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Diffusion controlled multiplicative process: typical versus average behaviour

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Abstract. We investigate the dynamics of the number of particles diffusing in a multiplicative medium. We show that the typical behaviour of the growth process is different from the average. We develop a new formalism to study the average growth process and extend it to the calculation of higher moments and finally of the probability distribution. We show that the fluctuations of the growth process increase exponentially with time. We describe the interesting features of the distribution.

Keywords. Random walk; multiplicative process; fluctuations; reaction kinetics; random growth process; diffusion.

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1. Introduction

The problem of the typical behaviour of a macroscopic quantity as compared to its average, has received considerable attention recently in the context of diffusion in disordered media (see Noskowicz and Goldhirsch (1988), Le Doussal (1989), Murthy and Kehr (1989), Kehr and Murthy (1990)). To motivate the basic issues involved we consider the following simple example.

Let Y denote the product of N identically distributed, positive definite, independent random variables, $x_1, x_2, \ldots, x_N$. Also, for simplicity, let x_i take only two values r_1 or r_2 with equal probability. It is clear that there are a total of 2^N configurations for the string $(x_1, x_2, \ldots, x_N)$. Amongst these, there are $n(Y)$ configurations, each leading to the same product Y. The value of Y for which $n(Y)$ is maximum is defined as the typical value, $\hat{Y}$. In other words, $\hat{Y}$ is the value of Y for which the entropy is maximum. In the present example, it is easily seen that $\hat{Y} = (r_1 r_2)^{N/2}$. On the other hand averaging over all configurations leads to $\langle Y \rangle = (r_1 + r_2)^N \cdot 2^{-N}$, which is different from $\hat{Y}$.

Let us consider the following experiment. Select randomly a string $(x_1, x_2, \ldots, x_N)$ by random sampling each x_i from its distribution. Calculate the value of Y for this string. Repeat the above independently M times and collect a set of values of Y, denoted by Y_i: $i = 1, M$. Let $\bar{Y}_M = (Y_1 + Y_2 + \cdots + Y_M)/M$. In the limit $M \to \infty$, $\bar{Y}_M = \langle Y \rangle$. However if M is not too large, $\bar{Y}_M$ is most likely to be close to $\hat{Y}$ rather than to $\langle Y \rangle$. Thus, the typical value $\hat{Y}$ represents more truly, the small sample average than does the true average. The task now is to obtain a 'working' expression for $\hat{Y}$.

Let $Z = \ln Y$. Then Z is a sum of N independent random variables $\{\ln x_i; i = 1, N\}$. For large N, by Central Limit Theorem (CLT), the distribution of Z is normal, with fluctuations σ (defined as the standard deviation divided by the mean) small and of the order of $N^{-1/2}$. Let $\hat{Z}$ denote the typical value of Z, that corresponds to maximum entropy. It is clear that for large N, $(\hat{Z} - \langle Z \rangle)/\langle Z \rangle \sim \pm \sigma$. Also the typical value of $\exp(Z)$ is the same as $\exp(\hat{Z})$, since both these quantities share the same configurations. It follows then that asymptotically (i.e., $N \to \infty$ and hence $\sigma \to 0$),

$$\hat{Y} = \exp[\langle \ln Y \rangle] \tag{1}$$

For the simple problem considered, it is easily verified that $\exp[\langle \ln Y \rangle] = (r_1 r_2)^{N/2}$, which is the same as the value of Y for which the entropy is maximum.

It is obvious that the definition of the typical value as given by (1) holds good trivially when the random variables $x_1, x_2, \ldots, x_N$ combine additively to yield Y. Even for problems involving sum over products of random variables, (1) is useful. For, it is usually possible to identify a dominant product in the sum; see Noskowicz and Goldhirsch (1988); Murthy and Kehr (1989).

When the random variables determining the macroscopic process Y are not all independent, (1) should still prove useful as a reasonably good description of the typical behaviour, if the fluctuations of $\ln Y$ go to zero asymptotically. Even for cases where fluctuations of $\ln Y$ do not go to zero but to a constant, (1) yields an estimate, of the typical value, with the correct asymptotic dependence on the 'system size' N. Thus (1) should prove useful for all problems except the ones for which the fluctuations of $\ln Y$ increase with the system size. An interesting problem that arises in this context is diffusion in a multiplicative environment, and this forms the subject matter of this paper.

Consider a particle executing a simple random walk on a one dimensional integer lattice. One site of the lattice is however defective, in the sense, whenever a particle visits that site, $1 + E$ particles emerge from the site. These particles diffuse independently in the lattice. The problem with $-1 \leqslant E < 0$, which implies that the defect is a trap, has been studied extensively by Rubin (1967), Montroll (1969), Weiss (1981), Donsker and Varadhan (1975), Grassberger and Proccacia (1982), Havlin *et al* (1984) and Politi and Schneider (1988), to name only a few. Hence in this paper we shall confine ourselves to the growth process, for which $E > 0$. ben-Avraham *et al* (1989) have shown that the average number of particles in the lattice grows exponentially with time. However the dynamics of the typical number of particles in the system has not been studied; nor, studies on characterizing the fluctuations and the distribution of the growth process have been reported in the literature. These constitute the principal objectives of the present work.

In §2 we make a brief statement of the problem under investigation and describe the different statistical parameters of interest. In §3 we study the dynamics of the typical number of particles in the system and show that it grows as $\exp(t^{1/2})$, slower than the average. In §4 we present a new formulation to obtain the average growth process and later extend the formulation to facilitate calculation of higher moments. In particular we show analytically, that the fluctuations of the growth process, increase exponentially with time. In §5 we calculate the distribution of the growth process and describe its interesting features. In §6 we briefly summarise the principal conclusions of the study, highlight the new findings and indicate possible applications and scope for future work.

2. Random walk on a lattice with a single defect: statement of the problem

Consider a one dimensional lattice extending from $-\infty$ to $+\infty$. A particle starts at origin and executes a simple random walk. Let $x(n)$ be the position of the random walk after n steps i.e. at discrete time n. The random walk process is defined by,

$$x(n) = x(n-1) + \lambda_n.$$
$$x(0) = 0. \tag{2}$$

In the above, λ_n is an independent random displacement whose distribution $\rho(\lambda)$ is given by,

$$\rho(\lambda) = 0{\cdot}5\{\delta(\lambda+1) + \delta(\lambda-1)\}. \tag{3}$$

Let us now declare one of the lattice sites, say S, to be defective and say that whenever the particle visits the site S, it spawns E new particles besides the original particle. These $(1+E)$ particles are non-interacting and they diffuse independently in the lattice. When any one of them returns to the defective site, E new particles are produced and the process goes on. Thus, soon, there is an avalanche of particles. Let $W(n, E)$ be the total number of particles in the system, at any time n. Our interest is to calculate the statistics of the stochastic process $W(n, E)$.

Alternatively, it is possible to interpret $W(n, E)$ as the mass of a single particle which starts at the origin with unit mass and which becomes more massive by a factor of $1 + E$ whenever it visits the defective site during its random walk. Thus we need to track only one particle and update its mass appropriately at every step. In what follows, we take, without loss of generality, the origin to be the defective site and set $W(0, E) = 1 + E$. The stochastic process $W(n, E)$ is formally defined by,

$$W(n, E) = (1+E)^{\psi(n)}, \tag{4}$$

where $\psi(n)$ is the number of times the particle visits the origin during its random walk. Note $\psi(0) = 1$ for all the random walks. $\psi(n)$, in turn, can be expressed as a sum of delta functions, as follows.

Let $\xi(j)$ be a random variable whose value is unity if the random walk is at the origin at time j, and zero otherwise. Formally,

$$\xi(j) = \delta_{x(j),0}, \tag{5}$$

where $x(j)$ is the random walk process defined by (2). Note that $\xi(0) = 1$. Thus we see,

$$\psi(n) = \sum_{j=0}^{n} \xi(j). \tag{6}$$

With the above, the growth process $W(n, E)$ is completely defined.

Before we proceed further, let us describe the relevant statistics of the basic process $\xi(j)$ that we shall require in the next three sections. From the definition, we see that $\xi^l(j) = \xi(j)$ for all $l \geqslant 1$. Since the particle visits the origin only at even numbered steps, $\xi(2j+1) = 0$, for all j, and for all the random walks. The average of $\xi(2j)$ is given by the conditional probability, that the random walk is at the origin at time $2j$ given that it started at the origin at time zero. Explicitly, see for example Chandrashekar (1943),

$$\langle \xi(2j) \rangle \equiv \langle \delta_{x(2j),0} \rangle = \frac{(2j)!}{j!j!}\frac{1}{2^{2j}}. \tag{7}$$

Exploiting the properties of delta functions, we can express the correlations of the process $\xi(2j)$ as,

$$\langle \xi(2j)\xi(2j+2k) \rangle = \langle \xi(2j) \rangle \langle \xi(2k) \rangle \tag{8}$$

Infact we can derive a more general result given by

$$\langle \xi(j) F(\xi(j+n_1), \xi(j+n_2), \ldots) \rangle = \langle \xi(j) \rangle \langle F(\xi(n_1), \xi(n_2), \ldots) \rangle \tag{9}$$

where F is an arbitrary function of non-negative powers of ξs.

In the next three sections, we shall investigate the typical and average behaviour of $W(n, E)$, the fluctuations of $\ln W(n, E)$ and $W(n, E)$ and finally the distribution of $W(n, E)$.

3. Typical behaviour of the growth process

It is clear that $W(2n+1, E) = W(2n, E)$ for all random walks of the particle. From (4) we get,

$$\ln W(2n, E) = \psi(2n) \ln(1+E). \tag{10}$$

Thus the statistical characteristics of the process $\ln W(2n, E)$ are the same as that of $\psi(2n)$ but for the constant factor $\ln(1+E)$. We can calculate the average of $\psi(2n)$ over the ensemble of all possible random walks, employing (6) and (7); it is given by,

$$\langle \psi(2n) \rangle = \sum_{j=0}^{n} \frac{(2j)!}{j!j!2^{2j}}. \tag{11}$$

We identify $\langle \psi(2n) \rangle$ as the coefficient of Z^{2n}, in the power series expansion of the function $(1 - Z^2)^{-3/2}$ and get

$$\langle \psi(2n) \rangle = \frac{(2n+1)!}{n!n!}\frac{1}{2^{2n}}. \tag{12}$$

Employing Stirling's approximation in the above, we get for large n,

$$\langle \psi(2n) \rangle = 2\pi^{-1/2} n^{1/2}. \tag{13}$$

Denoting the typical value of the growth process as $\hat{W}(n, E)$, we get its asymptotic behaviour as,

$$\hat{W}(n, E) \sim \exp[n^{1/2}(2/\pi)^{1/2} \ln(1+E)]. \tag{14}$$

It may be noted that we have employed (1) to describe the typical growth process. It is clear from (6) that $\psi(n)$ is a sum of correlated random variables $\xi(1), \xi(2), \ldots, \xi(n)$. Hence, use of (1) for characterizing the typical process is justified only if the fluctuations of $\psi(n)$ are small; see discussion in § 1. To calculate the fluctuations of $\psi(n)$, we proceed as follows.

From (6), we get

$$\langle \psi^2(2n) \rangle = \sum_{j=0}^{n} \sum_{k=0}^{n} \langle \xi(2j)\xi(2k) \rangle. \tag{15}$$

Separating the diagonal and off-diagonal terms and employing the statistical characteristics of $\xi(j)$, described by (8), we get

$$\langle \psi^2(2n) \rangle = 2 \sum_{j=0}^{n} \langle \xi(2j) \rangle \sum_{l=0}^{n-j} \langle \xi(2l) \rangle - \sum_{j=0}^{n} \langle \xi(2j) \rangle. \tag{16}$$

Average of $\xi(2j)$ is explicitly known and is given by (7), which when substituted in the above, yields, after a few simple algebraic manipulations,

$$\langle \psi^2(2n) \rangle = 2(n+1) - \frac{(2n+1)!}{n!\,n!\,2^{2n}}. \tag{17}$$

Employing Stirling's approximation in the above and using (13) we see that asymptotically the fluctuation of $\psi(n)$ is given by

$$\frac{\{\langle \psi^2(2n) \rangle - \langle \psi(2n) \rangle^2\}^{1/2}}{\langle \psi(2n) \rangle} \sim \{(\pi/2) - 1\}^{1/2}. \tag{18}$$

Thus the fluctuations of the process $\ln W(n, E)$ are constant asymptotically.

4. Average and higher moments of the growth process

Let us define $Q_1(n, E)$ as the average growth process. Thus $Q_1(n, E) = \langle W(n, E) \rangle$. In this section we present a new formulation to calculate $Q_1(n, E)$ making use of the statistical properties of the basic process $\xi(j)$. Later, we shall extend it to facilitate calculation of higher moments of $W(n, E)$.

From (4) and (6), we get

$$Q_1(n, E) = (1+E) \left\langle \prod_{j=1}^{n} [1 + E\xi(j)] \right\rangle. \tag{19}$$

The product inside the expectation sign is written as $\{1 + E\xi(1)\}$ times the same product, but now with the index j running from 2 to n. Expanding this and rearranging, we get

$$Q_1(n, E) = (1+E) \left\langle \prod_{j=2}^{n} [1 + E\xi(j)] \right\rangle + E(1+E) \left\langle \xi(1) \prod_{j=2}^{n} [1 + E\xi(j)] \right\rangle. \tag{20}$$

For calculating the correlations in the second terms on the RHS of the above, we employ (8) and (9) and get

$$Q_1(n,E) = E\langle\xi(1)\rangle Q_1(n-1,E) + (1+E)\left\langle \prod_{j=2}^{n} [1+E\xi(j)] \right\rangle. \tag{21}$$

Again we express the product in the RHS of (21) as two terms and repeat the above procedure to get an equation for $Q_1(n,E)$ in terms of $Q_1(n-1,E)$, $Q_1(n-2,E)$ and a product with j running from 3 to n. Iterating further, we get finally,

$$Q_1(n,E) = 1 + E + E\sum_{k=1}^{n} \langle\xi(k)\rangle Q_1(n-k,E). \tag{22}$$

Let us multiply both sides of the above by Z^n and then take the sum over $n=0$ to ∞, to get an expression for $Q_1(Z,E)$, the generating function of $Q_1(n,E)$ as

$$Q_1(Z,E) \equiv \sum_{n=0}^{\infty} Z^n Q_1(n,E) = \frac{1+E}{(1-Z)[1+E-Ef(Z)]}, \tag{23}$$

where $f(Z)$ is the generating function of $\langle\xi(j)\rangle$ which can be easily calculated, starting from (7) as,

$$f(Z) \equiv \sum_{n=0}^{\infty} Z^n \langle\xi(n)\rangle = (1-Z^2)^{-1/2}. \tag{24}$$

Equation (23) for $Q_1(Z,E)$ agrees with the expression derived by ben-Avraham *et al* (1989). Notice that (23) for $Q_1(Z,E)$ is general and describes the growth process of particles diffusing in a medium of arbitrary dimension. All we need is $f(Z)$ for the given lattice (say in d dimensions) which when inserted in (24) describes diffusion and growth in a d-dimensional lattice. This observation about the general nature of the formulation holds good even for the investigations reported in the next two sections. However, in the present paper we confine ourselves to diffusion and growth in one dimensional lattice.

Expanding $Q_1(Z,E)$ as a power series in Z, we get,

$$Q_1(2n,E) \equiv Q_1(2n+1,E) = \frac{1}{(1-\alpha^2)^{n+1}}\left\{1 + \frac{\alpha}{|\alpha|} - \alpha\sum_{l=n+1}^{\infty} \frac{(2l)!(1-\alpha^2)^l}{l!l!2^{2l}}\right\}, \tag{25}$$

where $\alpha = E/(1+E)$. Equation (25) is an exact and general result valid for all n and E. For the growth process with $E > 0$, and for large n, we get

$$Q_1(n,E) = \exp\left[n\ln\left(\frac{1+E}{(1+2E)^{1/2}}\right)\right] + 0(n^{-1/2}). \tag{26}$$

Thus we see that the average number of particles in the lattice grows exponentially with time, which is faster than the typical growth process.

Whenever the difference between the typical and average behaviour of a macroscopic quantity increases with respect to a parameter, then one would expect the fluctuations of the macroscopic quantity to increase. The parameter is the size of the system in the problem of Mean First Passage Time (MFPT) in a Sinai (1982) lattice; see Kehr and Murthy (1990). In the present problem, however, the parameter is time. Hence it would be of interest to calculate the dynamics of the higher moments of $W(n, E)$ and in particular its fluctuations.

The Kth moment of $W(n, E)$ is, by definition, given by,

$$Q_K(n, E) \equiv \langle W^K(n, E) \rangle = \langle \exp[\psi(n) K \ln(1 + E)] \rangle. \tag{27}$$

The expression for the first moment for the problem with E' as the growth parameter is known and is explicitly given by

$$Q_1(n, E') = \langle \exp[\psi(n) \ln(1 + E')] \rangle. \tag{28}$$

Substitute in the above $E' = (1 + E)^K - 1$. We then recover (27) which is the desired expression for the Kth moment. Thus the asymptotic behaviour of the Kth moment of $W(n, E)$ is obtained by simply replacing E by $(1 + E)^K - 1$ in the expression for $Q_1(n, E)$ given by (26). We get,

$$\begin{aligned} Q_K(n, E) &\equiv Q_1(n, E \to (1 + E)^K - 1) \\ &= \exp[n \ln\{(1 + E)^K (2(1 + E)^K - 1)^{-1/2}\}]. \end{aligned} \tag{29}$$

Fluctuations in $W(n, E)$ can now be calculated and we get asymptotically,

$$\frac{\{Q_2(n, E) - Q_1^2(n, E)\}^{1/2}}{Q_1(n, E)} \sim \exp(\gamma n) \tag{30}$$

$$\gamma = 0{\cdot}5 \ln[(1 + 2E)(1 + 4E + 2E^2)^{-1/2}]. \tag{30a}$$

Thus we see, as expected, that the fluctuations increase with time, exponentially. This shows that, in general CLT or its equivalent does not exist for describing such phenomena, wherein a large number of correlated random variables combine multiplicatively to determine the macroscopic process.

The difference between the typical and average growth process can be understood physically, as follows. Recall that we have been considering the random walk of a single particle on a lattice. The mass of the particle increases whenever it returns to the origin. It is clear from (6) that the increment in the number of returns to the origin at time n is given by $\xi(n)$. But, as per (7), $\langle \xi(n) \rangle \sim n^{-1/2}$ for large n. Thus return to the origin becomes less frequent at large time. But these rare events are precisely the ones that contribute significantly to the growth process. Hence these rare events influence dominantly the average and more dominantly the higher moments of the growth process. The typical behaviour, however, is not affected significantly by the fluctuations of the growth process. It is more or less determined by the typical random walks which constitute the majority of the ensemble of all possible random walks.

The above picture indicates that the distribution $\rho(W, n)$ of the process $W(n, E)$ should be characterized by a tail which dominates the average and higher moments of $W(n, E)$. In the context of diffusion in disordered system, Kehr and Murthy (1990) found that the distribution of the MFPT has a tail that decays algebraically. It would indeed be of interest to investigate, in the present problem, the nature of the distribution $\rho(W, n)$ and in particular the characteristics of the tail. To this, we turn our attention in the next section.

5. Probability distribution of the growth process

In this section, we shall derive an analytical expression for the probability density $\rho(W, n)$ of the growth process $W(n, E)$. To this end we first consider the process $\ln W(n, E)$ and obtain its distribution $P(n, \ln W)$. Then,

$$\rho(n, W) = W^{-1} P(n, \ln W). \tag{31}$$

Let $G(K, n)$ be the characteristic function (i.e. the Fourier transform) of $P(n, \ln W)$, defined as

$$G(K, n) = \langle \exp[iK \ln W(n, E)] \rangle. \tag{32}$$

The key step in the formulation is the identification of the RHS of the above, as the average of $W(n, E')$ with $E' = (1 + E)^{iK} - 1$ (see §4). The generating function of $\langle W(n, E') \rangle$ has been exactly calculated and is given by (23) with E replaced by E'. Thus,

$$G(K, Z) = \sum_{n=0}^{\infty} Z^n G(K, n) = (1 - Z)^{-1}\{1 - f(Z)[1 - (1 + E)^{-iK}]\}^{-1}, \tag{33}$$

where $f(Z)$ is given by (24). Let $\gamma = \ln(1 + E)$. We can rewrite $G(K, Z)$ in a convenient form, as given by

$$G(K, Z) = \frac{1}{(1 - Z) f(Z)} \sum_{l=0}^{\infty} \left[\frac{f(Z) - 1}{f(Z)}\right]^l \exp[iK\gamma(l + 1)]. \tag{34}$$

Inverting the above ($K \to \ln W$ and $Z \to n$) would yield the required expression for $P(n, \ln W)$. Thus formally, we have,

$$\sum_{n=0}^{\infty} Z^n P(n, \ln W) = (2\pi)^{-1} \int_{-\infty}^{+\infty} \mathrm{d}K\, G(K, Z) \exp(-iK \ln W) \tag{35}$$

Substituting in the above for $G(K, Z)$ (see (34)) we get

$$\sum_{n=0}^{\infty} Z^n P(n, \ln W) = \frac{1}{(1 - Z) f(Z)} \sum_{l=0}^{\infty} \left[\frac{f(Z) - 1}{f(Z)}\right]^l \times \int_{-\infty}^{+\infty} \frac{\mathrm{d}K}{2\pi} \exp[iK\{v(l + 1) - \ln W\}]. \tag{36}$$

The integral in the above is identified as a representation of the Dirac delta function $\delta[\ln W - \gamma(l + 1)]$. The RHS of (36) can be expressed as a power series in Z, and the coefficient of Z^n would yield the required value of $P(n, \ln W)$. Thus (36) can be rewritten as,

$$\sum_{n=0}^{\infty} Z^n P(n, \ln W) = \sum_{l=0}^{\infty} \delta(\ln W - \gamma(l + 1)) \Phi(Z, l), \tag{37a}$$

where,

$$\Phi(Z, l) = \frac{1}{(1 - Z) f(Z)} \left[\frac{f(Z) - 1}{f(Z)}\right]^l. \tag{37b}$$

Substituting for $f(Z)$ from (24) in the above, we get

$$\Phi(Z,l) = (1+Z)[\{1-(1-Z^2)^{1/2}\}^l(1-Z^2)^{-1/2}]. \tag{38}$$

The expression within the square bracket above is a function of Z^2. This implies that $P(2n+1, \ln W) \equiv P(2n, \ln W)$, a result, also expected on physical grounds. Thus, it is enough if we extract the coefficient of Z^{2n} of $\chi(Z,l) = \Phi(Z,l)/(1+Z)$, to obtain $P(2n, \ln W)$.

Let us formally write,

$$\chi(Z,l) = \{1-(1-Z^2)^{1/2}\}^l(1-Z^2)^{-1/2} = \sum_{n=0}^{\infty} Z^{2n} C_{2n,l}. \tag{39}$$

It is clear from the expression for $\chi(Z,l)$ that $C_{2n,l} = 0$ for all $l > n$, consistent with the definition of $\psi(n)$ given by (6). We get,

$$P(2n, \ln W) = \sum_{l=0}^{n} C_{2n,l}\,\delta[\ln W - (l+1)\ln(1+E)]. \tag{40}$$

Thus we find that $P(2n, \ln W)$ is a series of delta functions situated at values of $\ln W = (l+1)\ln(1+E)$, with $0 \leqslant l \leqslant n$. The strengths of the delta functions are given by $C_{2n,l}$, the coefficient of Z^{2n} in the power series expansion of $\chi(Z,l)$. The distribution $\rho(n, W)$ can now be easily written as,

$$\rho(2n, W) \equiv \rho(2n+1, W) = \sum_{l=0}^{n} C_{2n,l}\,\delta[W-(1+E)^{l+1}]. \tag{41}$$

$\rho(2n, W)$ is again a series of delta functions each of strength $C_{2n,l}$ situated at values of $W = (1+E)^{l+1}$, with l running from 0 to n.

Thus the strengths $C_{2n,l}$ contain complete information about the distribution of $W(n,E)$. Expression for $C_{2n,l}$ can be obtained by expanding $\chi(Z,l)$ given by (39), as a power series in Z^2. We get

$$C_{2n,l} = \frac{1}{2^{2n} n!} \sum_{K=0}^{[l/2]} \frac{(-1)^K l!(2n-2K)!}{K!(l-2K)!(n-K)!} \tag{42}$$

where $[l/2]$ represents the integer part of $l/2$. Thus we notice that all features of $\rho(n, W)$ can be obtained by investigating the behaviour of $C_{2n,l}$ for various values of n and l.

By applying Stirling's approximation to (42) we get for large n,

$$C_{2n,l} \sim (\pi n)^{-1/2} \sum_{K=0}^{[l/2]} \frac{(-1)^K}{K!}\left(\frac{l^2}{4n}\right)^K; \quad (n \gg 1). \tag{43}$$

First let us consider the case with l less than or of the order of $\sqrt{n}$. We get

$$C_{2n,l} \approx (\pi n)^{-1/2}; \quad (n \gg 1; l \lesssim \sqrt{n}). \tag{44}$$

Thus for small l, $C_{2n,l}$ is almost a constant. This shows that the distribution $\rho(n, W)$ is a slowly varying function of W for values of W up to the order of $\exp(\sqrt{n})$. Let us now consider the case with large n and $l \gg \sqrt{n}$. The expression for $C_{2n,l}$ for this case is given by,

$$C_{2n,l} \sim (\pi n)^{-1/2} \exp[-l^2/4n]; \quad (n \gg 1; l \gg \sqrt{n}). \tag{45}$$

This shows that the distribution of $\ln W$ has a Gaussian tail and hence the typical growth process or the higher moments of $\ln W$ are unaffected by the tail. For example the mth moment of $\ln W$ is completely determined by the nearly constant region of $\rho(W, n)$ and is given by

$$\langle [\ln W]^m \rangle \sim n^{m/2}, \tag{46}$$

which shows that $\hat{W} \sim \exp \sqrt{n}$ and the fluctuations of $\ln W(n, E)$ are a constant.

The discussions above indicates that (45) in conjuction with (41) implies that the distribution $\rho(W, n)$ has a log normal tail which makes a dominant contribution to the average and to the higher moments of the growth process.

6. Summary, conclusions and discussions

We have investigated the dynamics of the growth of the number of particles diffusing in a one dimensional lattice with a single defect at the origin. When the particle visits the origin the defect multiplies the particle by a factor of $(1 + E)$ when the growth parameter $E > 0$. The principal new findings of our work are listed below.

(i) We show that the typical number of particles in the lattice system grows much slower than the average number of particles.
(ii) We report a new formulation to calculate the average growth process. The advantage of our formulation is that its starting point is the process $W(n, E)$. Contrast this with the formulation of Redner and Kang (1984) which starts with $\langle W(n, E) \rangle$. It is precisely this, that has helped us see the possibility of extending the formulation to facilitate the calculation of higher moments and finally of the distribution of $W(n, E)$.
(iii) We show that the fluctuations of the growth process increase exponentially with time. This finding is significant since it explains succinctly the exponential growth of the difference between the typical and average processes.
(iv) The growth of fluctuations of $W(n, E)$ suggests that the distribution $\rho(n, W)$ must have a tail. We report for the first time analytical expression for the probability distribution of the growth process $W(n, E)$. Indeed we find that for large n, $\rho(n, W)$ is a slowly decreasing function of W for values of W up to the order of $\exp(\sqrt{n})$. For larger values of W, $\rho(W, n)$ decreases log normally. It is this tail that makes dominant contribution to the average and higher moments. But the typical growth process undermines the tail contribution, and hence it increases with time slower than the average process.

Applications of the formulation presented in this paper are many and we mention briefly a few below. Our investigations should prove useful in understanding the role of fluctuations in random multiplicative processes, and in providing a simple description of neutrons diffusing in a medium containing fissile materials. The model problem investigated bears close relation to self avoiding walks in random media, see Kardar and Zhang (1987); to the problems pertaining to interfaces in two dimensional Ising spin systems, see Huse *et al* (1985); and to the Burger's equation in the presence of a random force, see Medina *et al* (1989). In particular we plan to apply the formulation to the study of the kinetics of neutron population in a multiplying medium.

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