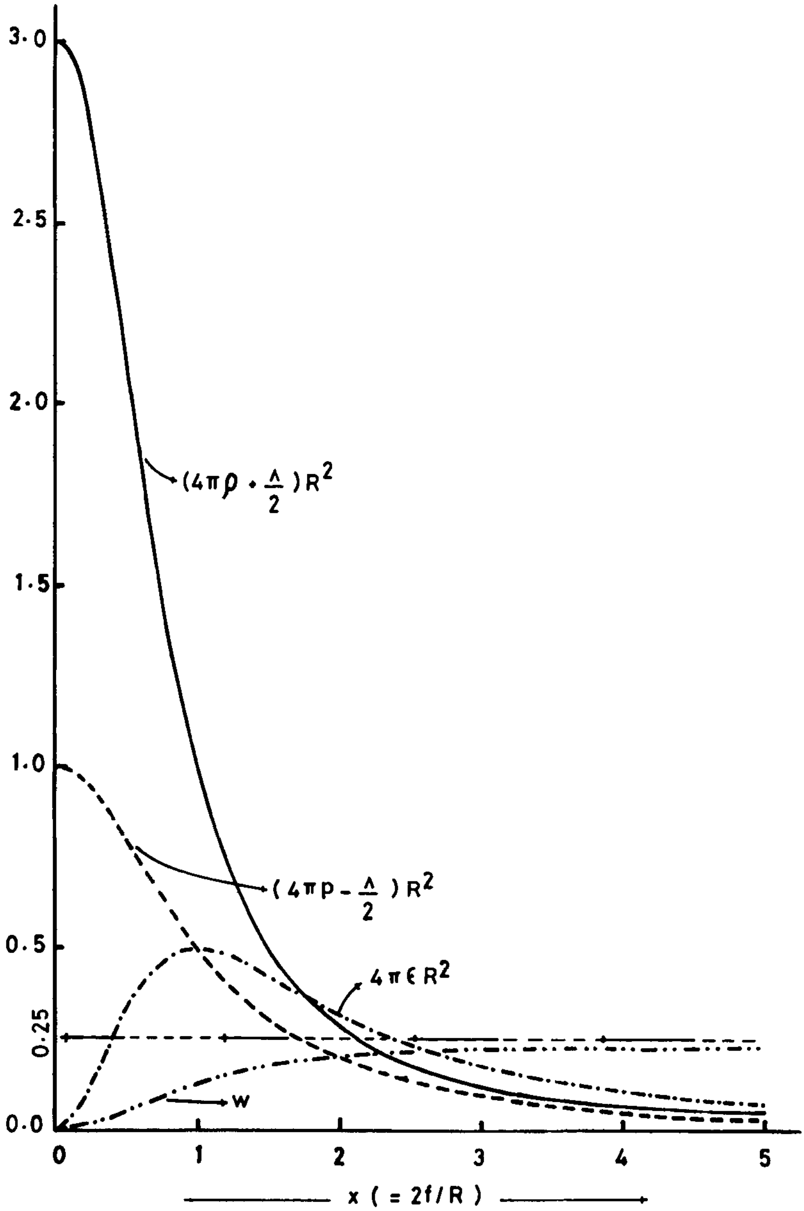

Figure 1. Graphs of $(4\pi\rho+(\Lambda/2))R^2$, $(4\pi p-(\Lambda/2))R^2$, $4\pi\varepsilon R^2$ and w drawn against $x(=2f/R)$.

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