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An analysis of the validity of local causality at the statistical level in Einstein-Podolsky-Rosen-type situations

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Abstract. We investigate the question of local causality at the statistical level in Einstein-Podolsky-Rosen (EPR) type situations, taking into account the most general class of measurements envisaged in quantum theory. The condition for local causality at the statistical level used in this paper pertains to the invariance of statistics of measurements on one sub-system with respect to the choice and type of measurements on its correlated partner in the EPR-type examples. Our analysis is based on a criterion for measurements performed on one of the EPR sub-systems, which is more general than the criterion used in the earlier treatments. We discuss both non-absorptive measurements (where the system is available for further observation after the measurement is performed) as well as absorptive measurements (where the system is absorbed in the process of a particular outcome being realized). We show that in the case of arbitrary non-absorptive measurements characterized by operation-valued measures, the requirement of local causality at the statistical level is satisfied and in the process we identify the key inputs in such a proof. We also obtain the specific conditions under which an absorptive measurement satisfies local causality at the statistical level.

Keywords. Local causality at statistical level; Einstein-Podolsky-Rosen argument; absorptive measurements.

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1. Introduction

The peculiarities of quantum correlations between spatially separated systems have been discussed eversince the work of Einstein *et al* (EPR) (1935) over fifty years ago. The standard illustration of an EPR type set up is that discussed by Bohm (1951) and involves a spin-zero system decaying into two correlated spin-1/2 particles. A key element in this example and all its variants is the non-separable character of the (non-factorizable) two particle wave function, which is a superposition of products of one particle wave functions. The intriguing feature of an EPR type set up is that the state of any one particle of the pair "collapses" (or changes due to measurement) depending on the measurement performed on its partner and the outcome realized, even though the members of the EPR pair are sufficiently separated so that they are mutually non-interacting (see Selleri (1988) for a recent overview).

There have indeed been several demonstrations that the above non-local feature of the state vector collapse does not lead to any observable violations of local causality at the statistical level (Ghirardi *et al* 1980, 1988; Kraus 1983). It is shown that the density operator characterizing the ensemble of either of the two particles in the EPR type set up, is actually independent of whatever measurement is (or is not) performed on the other particle. Thus all the statistical results pertaining to any observable measured on either of the particles are independent of whatever measurement is (or is not) performed on the other. However, recently the possibility of non-local effects even at a statistical level (when one considers more general class of measurements than allowed in conventional quantum theory) has been suggested (Datta *et al* 1988; Home 1989) and the issue is being widely debated (Clifton and Redhead 1988; Corbett 1988; Eberhard and Ross 1989; Finkelstein and Stapp 1987; Hall 1987; Home 1989, 1990; Squires 1988). Hence there is a need to investigate from first principles the validity of local causality at the statistical level for a general class of measurements. In this paper we shall address ourselves to this problem.

We first discuss various classes of measurements that generalize the usual collapse postulate of von Neumann-Lüders which characterizes the change in the state of a system due to the so called 'ideal' measurement of an observable with a purely discrete spectrum. If we confine ourselves to the standard Hilbert space formulation of quantum theory, then it is well known (see for instance, Davies 1976) that the most general measurement, when the system is available for further observation, is characterized by an operation-valued-measure. However there are a large class of physical experiments where the system is not always available for further observation. An important feature of the present paper is that we shall also consider such 'absorptive measurements' where the system is 'absorbed' (and is not available for further observations) in the very process of realisation of a particular outcome (or a set of outcomes) in the measurement.

In § 3, we consider measurements performed on a composite system such as a system composed of two particles in the typical EPR-type set up. Here we first propose an appropriate criterion as to when a measurement performed on the composite system can be considered actually as a measurement performed on one of the sub-systems. In § 4, we analyze the conditions under which local causality is satisfied at the statistical level in an EPR-type situation. We shall show that in any non-absorptive measurement characterized by an operation-valued measure the density operator of either of the sub-systems is totally independent of whatever measurement is (or is not) performed on the other sub-system. This result showing the validity of local causality at the statistical level for non-absorptive measurements generalizes the earlier such results due to Ghirardi *et al* (1980, 1988) and Kraus (1983). In § 5, we present a simple mathematical characterization of absorptive measurements. We find the specific conditions under which an absorptive measurement performed on one of the sub-systems of a composite system in an EPR-type set up, satisfies local causality at the statistical level. It turns out that such absorptive measurements can be in some sense characterized as minimal modifications of non-absorptive measurements.

2. Ideal and non-ideal measurements

In standard or conventional quantum theory, the state changes due to a measurement of an observable with a purely discrete spectrum are given by the von Neumann-Lüders collapse postulate. According to this if A is an observable with a purely discrete spectrum given by

$$A = \sum_i \alpha_i P^A(\alpha_i) \tag{1}$$

where $\{\alpha_i\}$ are the eigenvalues and $P^A(\alpha_i)$ are the eigen projectors, then the change in the state of a system in a measurement of A in which the outcome α_i is realized, given by

$$\rho \rightarrow \frac{P^A(\alpha_i)\rho P^A(\alpha_i)}{\mathrm{Tr}[P^A(\alpha_i)\rho]}, \tag{2}$$

where ρ is the density operator characterizing the state of the system immediately prior to the measurement. The measurements such as (2) are called 'ideal' measurements*. These have the property of 'repeatability' that an immediate repetition of the measurement is certain to yield the same outcome. More general class of measurements can be defined once we realize that the unnormalized measurement transformation

$$\rho \rightarrow P^A(\alpha_i)\rho P^A(\alpha_i) \tag{3}$$

defines a positive linear map of norm less that one defined on the space $\mathscr{T}(H)$ of all trace-class operators on the Hilbert space H. Such maps on $\mathscr{T}(H)$ are called 'operations'.

A general measurement with a discrete set of outcomes $\lambda_1, \lambda_2, .. \lambda_i, ...$ can be characterized by a set of operations $\{\mathscr{E}_1, \mathscr{E}_2, ... \mathscr{E}_i, ...\}$ where

(i) Each $\mathscr{E}_i$ is an operation, or linear positive norm-non-increasing map on $\mathscr{T}(H)$.

(ii)
$$\sum_i \mathrm{Tr}[\mathscr{E}_i(\rho)] = \mathrm{Tr}\,\rho \tag{4}$$

for each $\rho \in \mathscr{T}(H)$.

In any such measurement, if the outcome λ_i is realized, then the change in the state of system is given by

$$\rho \rightarrow \mathscr{E}_i(\rho)/\{\mathrm{Tr}[\mathscr{E}_i(\rho)]\} \tag{5}$$

and $\mathrm{Tr}[\mathscr{E}_i(\rho)]$ gives the probability that the outcome is λ_i. The conditions (4) ensures that the total probability that any outcome is realized is unity. If we allow for a continuous set of outcomes also, then we need to characterize a general experiment by an operation-valued measure on the real line, $\Delta \rightarrow \mathscr{E}(\Delta)$ defined for each Borel set $\Delta \in \mathscr{B}(R)$ such that

(i) For each $\Delta \in \mathscr{B}(R)$, $\mathscr{E}(\Delta)$ is an operation.
(ii) If $\{\Delta_i\}$ is a disjoint sequence of elements of $\mathscr{B}(R)$ then

$$\mathscr{E}(\cup \Delta_i) = \sum_i \mathscr{E}_i(\Delta_i) \tag{6}$$

where the right hand side converges in the strong operator topology.

(iii)
$$\mathrm{Tr}[\mathscr{E}(R)\rho] = 1 \tag{7}$$

for all density operators $\rho \in \mathscr{T}(H)$.

If ρ is the state of an ensemble of systems prior to the measurement then

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* An older, but not very standard terminology is 'measurement of I kind'. An important feature of this class of measurements is that each measurement is characterized by a countable set of mutually orthogonal projection operators $\{P_i\}$ such that $\Sigma_i P_i = I$.

$\mathscr{E}(\Delta)\rho/\mathrm{Tr}[\mathscr{E}(\Delta)\rho]$ is the state of (the subensemble composed of) all those systems which yielded an outcome in Δ.

At this stage we would like to recall a couple of mathematical facts concerning operations. Associated with each operation $\mathscr{E}$ defined on $\mathscr{T}(H)$ there is the adjoint map $\mathscr{E}^*$ defined on $\mathscr{B}(H)$ the set of all bounded operators on H such that

(i) $\mathscr{E}^*$ is a linear positive map on $\mathscr{B}(H)$ which is 'normal' or continuous under the ultra-weak topology

(ii) $$\mathscr{E}^*(I) = F \leqslant I. \tag{8}$$

F is called the 'effect' associated with the operation $\mathscr{E}$. Further we have that

$$\mathrm{Tr}[\mathscr{E}(\rho)] = \mathrm{Tr}(\rho) \quad \text{for all} \quad \rho \Rightarrow \mathscr{E}^*(I) = I. \tag{9}$$

An important result concerning operations is the theorem due to Kraus (1983) that if an operation $\mathscr{E}$ (or its adjoint $\mathscr{E}^*$) satisfies the so called requirement of complete positivity (see Davies 1976 for a definition of this concept) then there exists a countable collection of operators

$$A_i \in \mathscr{B}(H),$$

such that

$$\mathscr{E}(\rho) = \sum_i A_i \rho A_i^\dagger. \tag{10}$$

Further, the $\{A_i\}$ satisfy the condition

$$\mathscr{E}^*(I) = F = \sum_i A_i^\dagger A_i \leqslant I. \tag{11}$$

It may be noted that an operation-valued measure characterizes the most general measurement in which the system is available for further observation after the measurement. We refer to such measurements as '*non-absorptive measurements*'. However we often meet with experiments in physics in which the system is absorbed (or destroyed) in the very process of realization of a particular outcome or a set of outcomes. The standard examples is that of the measurement of polarization of a photon by a nicol prism. We shall outline a way of characterizing such '*absorptive measurements*' in §5.

3. Measurement on composite systems

The EPR set up involves a composite system $L \otimes R$ composed of two sub-systems (particles) L, R. If H_L, H_R are the Hilbert spaces associated with the two sub-systems then the Hilbert space associated with the composite system is given by the tensor product $H_{LR} = H_L \otimes H_R$.

A density operator state $\rho \in \mathscr{T}(H_{LR})$ is said to be an uncorrelated state, if there exist density operators

$$\rho_L \in \mathscr{T}(H_L), \quad \rho_R \in \mathscr{T}(H_R) \tag{12}$$

such that

$$\rho = \rho_L \otimes \rho_R.$$

Given any density operator $\rho\in\mathscr{T}(H_{LR})$ we can define the density operators $\mathrm{Tr}_R(\rho)\in\mathscr{T}(H_L)$ and $\mathrm{Tr}_L(\rho)\in\mathscr{T}(H_R)$ where the partial traces Tr_L, Tr_R are given by the equations

$$\mathrm{Tr}[\mathrm{Tr}_R(\rho)A_L]=\mathrm{Tr}[\rho(A_L\otimes I_R)] \tag{13}$$

$$\mathrm{Tr}[\mathrm{Tr}_L(\rho)A_R]=\mathrm{Tr}[\rho(I_L\otimes A_R)] \tag{14}$$

for all $A_L\in\mathscr{B}(H_L)$, $A_R\in\mathscr{B}(H_R)$ and where I_L, I_R denote the identity transformations on H_L and H_R respectively. In particular for an uncorrelated state $\rho_L\otimes\rho_R$, we have

$$\mathrm{Tr}_L(\rho_L\otimes\rho_R)=(\mathrm{Tr}\,\rho_L)\rho_R \tag{15}$$

$$\mathrm{Tr}_R(\rho_L\otimes\rho_R)=(\mathrm{Tr}\,\rho_R)\rho_L. \tag{16}$$

An important class of measurements performed on a composite system is that which can be viewed as being exclusively performed on one of the sub-systems. We now propose the following as the appropriate criterion as to when a measurement performed on a composite system can be viewed as one which is performed exclusively on one of the sub-systems.

Let $\Delta\to\mathscr{E}(\Delta)$ be an operation-valued measure on $\mathscr{T}(H_{LR})$ associated with a measurement performed on a composite system $L\otimes R$. We say that this measurement can be considered as a measurement performed (exclusively) on the sub-system L provided there exists an operation-valued measure $\Delta\to\mathscr{E}_L(\Delta)$ defined on $\mathscr{T}(H_L)$ such that, for each $\Delta\in\mathscr{B}(R)$

$$\mathscr{E}(\Delta)(\rho_L\otimes\rho_R)=\mathscr{E}_L(\Delta)(\rho_L)\otimes\rho_R \tag{17}$$

for all uncorrelated density operators $\rho_L\otimes\rho_R\in\mathscr{T}(H_{LR})$.

The condition (17) can be easily seen to be equivalent to the requirement that there exist $\{\mathscr{E}_L^*(\Delta)\}$ which are normal positive maps on $\mathscr{B}(H_L)$ such that for each $\Delta\in\mathscr{B}(R)$

$$\mathscr{E}^*(\Delta)(A_L\otimes A_R)=\mathscr{E}_L^*(\Delta)(A_L)\otimes A_R$$

for all

$$A_L\in\mathscr{B}(H_L),\quad A_R\in\mathscr{B}(H_R). \tag{18}$$

If in addition $\mathscr{E}(\Delta)$ is completely positive then according to result of Kraus alluded to earlier, there exist a countable set of operators $A_{L\alpha}\in\mathscr{B}(H_L)$ such that (17) reduces to

$$\mathscr{E}(\Delta)(\rho_L\otimes\rho_R)=\left(\sum_\alpha A_{L\alpha}\rho_L A_{L\alpha}^\dagger\right)\otimes\rho_R \tag{19}$$

or to

$$\mathscr{E}(\Delta)(\rho)=\sum_\alpha(A_{L\alpha}\otimes I_R)\rho(A_{L\alpha}^\dagger\otimes I_R) \tag{20}$$

for all $\rho\in\mathscr{T}(H_{LR})$. Equation (20) has been employed by Kraus as the criterion that the measurement can be considered to be performed (exclusive) on the sub-system L.

In any case, all that the condition (17) ensures is that if the state of a system is an uncorrelated state of the form $\rho_L\otimes\rho_R$ then in a measurement which may be considered as being performed on the sub-system L (the state will continue to be an uncorrelated state and) there will be a change only in ρ_L and ρ_R will remain unaffected. But this is only for the case of uncorrelated states. For states ρ which are correlated (as in the typical EPR example) in general $\mathrm{Tr}_R(\rho)$ as well as $\mathrm{Tr}_L(\rho)$ are both altered even though the measurement performed is such that it can be considered to be performed exclusively on one of the sub-systems. This indeed is the crucial non-local feature of the 'collapse postulate' prescribing the change in the state of a system consequent to the performance of a measurement, in quantum theory.

We are now in a position to show the following result concerning the 'partial traces' of the state of a composite system, whenever a measurement is performed on one of the sub-systems.

Theorem 1. Let $\Delta \to \mathscr{E}(\Delta)$ be an operation-valued measure on $\mathscr{T}(H_{LR})$ characterizing a non-absorptive measurement, which can be considered as a measurement performed on the sub-system L according to the criterion (17). Then, for all states $\rho \in \mathscr{T}(H_{LR})$ and for all $\Delta \in \mathscr{B}(R)$ we have the following partial trace relations

$$\mathrm{Tr}_R[\mathscr{E}(\Delta)(\rho)] = \mathscr{E}_L(\Delta)(\mathrm{Tr}_R(\rho)) \tag{21a}$$

$$\mathrm{Tr}_L[\mathscr{E}(\Delta)(\rho)] = \mathrm{Tr}_L[(F_L(\Delta) \otimes I_R)\rho] \tag{21b}$$

where

$$F_L(\Delta) = \mathscr{E}_L^*(\Delta)(I_L) \tag{22}$$

is the effect associated with the operation $\mathscr{E}_L(\Delta)$.

Proof. For any $A_R \in \mathscr{B}(H_R)$ we have from (18)

$$\begin{aligned} &\mathrm{Tr}[(I_L \otimes A_R)\mathscr{E}(\Delta)(\rho)] \\ &\quad = \mathrm{Tr}[\mathscr{E}^*(\Delta)(I_L \otimes A_R)] \\ &\quad = \mathrm{Tr}[\{\mathscr{E}_L^*(\Delta)(I_L) \otimes A_R\}]. \end{aligned} \tag{23}$$

From (22), we now obtain

$$\mathrm{Tr}[(I_L \otimes A_R)\mathscr{E}(\Delta)\rho] = \mathrm{Tr}[(I_L \otimes A_R)\{F_L(\Delta) \otimes I_R\}\rho] \tag{24}$$

for all $A_R \in \mathscr{B}(H_R)$ which straightaway leads to the partial trace relation (21b), as per the definition (14). The partial trace relation (21a) can be proved in a similar way.

The partial trace result (21a) is a generalization of a similar result proved by Kraus (1983) for completely positive operations (satisfying his condition (20)). This relation essentially states that in a measurement $\Delta \to \mathscr{E}(\Delta)$ performed on the sub-system L the reduced state of the sub-system L transforms according to the associated operation-valued measure $\Delta \to \mathscr{E}_L(\Delta)$ defined on $\mathscr{T}(H_L)$ given by (17).

The relation (21b) is indeed equally interesting in that it shows that in any measurement performed on the sub-system L the change in the reduced state of R is completely determined by the effect $F_L(\Delta)$ associated with the operation $\mathscr{E}_L(\Delta)$. Thus even if different measurements $\Delta \to \mathscr{E}_{L_1}(\Delta)$ and $\Delta \to \mathscr{E}_{L_2}(\Delta)$ are performed on the sub-system L, they induce the same changes in the reduced states of the sub-system R as long as the associated effects are identical, namely we have

$$\mathscr{E}_{L_1}^*(\Delta)(I_L) = F_{L_1}(\Delta) = F_{L_2}(\Delta) = \mathscr{E}_{L_2}^*(\Delta)(I_L). \tag{25}$$

4. The validity of local causality at the statistical level for general non-absorptive measurements

We shall now investigate the requirement of local causality at the statistical level in an EPR type set up. For the case of non-absorptive measurements characterized by operation-valued measures, we have the following result:

Theorem 2. Let $\Delta \to \mathscr{E}(\Delta)$ be an operation-valued measure on $\mathscr{T}(H_{LR})$ considered as a measurement performed on sub-system L according to the criterion (17). Then we have

$$\mathrm{Tr}_L[\mathscr{E}(R)(\rho)] = \mathrm{Tr}_L[\rho] \tag{26}$$

for all states $\rho \in \mathscr{T}(H_{LR})$.

Proof. From equation (18), which is equivalent to (17) we have

$$\mathscr{E}(\Delta)^*(I_L \otimes A_R) = \mathscr{E}_L(\Delta)^*(I_L) \otimes A_R \tag{27}$$

for all $A_R \in \mathscr{B}(H_R)$ where $\Delta \to \mathscr{E}_L(\Delta)$ is the operation-valued measure on $\mathscr{T}(H_L)$ associated with the operation-valued measure $\Delta \to \mathscr{E}(\Delta)$ as given by (17).

Further, it follows from (11) that

$$\mathscr{E}_L(R)^*(I_L) = I_L. \tag{28}$$

We therefore get

$$\mathscr{E}(R)^*(I_L \otimes A_R) = I_L \otimes A_R. \tag{29}$$

Taking the trace with ρ on both sides we get

$$\mathrm{Tr}[\{\mathscr{E}(R)(\rho)\}(I_L \otimes A_R)] = \mathrm{Tr}[\rho(I_L \otimes A_R)] \tag{30}$$

for all $A_R \in \mathscr{B}(H_R)$. Thus we arrive at the equation

$$\mathrm{Tr}_L[\mathscr{E}(R)(\rho)] = \mathrm{Tr}_L(\rho)$$

for all $\rho \in \mathscr{T}(H_{LR})$.

Equation (26) is the important result that the reduced density operator of the entire ensemble of sub-systems R is totally independent of whether or not a measurement characterized by the operation-valued measure $\Delta \to \mathscr{E}(\Delta)$ is performed on the ensemble of sub-systems L. Thus we have demonstrated the validity of local causality at the statistical level for the case of general non-absorptive measurements characterized by operation-valued measures. Our result is indeed more general than earlier proofs of local causality at the statistical level given by Kraus (1983) and Ghirardi *et al* (1988)* which assume that the measurement transformations are completely positive (and utilize eqs (12), (19)). Our demonstration also shows that the crucial inputs in proving

* A much earlier version of this result was the proof of local causality at the statistical level for the case of 'ideal' measurements, given by Ghirardi *et al* (1980).

the validity of the local causality at the statistical level are the total probability condition (7) and the condition (17) which specifies when a measurement performed on the composite system can be considered as one being performed on one of its sub-systems. It is of course assumed that the entire ensemble of sub-systems on either side is available for further measurements.

5. The requirement of local causality at the statistical level for absorptive measurements

Theorem 2 of §4 demonstrates that in an EPR type set up local causality at the statistical level is always satisfied whenever the measurement performed on the system is characterized by an operation-valued measure defined on the Hilbert space of the (composite) system. In the light of recent discussions on the validity of local causality at the statistical level in quantum theory it may be of interest to consider situations where the measurement performed on the system is not necessarily characterized by an operation-valued measure. One such situation appears to be that of an absorptive measurement, in which the system is absorbed or destroyed in the very process of realization of a particular outcome or a set of outcomes. Such measurements are commonly met with in physics and the standard example is that of measurement of the polarization of a photon by a nicol prism.

Since there does not appear to be a general theory of absorptive measurements, we shall present a simple mathematical characterization of such measurements. First let us consider a single system on which an absorptive measurement with a finite set of outcomes $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is performed and where we assume that λ_0 is the outcome corresponding to which the system gets absorbed. Clearly such a measurement is characterized by a set of operations $\{\mathscr{E}_0, \mathscr{E}_1, \ldots, \mathscr{E}_n\}$ which are linear positive norm-nonincreasing transformations of $\mathscr{T}(H)$, where H is the Hilbert space of the system, such that

$$\sum_{i=1}^{n} \mathrm{Tr}[\mathscr{E}_i(\rho)] \leqslant 1 \tag{31}$$

for all density operators ρ. If the outcome λ_i is realized then the change in the state of the system is given by

$$\rho \to \mathscr{E}_i(\rho)/\mathrm{Tr}[\mathscr{E}_i(\rho)]. \tag{32}$$

When the outcome λ_0 is realized, the system is absorbed and there is no post-measurement density operator. The probability that the system is absorbed $p_0(\rho)$ is given by

$$p_0(\rho) = 1 - \sum_{i=1}^{n} \mathrm{Tr}[\mathscr{E}_i(\rho)] \tag{33}$$

and the condition (31) ensures that (33) is indeed a probability.

We now proceed to consider absorptive measurements performed on a composite system in EPR type set up. The interesting case is when the absorptive measurement can be considered as a measurement performed on one of the sub-systems and where the sub-system gets absorbed in the very process of realization of one of the outcomes. We are clearly led to the following definition:

An absorptive measurement which can be considered to be performed on the sub-system L of the composite system $L\otimes R$ with outcomes $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is characterized by a set of operations $\{\mathscr{E}_0, \mathscr{E}_1, \ldots, \mathscr{E}_n\}$ where (i) $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_n$ are operations on $\mathscr{T}(H_{LR})$ such that there are operations $\mathscr{E}_{iL}(i = 1, \ldots, n)$ on $\mathscr{T}(H_L)$ such that

$$\mathscr{E}_i(\rho_L \otimes \rho_R) = \mathscr{E}_{iL}(\rho_L) \otimes \rho_R \tag{34}$$

for all $i = 1, 2, \ldots, n$ and for all $\rho_L \in \mathscr{T}(H_L)$, $\rho_R \in \mathscr{T}(H_R)$ (ii) $\mathscr{E}_0$ which corresponds to the absorptive outcome is an operation or a linear positive, norm-nonincreasing map

$$\mathscr{E}_0 : \mathscr{T}(H_{LR}) \to \mathscr{T}(H_R). \tag{35}$$

(iii) $$\sum_{i=1}^{n} \mathrm{Tr}[\mathscr{E}_i(\rho)] + \mathrm{Tr}[\mathscr{E}_0(\rho)] = 1 \tag{36}$$

for all density operators ρ of the composite system $L\otimes R$. We may note that (i) ensures that the measurement may be considered to be performed on the sub-system L and is the same as the condition (17) for non-absorptive measurements. The condition (ii) reflects the situation that when the absorptive outcome λ_0 is realized, the sub-system L gets absorbed in the process and only the sub-system R survives. The condition (iii) ensures that the total probability is 1. As before, the probability that the sub-system L is absorbed is given by

$$p_0(\rho) = \mathrm{Tr}[\mathscr{E}_0(\rho)] = 1 - \sum_{i=1}^{n} \mathrm{Tr}[\mathscr{E}_i(\rho)]. \tag{37}$$

Because of the fact that the measurement transformation $\mathscr{E}_0$ given by (35) which is associated with the absorptive outcome is not an operation on $\mathscr{T}(H_{LR})$ (in the usual sense that it maps elements of $\mathscr{T}(H_{LR})$ into elements $\mathscr{T}(H_{LR})$), an absorptive measurement performed on the composite system $L\otimes R$ is not characterized by an operation-valued measure on $\mathscr{T}(H_{LR})$. Therefore theorem 2 of §4 which demonstrates the validity of local causality at the statistical level for measurements characterized by operation-valued measures, is not applicable for absorptive measurements as defined above.

In this context, it should be remarked that, when one takes recourse to the formalism of quantum field theory, it is possible to characterize absorptive measurements also as operation-valued measures. For, if one takes the Hilbert spaces H_L and H_R as the appropriate Fock spaces, then the measurement transformation $\mathscr{E}_0$ (associated with the outcome λ_0 when the sub-system L gets absorbed) may be characterized by an operation on $\mathscr{T}(H_{LR})$ of the form

$$\mathscr{E}_0(\rho_L \otimes \rho_R) = \mathscr{E}_{0L}(\rho_L) \otimes \rho_R \tag{38}$$

for all uncorrelated states of the form $\rho_L \otimes \rho_R$, where $\mathscr{E}_{0L}(\rho_L)$ is taken to be the density operator, modulo normalization, of the vacuum state in H_L. Now the set of operations $\{\mathscr{E}_0, \mathscr{E}_1, \ldots, \mathscr{E}_n\}$ indeed define an operation-valued measure on $\mathscr{T}(H_{LR})$. When such is the case, the conditions of theorem 2 become applicable and local causality at the statistical level is automatically guaranteed.

But most of the discussions on local causality in an EPR set up are usually carried out without taking recourse to the formalism of quantum field theory. The kind of description of absorptive measurements that has been presented above is also in the spirit of such discussions. We now proceed to investigate the requirement of local causality at the statistical level, when an absorptive measurement as characterized by the above conditions (34), (35), (36) is performed on the subsystem L of a composite system $L \otimes R$. First, let us consider the (reduced) density operator state of the sub-system R after the absorptive measurement has been performed. Clearly when any of the non-absorptive outcomes $\lambda_i (i = 1, 2, \ldots, n)$ are realized, the post-measurement unnormalized state of the ensemble of subsystems R would be $\mathrm{Tr}_L[\mathscr{E}_i(\rho)]$ where ρ is the initial state of the composite system. According to the condition (35), when the absorptive out come λ_0 is realized, the post-measurement unnormalized state of the ensemble sub-systems R is just $\mathscr{E}_0(\rho)$. Since these unnormalized states are already multiplied by the associated probabilities as weight factors, the post-measurement state of the entire ensemble of sub-systems R is given by the density operator

$$\mathscr{E}_0(\rho) + \sum_{i=1}^{n} \mathrm{Tr}_L[\mathscr{E}_i(\rho)].$$

Now the condition of local causality at the Statistical level is precisely that the above density operator characterizing the entire ensemble of sub-systems R after an absorptive measurement is performed on L should be the same as the reduced density operator characterizing the state of the entire ensemble of sub-systems R prior to any measurement. Thus the local causality condition at the statistical level for absorptive measurements is given by the equation

$$\mathscr{E}_0(\rho) + \sum_{i=1}^{n} \mathrm{Tr}_L[\mathscr{E}_i(\rho)] = \mathrm{Tr}_L[\rho] \tag{39}$$

for all $\rho \in \mathscr{T}(H_{LR})$.

We now have the following result:

Theorem 3. An absorptive measurement with outcomes $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ (where λ_0 is the absorptive outcome) and associated measurement transformations $\{\mathscr{E}_0, \mathscr{E}_1, \ldots, \mathscr{E}_n\}$ which is performed on the sub-system L of a composite system and characterized by conditions (34)–(36), satisfies the requirement (39) of local causality at the statistical level, if and only if any of the following equivalent conditions I--III is satisfied.

(I) $$\mathscr{E}_0(\rho_L \otimes \rho_R) = p_0(\rho_L)\rho_R \tag{40}$$

for all uncorrelated states of the form $\rho_L \otimes \rho_R$ where $p_0(\rho_L)$ is a linear functional on $\mathscr{T}(H_L)$

(II) $$\mathscr{E}_0(\rho) = \mathrm{Tr}_L[(F_{0L} \otimes I_R)\rho] \tag{41}$$

for all $\rho \in \mathscr{T}(H_{LR})$, where

$$F_{0L} = I_L - \sum_{i=1}^{n} \mathscr{E}^*_{iL}(I_L) \tag{42}$$

where $\mathscr{E}_{iL}$ are as given by (34).

(III) There exists an operation-valued measure $\{\tilde{\mathscr{E}}_0, \mathscr{E}_1, \ldots, \mathscr{E}_n\}$ on $\mathscr{T}(H_{LR})$ characterizing a non-absorptive measurement performed on the sub-system L of the composite system, such that

$$\mathscr{E}_0(\rho) = \mathrm{Tr}_L[\tilde{\mathscr{E}}_0(\rho)] \tag{43}$$

for all $\rho \in \mathscr{T}(H_{LR})$.

Proof. We shall first show that condition II is equivalent to the statistical local causality condition (39) once (34)–(36) are satisfied. From (42) it follows that for arbitrary $\rho \in \mathscr{T}(H_{LR})$, $A_R \in \mathscr{B}(H_R)$

$$\begin{aligned}
&\mathrm{Tr}[\{\mathrm{Tr}_L(\rho(F_{0L} \otimes I_R))\} A_R] \\
&\quad = \mathrm{Tr}[\rho(F_{0L} \otimes I_R)(I_L \otimes A_R)] \\
&\quad = \mathrm{Tr}[\rho(F_{0L} \otimes A_R)] \\
&\quad = \mathrm{Tr}\left[\rho\left(\left\{I_L - \sum_{i=1}^{n} \mathscr{E}^{*}_{iL}(I_L)\right\} \otimes A_R\right)\right].
\end{aligned} \tag{44}$$

If we now employ (18), we get,

$$\begin{aligned}
&\mathrm{Tr}[\{\mathrm{Tr}_L(\rho(F_{0L} \otimes I_R))\} A_R] \\
&\quad = \mathrm{Tr}[\rho(I_L \otimes A_R)] - \mathrm{Tr}\left[\rho \sum_{i=1}^{n} \mathscr{E}^{*}_{i}(I_L \otimes A_R)\right] \\
&\quad = \mathrm{Tr}[\rho(I_L \otimes A_R)] - \mathrm{Tr}\left[\left(\sum_{i=1}^{n} \mathscr{E}_i(\rho)\right)(I_L \otimes A_R)\right] \\
&\quad = \mathrm{Tr}[\mathrm{Tr}_L(\rho) A_R] - \mathrm{Tr}\left[\mathrm{Tr}_L\left(\sum_{i=1}^{n} \mathscr{E}_i(\rho)\right) A_R\right].
\end{aligned} \tag{45}$$

Since (45) holds for all $A_R \in \mathscr{B}(H_R)$, it follows that the statistical local causality condition (39) is equivalent to

$$\mathrm{Tr}_L[(F_{0L} \otimes I_R)\rho] = \mathrm{Tr}_L(\rho) - \mathrm{Tr}_L\left(\sum_{i=1}^{n} \mathscr{E}_i(\rho)\right) = \mathscr{E}_0(\rho) \tag{46}$$

which is nothing but the condition II of theorem 3.

We now show that condition II is equivalent to condition I. For this purpose we need to note that condition II is equivalent to the condition

$$\mathscr{E}_0(\rho_L \otimes \rho_R) = \mathrm{Tr}_L[(F_{0L} \otimes I_R)(\rho_L \otimes \rho_R)] = \mathrm{Tr}[F_{0L}\rho_L]\rho_R \tag{47}$$

for all uncorrelated states $\rho_L \otimes \rho_R$. If we now set

$$p_0(\rho_L) = \mathrm{Tr}[F_{0L}\rho_L] \tag{48}$$

then we see straightaway that condition II is equivalent to the condition I.

To show that condition II is equivalent to condition III, let us consider an arbitrary non-absorptive measurement on the sub-system L characterized by an operation-valued measure $\{\tilde{\mathscr{E}}_0, \mathscr{E}_1, \ldots, \mathscr{E}_n\}$. We then have (the analogue of (4))

$$\mathrm{Tr}[\tilde{\mathscr{E}}_0(\rho)] + \sum_{i=1}^{n} \mathrm{Tr}[\mathscr{E}_i(\rho)] = \mathrm{Tr}\,\rho \tag{49}$$

and for a measurement performed on the sub-system L, we have from (21b) of theorem 1 that

$$\mathrm{Tr}_L[\tilde{\mathscr{E}}_0(\rho)] = \mathrm{Tr}_L[(\tilde{\mathscr{E}}^*_{0L}(I_L)\otimes I_R)\rho]. \tag{50}$$

From (49) and (50) it follows that

$$\mathrm{Tr}_L[\tilde{\mathscr{E}}_0(\rho)] = \mathrm{Tr}_L\left[\left\{\left(I_L - \sum_{i=1}^{n}\mathscr{E}^*_{iL}(I_L)\right)\otimes I_R\right\}\rho\right]. \tag{51}$$

From (42) it then follows that

$$\mathrm{Tr}_L[\tilde{\mathscr{E}}_0(\rho)] = \mathrm{Tr}_L[(F_{0L}\otimes I_R)\rho] = \mathscr{E}_0(\rho) \tag{52}$$

thereby establishing the equivalence of conditions II and III. We have thus established the above theorem 3.

What is the significance of the conditions (I)–(III), each of which ensure that the absorptive measurement satisfies the requirement of local causality at the statistical level? Condition I indeed seems reasonable and merely expresses the requirement that in a measurement performed on sub-system L the measurement transformation associated with the absorptive outcome also leaves the state ρ_R of the sub-system R unaffected whenever the initial state happens to be an uncorrelated state of the form $\rho_L\otimes\rho_R$. In fact it would be entirely justified to include the condition I (or equation (40)) as an integral part of the very definition of, or what characterizes, an absorptive measurement performed on sub-system L, in addition to the conditions (34)–(36).

Condition II of theorem 3 reveals another interesting aspect of absorptive measurements. When local causality at the statistical level is satisfied, the measurement transformation $\mathscr{E}_0$ associated with the absorptive outcome is completely determined via equations (41), (42) in terms of the measurement transformations $\{\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_n\}$ associated with non-absorptive outcomes. This is in marked contrast with the case of non-absorptive measurements characterized by operation-valued measures, as it is well known that given any set of operations $\{\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_n\}$ satisfying

$$\sum_{i=1}^{n}\mathrm{Tr}[\mathscr{E}_i(\rho)] < \mathrm{Tr}\,\rho$$

one can find innumerable operations $\tilde{\mathscr{E}}_0$ such that $\{\tilde{\mathscr{E}}_0, \mathscr{E}_1, \ldots, \mathscr{E}_n\}$ is an operation-valued measure. Finally condition III expresses yet another aspect of statistically locally-causal absorptive measurements that they can be obtained in some sense as minimal modifications of non-absorptive measurements. Further insight into these conditions is necessary, to examine in more detail, to what extent they are generally valid.

The important question which needs to be investigated in the light of the above discussion is whether the various theoretical models of absorptive measurements, as realized in practice in typical EPR-type situations, do indeed satisfy the condition (I)–(III) of theorem 3 for local causality at the statistical level. For instance, in the example discussed by Datta *et al* (1987, 1988), what are indeed involved are absorptive measurements pertaining to decay products. It would be interesting to explore the nature of the absorptive measurements in such examples, especially with a view to testing the validity of local causality at the statistical level.

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References

Bohm D 1951 *Quantum theory* (New Jersey: Prentice Hall) p. 614
Clifton R K and Redhead M L C 1988 *Phys. Lett.* **A126** 295
Corbett J V 1988 *Phys. Lett.* **A130** 419
Datta A, Home D and Ray Chaudhri A 1987 *Phys. Lett.* **A123** 4
Datta A, Home D and Ray Chaudhri A 1988 *Phys. Lett.* **A130** 187
Davies E B 1976 *Quantum theory of open systems* (New York: Academic Press)
Eberhard P H and Ross P R 1989 *Found. Phys. Lett.* **12** 127
Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
Finkelstein J and Stapp H P 1987 *Phys. Lett.* **A126** 159
Ghirardi G C, Rimini A and Weber T 1980 *Lett. Nuovo Cimento.* **27** 293
Ghirardi G C, Grassi R, Rimini A and Weber T 1988 *Europhys. Lett.* **6** 95
Hall M J W 1987 *Phys. Lett.* **A125** 89
Home D 1989 in *The Concept of probability* (eds) E T Bitsakis and C A Nicolaides (Dordrecht: Kluwer)
Home D 1990 in *Proceedings of III. International Symposium on foundations of quantum mechanics* (Tokyo: Physical Soc. of Japan) (in Press)
Kraus K 1983 *States, effects and operations* (Berlin: Springer)
Selleri F (ed.) 1988 *Quantum mechanics versus local realism: The Einstein-Podolsky-Rosen problem* (NewYork: Plenum)
Squires E J and Siegward D 1987 *Phys. Lett.* **A130** 192
Squires E J 1988 *Phys. Lett.* **A130** 192