Pramāṇa – J. Phys., Vol. 35, No. 4, October 1990, pp. 349–354. © Printed in India.

Fluctuating hydrodynamics of a fluid with internal rotation

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MS received 18 May 1990

Abstract. The fluctuating hydrodynamics theory of a fluid possessing internal rotation is set up following the Landau-Lifshitz approach.

Keywords. Fluctuating hydrodynamics; Landau-Lifshitz theory; internal rotation.

PACS Nos 47·90; 05·40

1. Introduction

Landau and Lifshitz (1957) were the first to develop the theory of fluctuating hydrodynamics of a simple liquid (see also: Lifshitz and Pitaevskii 1980a; together with Landau and Lifshitz (1957), jointly referred to hereafter as I). In this classic work they added random 'outside' stress to the stress tensor of the fluid and random 'outside' heat flux to the heat flux vector of the fluid. They wrote down the linearised equations of flutuating hydrodynamics. Treating the fluctuations as Gaussian Markov processes, they then derived the fluctuation-dissipation relations of the random forces in the fluid. The formalism of I has been widely used in the literature for dealing with practical problems such as scattering of light from the fluid. Extension of the formalism to nonlinear hydrodynamics and also to nonequilibrium situations have received much attention (Fox 1978).

In the case of a fluid possessing internal rotation in the hydrodynamic sense, the Landau-Lifshitz formalism has not yet been extended. We have in mind fluids whose molecules may possess structure and thus are capable of rotation as well as translation. The real fluid may consist of two (or more) species of molecules, some possessing rotation and some not being able to rotate. In this case too, the fluid 'particle' in the context of hydrodynamics would possess some 'average' rotation. The deterministic hydrodynamic equations of such fluids have been obtained by Shliomis (1976) (to be referred to hereafter as II) in a penetrating analysis of the problem. It is the purpose of this paper to add fluctuating forces to the hydrodynamic equtions set up in II and to derive the fluctuation-dissipation relations for these random forces.

In this paper, we shall first recapitulate the essential equations of II, following the notation of II. Next we shall set up the formalism of fluctuating hydrodynamics following the procedure of I.

2. Hydrodynamics of the Shliomis fluid

The hydrodynamic equations of the fluid have been derived in II. Following are the local conservation equations for mass, energy, linear momenta, angular momenta, intrinsic angular momenta and entropy:

$$\partial_t \rho + \partial_i(\rho v_i) = 0, \tag{1}$$

$$\partial_t E + \partial_i Q_i = 0, \tag{2}$$

$$\partial_t(\rho v_i) + \partial_k \Pi_{ik} = 0, \tag{3}$$

$$\partial_t(L_{ik} + M_{ik}) + \partial_l G_{ikl} = 0, \tag{4}$$

$$\partial_t M_{ik} + \partial_l(v_l M_{ik}) = \sigma_{ki} - \sigma_{ik} - \partial_l g_{ikl}, \tag{5}$$

$$\partial_t S + \partial_i(v_i S) = R/T, \tag{6}$$

$$g_{ikl} = G_{ikl} - v_l(L_{ik} + M_{ik}) + x_i\sigma_{kl} - x_k\sigma_{il}. \tag{7}$$

Here the notations used are the following. Roman indices denote Cartesian components and summation convention is adopted.

x_i = position, t = time, $\partial_i = \partial/\partial x_i$, $\partial_t = \partial/\partial t$,

ρ = mass density,

v_i = velocity,

E = energy density,

Π_{ik} = momentum flux tensor,

$L_{ik} = \rho(x_i v_k - x_k v_i)$ = angular momentum density tensor,

M_{ik} = intrinsic (or internal) angular momentum density tensor,

$G_{ikl} = -G_{kil}$ = angular momentum flux tensor,

σ_{ik} = stress tensor of the fluid,

S = entropy density,

R = dissipation function or entropy production density, and

T = temperature. (8)

The expression for the various fluxes etc. are

$$\Pi_{ik} = \rho v_i v_k - \sigma_{ik}, \tag{9}$$

$$\sigma_{ik} = -p^* \delta_{ik} + \lambda_{ik}, \tag{10}$$

$$p^* = p + (\alpha M_i - \Omega_i) M_i, \tag{11}$$

$$\lambda_{ik} = \eta\{\partial_i v_k + \partial_k v_i - (2/3)\delta_{ik}\partial_l v_l\} + \zeta\delta_{ik}\partial_l v_l + \tfrac{1}{2}\gamma(\alpha M_{ik} - \Omega_{ik}), \tag{12}$$

where M_l and $\Omega_l (= \frac{1}{2} e_{lik}\partial_i v_k$, the vorticity of the fluid, e_{lik} being the Levi–Civita permutation tensor density) are dual pseudovectors corresponding to M_{ik} and L_{ik}

respectively, i.e.

$$M_l = \tfrac{1}{2} e_{lik} M_{ik}, \quad M_{ik} = e_{lik} M_l \tag{13}$$

etc., p is the pressure of the fluid, α is the reciprocal of the moment of inertia density (assumed to be a diagonal tensor), η is the shear viscosity, ζ, the bulk viscosity and γ is the rotational viscosity. The energy flux vector is

$$Q_k = \rho v_k(\tfrac{1}{2}v^2 + w) - v_i \lambda_{ik} + \alpha v_k M^2 + \alpha M_i g_{ik}, \tag{14}$$

where

$$g_{ik} = \tfrac{1}{2} e_{ilm} g_{lmk} \tag{15}$$

[i.e., g_{ik} and g_{ikl} are duals; this tensor is not defined properly in II]. The entropy production density is given by

$$R = \lambda_{ik}[\tfrac{1}{2}(\partial_i v_k + \partial_k v_i) + (\alpha M_{ik} - \Omega_{ik})] - \alpha g_{ik} \partial_k M_i. \tag{16}$$

[Note the absence of α in the coefficient of the last term in II.] The Onsager relation relating the tensor g_{ik} to the gradient of the vector M_i is

$$g_{ik} = -\mu \partial_k M_i, \tag{17}$$

where $\mu(>0)$ is a new transport coefficient which may be callled as the diffusion coefficient of **M**. Each term of R is separately positive definite. Note that ref. II does not postulate any conservation equation for moment of inertia. It implicitly assumes that the moment of inertia θ to be associated with a fluid "particle" is constant. Thus the moment of inertia density is assumed to be $\rho\theta$ leading to the identification

$$\alpha = (\rho\theta)^{-1}. \tag{18}$$

Due to the conservation of mass (eq. 1), the moment of inertia is automatically conserved. Alternate forms of equation of motion for v_i and M_i are

$$\begin{aligned} \rho d_t v_i = & -\partial_i[p + M_k(\alpha M_k - \Omega_k)] + [\eta + (\gamma/4)]\partial_{kk} v_i \\ & + [\zeta + (\eta/3) - (\gamma/4)]\partial_{ik} v_k + \tfrac{1}{2}\alpha\gamma e_{ikl} \partial_k M_l, \end{aligned} \tag{19}$$

$$d_t M_i = \gamma(\Omega_i - \alpha M_i) + \mu \partial_{kk} M_i - M_i \partial_k v_k, \tag{20}$$

where

$$d_t = \partial_t + v_k \partial_k \tag{21}$$

is the material derivative. Using relation (1) and introducing the intrinsic angular velocity variable via the relation

$$M_i = \rho\theta\omega_i, \tag{22}$$

eqs (19) and (20) can also be written in the form

$$\begin{aligned} \rho d_t v_i = & -\partial_i[p + \rho I \omega_p(\omega_p - \tfrac{1}{2} e_{pkl} \partial_k v_l)] + [\eta + (\gamma/4)]\partial_{kk} v_i \\ & + [\zeta + (\eta/3) - (\gamma/4)]\partial_{ik} v_k + \tfrac{1}{2}\gamma e_{ikl} \partial_k \omega_l. \end{aligned} \tag{23}$$

$$d_t(\rho\theta\omega_i) = -\gamma\omega_i + \mu\theta\partial_{kk}(\rho\omega_i) + \tfrac{1}{2}\gamma e_{ikl}\partial_k v_l - \rho\theta\omega_i \partial_k v_k. \tag{24}$$

The linearized forms of the equations are obtained by substituting

$$\rho = \rho_0 + \Delta\rho, \mathbf{v} = \Delta\mathbf{v}, p = p_0 + \Delta p, \alpha = \alpha_0 + \Delta\alpha, \tag{25}$$

where ρ_0, p_0, α_0 are equilibrium values of ρ, p and α respectively and $\Delta\rho$ etc. are derivation from equilibrium. The linearized equations (after relabelling $\Delta\rho \to \rho$, $\Delta p \to p$ etc.) are

$$\rho\partial_t v_i = [\eta + (\gamma/4)]\partial_{kk} v_i + [\zeta + (\eta/3) - (\gamma/4)]\partial_{ik} v_k - \partial_i p + \tfrac{1}{2}\gamma e_{ikl}\partial_k \omega_l. \tag{26}$$

$$\rho_0 \theta \partial_t \omega_i = -\gamma\omega_i + \mu\rho_0\theta\partial_{kk}\omega_i + \tfrac{1}{2}\gamma e_{ikl}\partial_k v_l. \tag{27}$$

3. Fluctuating hydrodynamics

Reference I has given the following prescription for working out the fluctuation-dissipation relations for a system described by a set of macroscopic variables $\{x_i | i = 1, \ldots, n\}$. If the system is in a state close to thermodynamic equilibrium, then this time evolution is given by equations of the form

$$\dot{x}_i = -\lambda_{ik} x_k + y_i \tag{28}$$

with constant coefficients λ_{ik} (the equilibrium values $x^0{}_i$ are assumed to be zeros). Here y_i are the random 'forces'. The deviation of entropy S from its equilibrium value S_0 is given by

$$S - S_0 = -\tfrac{1}{2}\beta_{ik} x_i x_k. \tag{29}$$

The variable thermodynamically conjugate to x_i is

$$X_i = -\partial S/\partial x_i = \beta_{ik} x_k \tag{30}$$

(clearly $\beta_{ik} = \beta_{ki}$). If we express the quantities x_i in terms of X_i from (30) and substitute in (28), we obtain the relaxation equations in the form

$$\dot{x}_i = -\gamma_{ik} X_k \tag{31}$$

where

$$\gamma_{ik} = \lambda_{il}\beta^{-1}{}_{lk} = \gamma_{ki} \tag{32}$$

are the constants known as kinetic coefficients, obeying the Onsager symmetry relations. [These relations hold for variables x_i and x_k which do not change sign under time reversal. If one of the (x_i, x_k) changes sign under time reversal then $\gamma_{ik} = -\gamma_{ki}$]. The entropy production rate in the system is

$$\dot{S} = (\partial S/\partial x_i)\dot{x}_i = -X_i \dot{x}_i = \gamma_{ik} X_i X_k. \tag{33}$$

The fluctuation-dissipation relations for the random forces $y_i(t)$ are then of the form

$$\langle y_i(t) y_k(t') \rangle = k(\gamma_{ik} + \gamma_{ki})\delta(t - t') \tag{34}$$

where k is the Boltzmann constant and the angular bracket denotes statistical thermal equilibrium average. The above procedure being very standard, we refer the reader for a detailed exposition to the work of Lifshitz and Pitaevskii (1980b).

Application of the above procedure to fluctuations in a simple fluid has been illustrated in I. Following I, we regard the constitutive relations (12) and (17) as analogous to (28), i.e. we add random 'forces' λ_{ik}^R and g_{ik}^R to these equations and obtain

$$\lambda_{ik} = \eta\{\partial_i v_k + \partial_k v_i - (2/3)\delta_{ik}\partial_l v_l\} + \zeta\delta_{ik}\partial_l v_l + \tfrac{1}{2}\gamma(\alpha_0 M_{ik} - \Omega_{ik}) + \lambda_{ik}^R. \quad (35)$$

$$g_{ik} = -\mu\partial_k M_i + g_{ik}^R. \quad (36)$$

The unimportant complication arising due to the fluid being a continuous medium is tackled by discretizing the fluid into small volume elements $\Delta V^{(a)}$ and defining average quantities such as $\lambda_{ik}^{(a)}$, $v_i^{(a)}$ etc, defined within each cell. Equations (35) and (36) are analogous to eq. (28). Here $\dot{x}_i \to \lambda_{ik}$, g_{ik} and $y_i \to \lambda_{ik}^R$, g_{ik}^R. To identify the variables X_i, we consider the rate of entropy production

$$d_t S = \int dV\dot{S} = \int dV(\partial_t S + v_k\partial_k S) = \int dV(R/T)$$

$$\to \sum \Delta V[\lambda_{ik}\{\tfrac{1}{2}(\partial_k v_i + \partial_i v_k) + (\alpha M_{ik} - \Omega_{ik})\} - \alpha_0 g_{ik}\partial_k M_i], \quad (37)$$

where the arrow represents the discretization process and the cell indices (such as $\Delta V^{(a)}$, $v_i^{(a)}$, etc.) have been dropped. Comparing with (33), we obtain the following conjugate variables correspondence $x_i \to X_i$:

$$\lambda_{ik} \to \Lambda_{ik} \equiv -(\Delta V/T)[\tfrac{1}{2}(\partial_k v_i + \partial_i v_k) + (\alpha M_{ik} - \Omega_{ik})].$$

$$g_{ik} \to \Gamma_{ik} \equiv (\Delta V/T)\alpha_0\partial_k M_i. \quad (38)$$

We now write the equations analogous to (31) so as to extract the kinetic coefficients of the present problem defined by the following equations

$$\lambda_{ik} = \gamma_{ik,lm}\Lambda_{lm}$$

$$g_{ik} = \bar{\gamma}_{ik,lm}\Gamma_{lm}. \quad (39)$$

It is a trivial matter to deduce that

$$\gamma_{ik,lm} = (T/\Delta V)[\eta(\delta_{il}\delta_{km} + \delta_{kl}\delta_{im}) + \{\zeta - (2/3)\eta\}\delta_{il}\delta_{km} + \tfrac{1}{2}\gamma\delta_{il}\delta_{km}].$$

$$\bar{\gamma}_{ik,lm} = (T/\Delta V)(\mu/\alpha_0)\delta_{li}\delta_{km}. \quad (40)$$

We note that in (38), there are no terms which would relate λ_{ik} to $\partial_k M_i$ or g_{ik} to the velocity gradients. This means that there are no cross kinetic coefficients relating λ_{ik} to Γ_{lm} or g_{ik} to Λ_{lm}. Hence we obtain from (34) the relations

$$\langle \lambda_{ik}^R(\mathbf{r}, t) g_{lm}^R(\mathbf{r}', t')\rangle = 0, \quad (41)$$

i.e. λ_{ik}^R and g_{lm}^R are uncorrelated. Next, the coefficients relating to values of λ_{ik} (g_{ik}) to those of $\Lambda_{lm}(\Gamma_{lm})$ are zero, if these quantities are taken in different cells ΔV. Equations corresponding to (34) are now obtained after taking the limit $\Delta V \to 0$,

$$\langle \lambda_{ik}^R(\mathbf{r}, t)\lambda_{lm}^R(\mathbf{r}', t')\rangle = 2kT[\eta(\delta_{il}\delta_{km} + \delta_{kl}\delta_{im}) + \{\zeta - (2/3)\eta\}\delta_{il}\delta_{km} + \tfrac{1}{2}\gamma\delta_{il}\delta_{km}]\delta(\mathbf{r} - \mathbf{r}')\delta(t - t') \quad (42)$$

$$\langle g_{ik}^R(\mathbf{r}, t) g_{lm}^R(\mathbf{r}', t')\rangle = 2kT(\mu/\alpha_0)\delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$$

$$= 2kT\rho_0\theta\mu\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \quad (43)$$

Equations (42) and (43) are the fluctuation dissipation relations we had set as our task to derive. The fluctuating hydrodynamics equations are obtained by the replacements $\lambda_{ik} \to \lambda_{ik} + \lambda_{ik}{}^R$ and $g_{ik} \to g_{ik} + g_{ik}{}^R$ in the deterministic equations (3)–(5). These lead to the following fluctuating versions of (23) and (24):

$$\rho_0 \partial_t v_i - [\eta + (\gamma/4)] \partial_{kk} v_i - [\zeta + (\eta/3) - (\gamma/4)] \partial_{ik} v_k$$
$$+ \partial_i p + \tfrac{1}{2}\gamma e_{ikl} \partial_k \omega_l = - \partial_k \lambda_{ik}{}^R, \tag{44}$$
$$\rho_0 \theta \partial_t \omega_i + \gamma \omega_i - \mu \rho_0 \theta \partial_{kk} \omega_i - \tfrac{1}{2}\gamma e_{ikl} \partial_k v_l = - \partial_k g_{ik}{}^R - e_{ikl} \lambda_{kl}{}^R. \tag{45}$$

4. Conclusion

The fluctuation-dissipation relations (41)–(43) are the new results of this paper. Together with (44) and (45), and an appropriate equation of state for the pressure $p = p(\rho, T)$ we have a closed set of hydrodynamic equations. Thermal fluctuations in the heat flux **Q**, in the spirit of II, can be incorporated trivially. Possible applications of the present formalism are numerous. To indicate a few, the force acting on a Brownian particle in this fluid can be derived following a similar derivation (in the case of simple fluid) of Bedeaux and Mazur (1974). The rotational Brownian motion problem is another straightforward application. It would be interesting to consider the problem of light scattering under nonequilibrium steady state conditions such as under a constant (small) temperature gradient or under shear flow. Some such problems will be considered in future publications.

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