Pramāna - J. Phys, Vol. 34, No. 5, May 1990, pp. 403–414. © Printed in India.

Wave function with second order correction and inflationary solutions in quantum cosmology

SUBENOY CHAKRABORTY
Department of Mathematics, Jadavpur University, Calcutta 700032, India

MS received 3 August 1989

Abstract. A brief account for the higher order wave function in Hartle-Hawking (H–H) proposal is given which is compared with the tunneling wave function due to Vilenkin. The probability distributions are determined for both types of wave functions Also a class of solutions are evaluated using H–H approach for Kantowski–Sachs metric with a scalar field and inflation is observed

Keywords. Path-integral; tunneling; wave function; inflation; saddle point.

PACS Nos **04·20; 98·90**

1. Introduction

In recent years exciting developments have taken place in quantum cosmology. To know the early stages of the evolution of the universe (where quantum effects are important) two appealing approaches have been suggested. In the first proposal, Vilenkin (1988) has considered quantum tunneling from nothing, i.e. creation of the universe from nothing to describe the spontaneous nucleation of the universe and then evolves in an inflationary manner using the idea of Linde (1984). The second method has been developed by Hartle and Hawking (1983), combining ideas of canonical and path-integral quantization of general relativity. In this proposal, the wave function of the universe is given by a path integral over compact Euclidean geometries. They showed that for conformally non-invariant field, the desirable cosmological phenomenon of inflation can occur at early stages of the evolution without any phase transition. A comparison between these two approaches has been studied by Vilenkin (1988). The tunneling boundary conditions can be written as "at singular boundaries of superspace, includes only outgoing modes". On the other hand, in the H–H approach the boundary condition is that it has no boundary.

In this paper I consider the H–H approach with higher order correction to the wave function, by the saddle point method (Morse and Feshback 1953) and discuss the cosmological predictions. The tunneling wave functions are also evaluated using the approach of Vilenkin by analytic continuation and the probability distributions are calculated for both types of wave functions. An exhaustive set of solutions for Kantowski–Sachs model are evaluated using H–H proposal for a scalar potential in the field $\frac{1}{2}\lambda\phi^n$. The paper is organised as follows:

Section 2 reviews the basic formalism of H–H proposal and compared with tunneling approach. A collection of solutions for both classical and classically forbidden region are given in §3. The conclusions are given in §4.

2. Wave function with second order correction in H–H approach and comparison with tunneling wave function

The wave function of the universe is defined by the Euclidean path integral (Fang and Ruffinm 1987).

$$\psi_{HH}(h_{ij}, \phi) = \int d[g_{\mu\nu}]\, d[\phi] \exp[-I_E(g_{\mu\nu}, \phi)]. \tag{1}$$

The integral is taken over all compact-Euclidean 4-geometries $g_{\mu\nu}$ and regular matter fields ϕ. The three metric h_{ij} and ϕ, appearing in the argument of ψ_{HH} are boundary condition of the Wheeler–De-Witt equation

$$\hat{H}_{\perp}\psi = 0. \tag{2}$$

It also satisfies the supermomentum constraints

$$\hat{H}_i\psi = 0. \tag{3}$$

As the path integral (1) has no precise definition so its qualitative behaviour can be obtained using semi-classical techniques in simple cosmological models. Equivalently the Wheeler–De-Witt equation is a second order hyperbolic differential equation on superspace. To solve it one restricts to minisuperspace. So the boundary conditions in the semi classical approximation to the wave function is in the form (Fang and Ruffinm 1987).

$$\psi[h_{ij}, \phi] = N_0 \sum A_i \exp(-B_i), \tag{4}$$

where N_0 is a normalization constant and B_i is the action of the classical solution with arguments of ψ on the boundary. The prefactors A_i denote fluctuations about classical solutions.

Hartle and Hawking has started with four geometrics to be spatially homogeneous, isotropic and closed and is given in a useful co-ordinate system by

$$ds^2 = \sigma^2[-N^2(t)\cdot dt^2 + a^2(t)\cdot d\Omega_3^2], \tag{5}$$

where $d\Omega_3^2$ is the metric on the unit three sphere S^3. The overall prefactor is taken to be

$$\sigma^2 = 2G/3\pi,$$

where G is Newton's constant ($\hbar = c = 1$), $a(t)$ is the cosmic scale factor. For matter degrees of freedom they consider a single conformally invariant scalar field $\phi = \phi(t)$.

The Lorentzian action with positive cosmological constant Λ is given by

$$S = \tfrac{1}{2}\int Na^3\left(-\frac{\dot{a}^2}{N^2a^2} + \frac{1}{a^2} + \lambda + \frac{\dot{\chi}^2}{N^2a^2} - \frac{\chi^2}{a^4}\right) dt, \tag{6}$$

where

$$\chi = (2\pi^2\sigma^2)^{1/2} a\phi, \quad \text{and } \lambda = \tfrac{1}{3}\sigma^2\Lambda.$$

The Wheeler–De-Witt equation is

$$\frac{1}{2}\left[\frac{1}{a^p}\frac{\partial}{\partial a}\left(a^p\cdot\frac{\partial}{\partial a}\right) - a^2 + \lambda a^4 - \frac{\partial^2}{\partial\chi^2} + \chi^2\right]\psi(a, \chi) = 0. \tag{7}$$

The index p represents some (not all) of the factor ordering ambiguity in the above equation. Its value will not make great difference in the nature of the solution. Then according to H–H proposal the wave function is given by the path integral

$$\psi_0(a_0, \chi_0) = \int \delta a \delta\chi \exp(-I_E(a, \chi)), \tag{8}$$

over compact 4-geometrics and matter field configuration $\chi(\tau)$, which has the value χ_0 on the boundary. The compact 4-geometry is given by

$$ds^2 = \sigma^2[d\tau^2 + a^2(\tau)\, d\Omega_3^2], \tag{9}$$

for which $a(\tau)$ matches a_0 on the given boundary. The Euclidean action I_E is given by

$$I_E = \tfrac{1}{2}\int d\eta \left[-\left(\frac{da}{d\eta}\right)^2 + a^2 + \lambda a^4 + \left(\frac{d\chi}{d\eta}\right)^2 + \chi^2\right], \tag{10}$$

with

$$\eta = \int \frac{d\tau}{a}.$$

In the K-representation (Hartle *et al* (1983)) the wave function is given by

$$\phi_0(k_0, \chi_0) = \int \delta a \delta\chi \exp(-I^k(a, \chi)). \tag{11}$$

The sum is over the same class of 4-geometry and matter field but on the boundary $k = k_0$ is prescribed instead of $a = a_0$. Here $k = \sigma K/9$ is the simplifying measure of K (extrinsic curvature) and is related to a (τ) by the relation

$$k = \frac{1}{3a}\frac{da}{d\tau}. \tag{12}$$

The action I^k is related to I_E as

$$I^k = K_0 a_0^3 + I_E. \tag{13}$$

So the ground state wave function may be recovered by inverse Laplace transform as

$$\psi_0(a_0, \chi_0) = -\frac{1}{2\pi^2}\int_c dk \exp(k a_0^3)\phi_0(k, \chi_0), \tag{14}$$

where the contour runs from $-i\infty$ to $+i\infty$ to the right of any singularity of $\phi_0(k, \chi_0)$. Now separating the gravitational and field part in the action (10) the wave function $\psi_0(a)$ for pure gravitational part can be written as

$$\psi_0(a) = -\frac{N}{2\pi i}\int_c dK \exp[Ka^3 - I_E^k(k)], \tag{15}$$

with

$$I_E^k(k) = -\frac{1}{3H^2}\left(1 - \frac{3K}{(9K^2 + H^2)^{1/2}}\right) \text{ and } H^2 = \lambda. \tag{16}$$

The contour integral in (15) is now evaluated by the saddle point method retaining terms up to 2nd order (which indicates fluctuation about classical solution).

For $aH < 1$, the dominant contribution to the integral (15) comes from the stationary phase point

$$K = \frac{H}{3}\tan\alpha, \quad \text{with } \cos\alpha = aH.$$

(There are also complex extrema but all have real part less than $(H/3)\tan\alpha$).

So by distorting the contour (as H–H) into the steepest-descent path $u = (H/3)\tan\alpha$ (near the saddle point) in the k-plane ($k = u + iv$) (see figure 1(a)), we obtain the ground state wave function in the classically forbidden region ($aH < 1$) as

$$\psi_{HH}(a) = \frac{N}{3\sqrt{2\pi a^2}(1 - a^2H^2)^{1/4}}\exp[\{1 - (1 - a^2H^2)^{3/2}\}/3H^2]. \tag{17}$$

Moreover, at the tunneling point ($aH = 1$) we have

$$\psi_{HH}(a) = \frac{N}{\sqrt{2\pi}}\exp(-1/3H^2), \tag{18}$$

where $k = 0$ is the saddle point with dominant real part. In this case the whole of $u = 0$ can be taken as the path of integration (see figure 1(b)). Lastly, in the classical region i.e. for $aH > 1$ there are no real extrema. This is expected as it is not possible to bound a 4-sphere of radius $1/H$ by a 3-sphere of radius a. Hence we obtain two complex saddle points with larger real part and are given by

$$k = \pm(ih/3)\tanh\alpha, \text{ where } \cosh\alpha = aH.$$

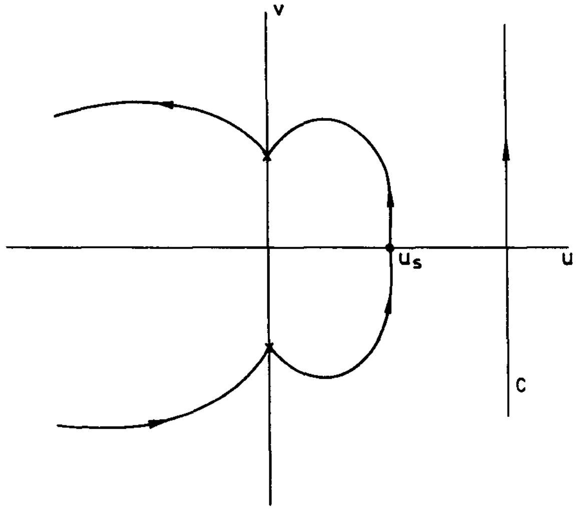

Figure 1(a). The steepest descent contour for the ground-state wave function of FRW model in $Ha < 1$ Near the saddle point $k = (H/3)\tan\alpha$ on the real axis of k-plane, the path is the straight line $u = (H/3)\tan\alpha$ The contour also passes through the branch points $k = \pm i$, located by crosses

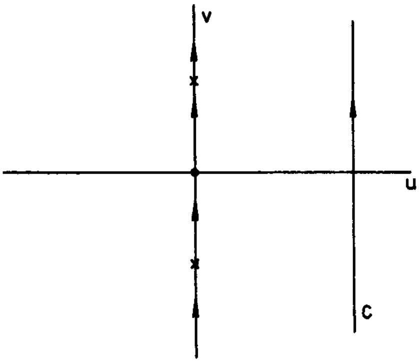

Figure 1(b). For $aH=1$, the whole of v-axis is the path of the saddle point method with $k=0$ as the saddle point.

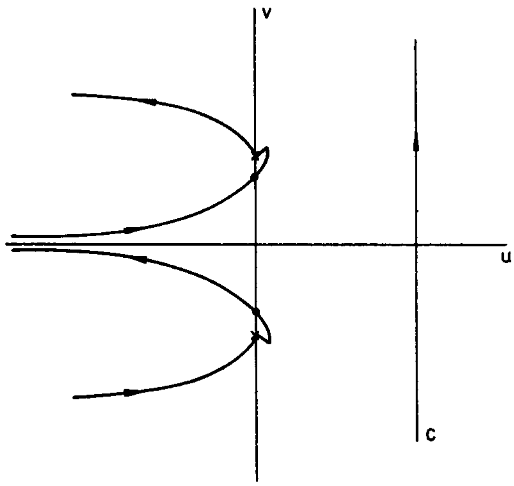

Figure 1(c). There are two complex extrema at $k=\pm i(H/3)\tanh\alpha$ for $aH>1$. The paths near the saddle points are the straight lines $v\mp(H/3)\tanh\alpha=u$ making 45° with real axis and also passes through the branch points $\pm i$.

The deformed contours near the saddle points are given by

$v\mp(H/3)\tanh\alpha=u$, as shown in figure 1(c).

So the wave function using these steepest-descent path can be obtained from (15) as

$$\psi_{HH}(a)=\frac{2N}{3(2\pi)^{1/2}a^2(a^2H^2-1)^{1/4}}\cos\left[\frac{(a^2H^2-1)^{3/2}}{3H^2}-\frac{\pi}{4}\right]. \qquad (19)$$

Now we shall evaluate the tunneling wave functions using the analytic continuation technique employed by Vilenkin (1988). According to him the tunneling wave function ψ_T obtained from ψ_{HH} by the relation

$$\psi_T=\psi_{HH}(H^2\to\exp(i\pi)H^2,\ a\to\exp(-i\pi/2)a).$$

Thus we obtained,

$$\psi_T = \frac{N\exp(i\pi)}{3(2\pi)^{1/2}a^2(1-a^2H^2)^{1/4}}\exp\left[\frac{(1-a^2H^2)^{3/2}-1}{3H^2}\right], \quad \text{for } aH<1, \tag{20}$$

$$= \frac{N}{(2\pi)^{1/2}}\exp(1/3H^2), \quad \text{for } aH=1, \tag{21}$$

$$= \frac{N\exp(-3i\pi/4)}{3(2\pi)^{1/2}a^2(a^2H^2-1)^{1/4}}\exp\left[-\frac{1+i(a^2H^2-1)^{3/2}}{3H^2}\right], \quad \text{for } aH>1. \tag{22}$$

An estimate of factor ordering is obtained by comparison of wave function in the region $a \gg H^{-1}$. According to Hawking (1983) the prefactor is

$$A \simeq a^{-(1+p/2)}, \quad \text{for } a \gg H^{-1}. \tag{23}$$

But from (19) the prefactor is of the order of $a^{-5/2}$ for $a \gg H^{-1}$. So we can take p to be 3. Hence a partial information about the factor ordering ambiguity in Wheeler–De-Witt equation is obtained.

Moreover it is evident from (17) that the wave function diverges in the limit of small three volume, which is a direct contradiction to the Hartle-Hawking boundary condition. A zeta function regularization (Hawking 1977) scheme has been employed by Schleich (1985) and Louko (1988) separately to determine the prefactor in higher order correction by adopting a scale invariant measure. However, none of them have given any explicit wave function form.

An interesting observation is that instead of conformally invariant scalar field with a positive cosmological constant if we consider a homogeneous and isotropic scalar field $\tilde{\phi}$ in a potential field $V(\tilde{\phi})$ the identical wave functions will be obtained except here

$$H^2 = V(\phi) = (2\pi\sigma^2)^2\tilde{V},$$

and

$$\phi = (4\pi\sigma/3)^{1/2}\tilde{\phi}.$$

Finally we determine the probability distribution for the initial states of the universe. The current vector

$$\mathbf{J} = \frac{i}{2}a^p(\psi^*\nabla\psi - \psi\nabla\psi^*),$$

has two components

$$J^a = \frac{i}{2}a^p(\psi^*\partial_a\psi - \psi\partial_a\psi^*),$$

$$J^\phi = -\frac{i}{2}a^{p-2}(\psi^*\partial_\phi\psi - \psi\partial_\phi\psi^*),$$

and the continuity equation

$$\operatorname{div}\mathbf{J} = 0,$$

becomes $\partial_a J_a + \partial_\phi J_\phi = 0$, for the minisuperspace model given by (5). So J^a is the probability density for ϕ at a particular value of the scale factor a. The conservation of probability is guaranteed by the relation

$$\partial_a \int J^a \, d\phi = 0.$$

So with our estimated value of p (i.e. $p = 3$) the probability density $\rho(a, \phi)$ (i.e. $\rho(a, \phi)\, d\phi$ is the probability for the scalar field to be between ϕ and $\phi + d\phi$ and scale factor be a) is given by

$$\rho(a, \phi) = N_T \exp\left[-\frac{2}{3H^2} \right], \text{ for tunneling wave function,}$$

$$= N_H \exp\left[+\frac{2}{3H^2} \right] \text{ for H–H wave function.}$$

where the normalization constants have the following expressions

$$N_T = \left\{ \int_{H>0} d\phi \exp[-2/3H^2] \right\}^{-1},$$

$$N_H = \left\{ \int_{H>0} d\phi \exp[2/3H^2] \right\}^{-1}$$

These results for probability distribution are identical to those obtained by Vilenkin (1988). So all the conclusions regarding convergence of the integrals and inflation in H–H wave function obtained by Vilenkin will also hold here.

3. Inflationary solutions in Kantowski-Sachs model

The 4-metric in Kantowski-Sachs model is

$$ds^2 = -N^2(t)\, dt^2 + a^2(t)\, dr^2 + b^2(t)\, d\Omega^2, \tag{24}$$

where $d\Omega^2$ is the metric on the unit 2-sphere and r is identified periodically.

In this section, I shall give a series of solutions for Kantowski–Sachs metric using H–H approach. The results are similar to the solutions for FRW-model by Esposito *et al* (1988). If we consider a scalar field $\phi'(t)$ having a potential of the form $V(\phi)\ \frac{1}{2}\lambda'(\phi')^n$, then the action is

$$I = -\frac{1}{2} \int N a b^2 \left(\frac{2}{N^2} \frac{\dot{a}}{a} \frac{\dot{b}}{b} + \frac{\dot{b}^2}{N^2 b^2} - \frac{1}{b^2} - \frac{\dot{\phi}^2}{N^2} + \lambda \phi^n \right) dt$$

$$= \int L \, dt, \tag{25}$$

where $\phi(t) = \phi'(t)\sqrt{2}\pi\sigma$, is dimensionless and

$$\lambda = \lambda' \sigma^{-n+4} (\pi\sqrt{2})^{-n+2}.$$

The Lorentzian field equations are

$$2\frac{\ddot{b}}{b}+\frac{\dot{b}^2}{b^2}+\frac{1}{b^2}+\dot{\phi}^2-\lambda\phi^n=0, \tag{26}$$

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\dot{a}\dot{b}}{ab}+\dot{\phi}^2-\lambda\phi^n=0, \tag{27}$$

and

$$\ddot{\phi}+\left(\frac{\dot{a}}{a}+\frac{2\dot{b}}{b}\right)\dot{\phi}+\lambda\cdot\phi^n=0, \tag{28}$$

which is the equation of continuity. The Wheeler-De-Witt equation is

$$\left[2\frac{\partial^2}{\partial\alpha\partial\beta}-\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\phi^2}+(\lambda\phi^n\exp(2(\alpha+2\beta)))\right.$$
$$\left.-\exp(2(\alpha+\beta))\right]\psi(\alpha,\beta,\phi)=0, \tag{29}$$

where $a=e^\alpha$, $b=e^\beta$ and the operator ordering for P_a^2 is taken to be $+1$.

According to Hartle-Hawking proposal there should be an effective cosmological constant at early stages of the universe leading to inflation and also the parameter (cosmological constant) has to decay and finally vanishes with the evolution of the universe as our universe is not expanding exponentially today. In the present case the scalar field ϕ is initially very large and almost constant. So $\phi\simeq$ constant $=\phi_0$ at very early stages of the evolution. Then the potential term can be viewed as an effective cosmological constant. This region is known as Euclidean or classically forbidden. The Euclidean field equations are

$$2\frac{\ddot{b}}{b}+\frac{\dot{b}^2}{b^2}+\dot{\phi}^2-\frac{1}{b^2}+\lambda\phi^n=0, \tag{30}$$

$$\frac{\ddot{b}}{b}+\frac{\ddot{a}}{a}+\frac{\dot{a}\dot{b}}{ab}+\dot{\phi}^2+\lambda\phi^n=0, \tag{31}$$

and the constraint equation is

$$2\frac{\dot{a}\dot{b}}{ab}+\frac{\dot{b}^2}{b^2}-\frac{1}{b^2}-\dot{\phi}^2+\lambda^2\phi^n=0.$$

$$\text{(the dot}=\mathrm{d}/\mathrm{d}\tau). \tag{32}$$

The initial conditions for the solution of (30)–(32) is implied by H–H proposal. According to this proposal the ground state is a Euclidean functional integral taken over compact 4-metrics and regular matter fields. Hence the semi classical approximation for the wave function reduces to

$$\psi=C\cdot\exp(I_E), \tag{33}$$

with

$$I_E=-ab(1-\tfrac{1}{3}H^2b^2)^{1/2}, \tag{34}$$

and

$$H = (\lambda \cdot \phi_0^n)^{1/2}. \tag{35}$$

So the initial conditions are

$$\begin{aligned} &b(0) = 0, \quad \phi(0) = \phi_0, \\ &\dot{b}(0) = 1, \quad \dot{\phi}(0) = 0. \end{aligned} \tag{36}$$

(the argument '0' denote $\tau = 0$). So from (30) neglecting $\dot{\phi}$ term and substituting $H^2 = \lambda\phi^n$ we obtain the solution

$$b(\tau) = \frac{1}{\delta}\sin(\delta\tau), \quad \delta = \frac{H}{\sqrt{3}}. \tag{37}$$

Also from (30) and (32) we obtain the relation

$$a = \mu\dot{b}, \ \mu \text{ is arbitrary}. \tag{38}$$

So we can write

$$a = \mu\cos(\delta\tau), \tag{39}$$

with initial conditions $a(0) = \mu$, $\dot{a}(0) = 0$.

Now for the classically allowed region in which the wave function is oscillatory, the WKB ansatz gives

$$\psi = \mathrm{Re}(C\exp(is)), \tag{40}$$

where we assume

$$\frac{\nabla^2 C}{C} \ll (\nabla S)^2. \tag{41}$$

So that the prefactor varies slowly compared to S. Substituting (40) in Wheeler-De-Witt equation one obtains

$$(\nabla S)^2 + V \simeq 0$$

i.e.

$$-2\left(\frac{\partial S}{\partial\alpha}\right)\left(\frac{\partial S}{\partial\beta}\right) + \left(\frac{\partial S}{\partial\alpha}\right)^2 + \left(\frac{\partial S}{\partial\phi}\right)^2 = \exp 2(\alpha+\beta) - \lambda\phi^n \cdot \exp 2(\alpha + 2\beta), \tag{42}$$

and

$$2\nabla S \cdot \nabla C + \nabla^2 S = 0. \tag{43}$$

Equation (42) is the Hamilton-Jacobi equation for general relativity and (43) is the continuity equation. The gradient S will be the direction of the classical trajectories in superspace. The particular Hamilton-Jacobi function which is the analytic continuation of Euclidean action, will be the desire action according to H–H proposal. The analytic continuation is obtained by rotating to imaginary time,

$$\tau = \frac{\pi}{2H} + it,$$

and we obtain

$$b(t) = \frac{1}{\delta}\cosh(\delta t),$$
$$a(t) = \mu \sinh(\delta t),$$
$$\phi(t) = \phi_0, \tag{44}$$

for small time t.

Thus the initial conditions for the Lorentzian trajectories which are analytic continuation of Euclidean paths are

$$\begin{aligned} &a(0) = 0, \quad b(0) = 1/\delta, \quad \phi(0) = \phi_0, \\ &\dot{a}(0) = \mu\delta, \quad \dot{b}(0) = 0, \quad \dot{\phi}(0) = 0. \end{aligned} \tag{45}$$

(here argument '0' stands for $t = 0$). These are identical to those of Laflamme and Shellard (1987).

Now by analytic continuation of (34), the action for classical region is given by

$$S \simeq -ab(\tfrac{1}{3}\lambda\phi^n b^2 - 1)^{1/2}. \tag{46}$$

Since S is real, so negative ϕ for odd n is not possible. Expressing momenta in terms of H–J function S, the 1st integral of the system is

$$p_a = \frac{\partial S}{\partial a} = \frac{\partial L}{\partial \dot{a}}, \quad p_b = \frac{\partial S}{\partial b} = \frac{\partial L}{\partial \dot{b}}, \quad p_\phi = \frac{\partial S}{\partial \phi} = \frac{\partial L}{\partial \dot{\phi}}, \tag{47}$$

where L is given by (25). So we obtain ($N = 1$) the differential equations

$$\dot{b} = (\tfrac{1}{3}\lambda\phi^n b^2 - 1)^{1/2}, \tag{48}$$
$$\dot{a} = \tfrac{1}{3}(\lambda\phi^n ab)/(\tfrac{1}{3}\lambda\phi^n b^2 - 1)^{1/2}, \tag{49}$$
$$\dot{\phi} = -n\dot{a}/2a\phi, \tag{50}$$

which are in agreement with the boundary conditions (45).

As t increases, exponential with positive t becomes dominant and $a(t)$ and $b(t)$ become identical (except for prefactor) (see (44)). In this situation the action (46) takes the approximate form

$$S \simeq -\frac{1}{\sqrt{3}}\lambda^{1/2}\phi^{n/2}ab^2, \tag{51}$$

which holds when either $\dot{\phi} \gg 1$ or b is very large.

Let $[0, t_1]$ be the time interval during which the Lorentzian action is given by (46). So the solutions are represented by (44). We shall now try to find out the solutions in the interval $[t_1, t_2]$ during which S can be approximated by (51). Thus the system of eqs (48)–(50) now give

$$\dot{b} = \frac{b\lambda^{1/2}}{\sqrt{3}}\phi^{n/2}, \tag{52}$$

$$\dot{a} = \frac{a\lambda^{1/2}}{\sqrt{3}}\phi^{n/2}, \tag{53}$$

$$\dot{\phi} = -\frac{n}{2\sqrt{3}}\lambda^{1/2}\phi^{(n/2-1)}. \tag{54}$$

From (54) we have on integration

$$\phi(t) = \left[\phi_0^{(4-n)/2} - \left(\frac{4-n}{2}\right)\frac{n\lambda^{1/2}}{2\sqrt{3}}(t-t_1)\right]^{2/(4-n)}, \quad n \neq 4 \tag{55}$$

$$= \phi_0 \exp\left[-\frac{2}{\sqrt{3}}\lambda^{1/2}(t-t_1)\right], \quad n = 4 \tag{56}$$

where we have assumed $\phi(t = t_1) = \phi_0$. So

$$b(t) = b_1 \exp\left(\frac{\lambda^{1/2}}{\sqrt{3}}\int_{t_1}^{t}[\phi(t')]^{n/2}\,\mathrm{d}t'\right), \tag{57}$$

$$a(t) = a_1 \exp\left(\frac{\lambda^{1/2}}{\sqrt{3}}\int_{t_1}^{t}[\phi(t')]^{n/2}\,\mathrm{d}t'\right), \tag{58}$$

where,

$$a_1 = \mu \sinh(\delta t_1) \simeq \frac{\mu}{2}\exp(\delta t_1),$$

$$b_1 = \frac{1}{\delta}\cosh(\delta t_1) \simeq \frac{1}{2\delta}\exp(\delta t_1).$$

Thus for $t > t_1$, a and b have same expression (except for arbitrary multiplicative constant) and satisfy identical differential equation. Further we can approximate (57) and (58) as

$$b(t) \simeq b_1 \exp\left(\frac{\lambda^{1/2}}{\sqrt{3}}\phi_0^{n/2}t\right),$$

$$a(t) \simeq a_1 \exp\left(\frac{\lambda^{1/2}}{\sqrt{3}}\phi_0^{n/2}t\right),$$

for small $(t - t_1)$. These are the correct inflationary form as expected. Therefore the Lorentzian trajectories in H–H proposal gives inflationary solutions by the scalar field.

4. Conclusions

In this paper we have considered the 2nd order corrections to the wave function by the method of steepest descent along the line of approach of Hartle–Hawking and compare with tunneling wave function. The probability distributions are calculated for both types of wave functions. The results show an initial singularity at $a = 0$ in both the wave functions. So Hartle–Hawking boundary conditions seem to be contradictory and initial singularity cannot be avoided. Also this 2nd order correction (which gives the initial singularity) indicates fluctuation about classical solutions and is important at very early stages of the universe. The factor ordering problems is partially overcome and there are no definite reasons to take $p = +1$ (in H–H approach) or $p = -1$ (by Vilenkin).

Lastly from the series of solutions in §3 we observe that inflation can be originated by the introduction of a scalar field but there are no means to stop it. So universe expands exponentially with time forever, which is contrary to our present state of universe. This point was also discussed by Vilenkin and according to him. "The universe had a beginning but it will have no end, and we live in a region which thermalized about 10^{10} years ago."

Acknowledgements

This research was supported in part by the UGC–DSA program in Mathematics Department of Jadavpur University.

References

Esposito G and Platania G 1988 *Class. Quan Grav* **5** 937

Fang L Z and Ruffinm R 1987 *"Quantum Cosmology, Advanced Series in Astrophysics and Cosmology"* (World Scientific) **3**

Hartle J B and Hawking S W 1983 *Phys Rev* **D28** 2960

Hawking S W 1977 *Commun. Math. Phys* **55** 133

Hawking S W 1983 *Relativity, groups and topology II* Les Houches (eds) B S Dewitt and R Stora, p. 336

Laflamme R and Shellard P E 1987 *Phys. Rev.* **D35** 2315

Linde A D 1984 *Nuovo Cimento Lett.* **39** 401

Louko Jorma 1988 *Ann Phys.* **181** 318

Morse P M and Feshback 1953 *Methods of Theoretical Physics* (N. York, Toronto, London: McGraw-Hill book Company) INC Vol. 1.

Schleich Kristin 1985 *Phys Rev* **D32** 1889

Vilenkin A 1988 *Phys Rev.* **D37** 888