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Spherical shells of positive density – can they be of non-positive mass?†

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Abstract. Spherical shells of fluid in general relativity are considered The density is assumed to be spatially uniform and it is found that there may be three cases of positive, negative and vanishing Schwarzschild mass of the shell although the density and the pressure are both positive throughout. However the negative mass case has to be associated with a singularity representing a negative mass particle and so is unphysical. The zero mass solution has the intriguing feature that the geometry on either side of the shell is Minkowskian and the space is closed This closure of the space saves the present result from being in contradiction with the positive energy theorems. Earlier investigations claiming zero-mass distributions are also discussed

Keywords. Zero mass, negative mass, spherical shells

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1. Introduction

The problem arose out of a consideration of the fit between the Schwarzschild metric of empty space

$$ds^2 = \frac{(1 - 2m/\hat{r})^2}{(1 + 2m/\hat{r})^2}\, d\hat{t}^2 - \left(1 + \frac{m}{2\hat{r}}\right)^4 (d\hat{r}^2 + \hat{r}^2\, d\theta^2 + \hat{r}^2 \sin^2\theta\, d\phi^2) \tag{1}$$

and the Friedmann metric

$$ds^2 = dt^2 - \frac{e^g}{\left[1 + \dfrac{kr^2}{4}\right]^2} (dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2) \tag{2}$$

where $g = g(t)$ and $k = 0$, $+1$ or -1 and the constant m signifies the mass of the Schwarzschild field source. The particular coordinate systems are characterized by the condition that in both (1) and (2), the three spaces have a conformally flat metric.

The problem of fitting a Schwarzschild metric with the Friedmann metric was first considered by Einstein and Straus (1945, 1946) who showed the existence of a transformation which renders the two metrics continuous. While in their case, the Schwarzschild vacuum was a hole in the universe, Raychaudhuri (1953) considered a dust sphere cut out of a homogeneous universe placed in a vacuum and showed

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that the metric (2) valid within the dust sphere may be fitted to the metric (1) at $r=r_1$, if,

$$m=\tfrac{4}{3}\pi\rho\frac{r_1^3\exp(3g/2)}{\left[1+\dfrac{kr_1^2}{4}\right]^3}. \tag{3}$$

where ρ is mass density of the dust in the sphere. The relation (3) shows that in $k=1$, the mass m of the dust sphere has a maximum for $r_1=2$ and then decreases to zero as $r_1\to\infty$. Thus one cannot obtain in this way a sphere of negative mass and in case $r_1\to\infty$, the sphere fills up the entire space leaving no empty space outside. However the decrease of m with increasing r_1, even though the density is positive everywhere, seems somewhat intriguing and here we investigate the problem whether we can obtain a spherical shell of positive density everywhere but nevertheless having a Schwarzschild field of negative or vanishing mass in the neighbouring vacuum.

Zero mass solutions have previously been proposed by Zel'dovich (1962) and Harrison *et al* (1965). But these models have been criticized by Leibovitz and Israel (1970) which we discuss in concluding section.

2. Field equations

We consider three regions (a) an internal vacuous region extending from $r=0$ to $r=r_1$ (i.e. $0\leqslant r\leqslant r_1$). (b) A shell of uniform positive density and (non-uniform) positive pressure from $r=r_1$ to r_2. (c) An external vacuous region extending beyond $r_2(r>r_2)$.

We consider a spherically symmetric system and assume that the matter which is a perfect fluid has a shearfree motion. Thus we can introduce a comoving coordinate system in which the metric assumes the form

$$ds^2=e^{\nu}dt^2-e^{\mu}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2) \tag{4}$$

where ν and μ are functions of r and t. The velocity vector v^μ of the fluid is given by $v^\mu=\exp(-\nu/2)\,\delta^\mu_0$ (the coordinates t, r, θ, ϕ being numbered 0, 1, 2, 3 respectively). In the above metric the coordinate r is uniquely defined up to a trivial constant multiplier in the fluid filled region and in the empty spaces by the conditions of continuity of $\exp(\nu)$ and $\exp(\mu)$ and their first derivatives.

The Einstein field equations

$$R^\mu{}_\nu-\tfrac{1}{2}R\delta^\mu{}_\nu=-8\pi T^\mu{}_\nu \tag{5}$$

give four non-trivial equations with the metric (4):

$$8\pi p\equiv-8\pi T^1{}_1=\exp(-\mu)\left[\frac{\mu'^2}{4}+\frac{\mu'\nu'}{2}+\frac{\mu'+\nu'}{r}\right]-\exp(\nu)\left(\ddot{\mu}+\frac{3\dot{\mu}^2}{4}-\frac{\dot{\mu}\dot{\nu}}{2}\right) \tag{6}$$

$$8\pi p \equiv -8\pi T^2{}_2 = -8\pi T^3{}_3 = \exp(-\mu)\left[\frac{\mu''+\nu''}{2}+\frac{\nu'^2}{4}+\frac{\mu'+\nu'}{2r}\right] - \exp(\nu)\left(\ddot{\mu}+\frac{3\dot{\mu}^2}{4}-\frac{\dot{\mu}\dot{\nu}}{2}\right) \tag{7}$$

$$8\pi\rho \equiv 8\pi T^0{}_0 = \exp(-\mu)\left(\mu''+\frac{\mu'^2}{4}+\frac{2\mu'}{r}\right)+\tfrac{3}{4}\exp(-\nu)\dot{\mu}^2 \tag{8}$$

$$0 = 8\pi T_{10} = -\left(\dot{\mu}'-\frac{\dot{\mu}\nu'}{2}\right). \tag{9}$$

We shall assume that the density ρ is uniform throughout the shell but leave the pressure free except that we require it to vanish at the two boundaries of the shell ($r=r_1$ and $r=r_2$—this is necessary if the continuity conditions at the boundary are to be satisfied) and further that the pressure of the fluid is positive everywhere else. Still, the association of a non-uniform pressure with a uniform density appears to be somewhat unphysical but the assumption of a uniform density has been introduced to make the mathematics tractable and presumably the assumption does not vitiate the final results.

Equation (9) is readily integrated to give

$$\exp(-\nu)\dot{\mu}^2 = \dot{g}^2 \tag{10}$$

where g is an arbitrary function of t alone. We now recall some results in Raychaudhuri's paper (1953). Introducing new variables ξ and x defined by

$$\exp(\mu)r^2 = \xi^4, \quad x = \ln r \tag{11}$$

equation (8) reduces to

$$\xi_{xx} \equiv \frac{\partial^2\xi}{\partial x^2} = \tfrac{1}{4}\xi + \tfrac{3}{16}\dot{g}^2\xi^5 \quad \text{(in vacuo)} \tag{12a}$$

$$= \tfrac{1}{4}\xi + \tfrac{3}{16}\left(\dot{g}^2 - \frac{32\pi\rho}{3}\right)\xi^5 \text{ (for fluid filled region).} \tag{12b}$$

The first integrals of equation (12a) are

$$\xi_x^2 \equiv \left(\frac{\partial\xi}{\partial x}\right)^2 = \frac{\xi^2}{4} + \tfrac{1}{16}\dot{g}^2\xi^6 \quad (r \leqslant r_1) \tag{13a}$$

$$= \frac{\xi^2}{4} + \frac{1}{16}\left(\dot{g}^2 - \frac{32\pi\rho}{3}\right)\xi^6 + K_1 \quad (r_1 \leqslant r \leqslant r_2) \tag{13b}$$

$$= \frac{\xi^2}{4} + \frac{1}{16}\dot{g}^2\xi^6 + K_2 \quad (r \geqslant r_2) \tag{13c}$$

where the subscript x indicates partial differentiation with respect to x and K_1, K_2

are integration 'constants' which are at most functions of t. However the condition of isotropy of pressure ($T^1{}_1 = T^2{}_2$) gives from (6) and (7) on eliminating ν' by (9):

$$\frac{\mu'^2}{4} + \frac{\mu'\dot{\mu}'}{\dot{\mu}} + \frac{\mu'}{2r} + \frac{\dot{\mu}'}{\dot{\mu}r} - \frac{\mu''}{2} - \frac{\dot{\mu}''}{\dot{\mu}} = 0.$$

This on integration yields

$$\exp(\mu/2)\left(\frac{\mu'^2}{2} + \frac{\mu'}{r} - \mu''\right) = f(r) \tag{14}$$

where $f(r)$ is an undetermined function of r alone. Now using (11), (12) and (13), eq. (14) reduces to

$$12K_1/r^3 = f(r), \quad 12K_2/r^3 = g(r)$$

or K_1 and K_2 are constants independent of t as well. The conditions of continuity of ξ and ξ_x at r_1 and r_2 give

$$\begin{aligned} K_1 &= \tfrac{2}{3}\pi\rho_1\xi_1{}^6 \\ K_2 &= \tfrac{2}{3}\pi\rho(\xi_1{}^6 - \xi_2{}^6) \end{aligned} \tag{15}$$

where ξ_1 and ξ_2 are the values of ξ at r_1 and r_2 respectively. From the divergence relation $T^{0\mu}{}_{;\mu} = 0$, one gets:

$$\dot{\rho} = -\tfrac{3}{2}(p + \rho)\dot{\mu}$$

so that at r_1 and r_2 where the pressure is to vanish,

$$\frac{\partial}{\partial t}(\rho \exp(3\mu/2)) = 0.$$

Thus $\rho\xi_1{}^6$ and $\rho\xi_2{}^6$ are constants —a conclusion which follows from the constancy of K_1 and K_2 also. We have omitted a constant in (13a) to avoid a singularity at the origin $r=0$. We may now write the solution as

$$\exp(\mu) = \xi^4 r^{-2}, \quad \exp(\nu) = 16\dot{\xi}^2/(\xi^2\dot{g}^2) \tag{16}$$

with ξ determined by (13). Of course an additional function of time will come from the integration of (13). This may be fixed up from the following consideration.

The divergence relation $T^{1\nu}{}_{;\nu} = 0$ gives

$$p' = -(p + \rho)\nu'/2 \tag{17}$$

and as ρ is independent of r, we may integrate the above equation to get

$$(p + \rho)\exp(\nu/2) = \text{function of } t \text{ alone.}$$

Now at r_1 and r_2, $p = 0$, and ρ is independent of r. Hence, from the above

$$\exp(\nu/2)_{r_1} = \exp(\nu/2)_{r_2}$$

Or, using (16)

$$(\dot{\xi}/\xi)_{r_1} = (\dot{\xi}/\xi)_{r_2}. \tag{18}$$

From (13b), it is obvious that the integral will be of the form

$$\xi = \xi[(x + C), \eta, K_1]$$

where C is a function of t alone and we have written η for

$$\frac{1}{16}\left(\dot{g}^2 - \frac{32\pi\rho}{3}\right)$$

so that η also is a function of t alone. Hence

$$\frac{\dot{\xi}}{\xi} = \frac{1}{\xi}\left[\frac{\partial\xi}{\partial(x + C)}\dot{C} + \frac{\partial\xi}{\partial\eta}\dot{\eta}\right].$$

$$= \frac{1}{\xi}\left[\xi_x\dot{C} + \frac{\partial\xi}{\partial\eta}\dot{\eta}\right].$$

Consequently (18) gives

$$\left[\left(\frac{\xi_x}{\xi}\right)_{r_1} - \left(\frac{\xi_x}{\xi}\right)_{r_2}\right]\dot{C} + \left[\left(\frac{1}{\xi}\frac{\partial\xi}{\partial\eta}\right)_{r_1} - \left(\frac{1}{\xi}\frac{\partial\xi}{\partial\eta}\right)_{r_2}\right]\dot{\eta} = 0. \tag{19}$$

This equation will determine C (except when the coefficient of C vanishes) if η is given and ensure that (18) is satisfied. We shall see in the next section that $(\xi_x/\xi)_{r_1}$ and $(\xi_x/\xi)_{r_2}$ are of opposite signs and the question of vanishing of the coefficient of C does not arise. Note that the constancy of $\rho\xi_1{}^6$ and $\rho\xi_2{}^6$ as follows from (15) automatically leads to (18). Equation (19) serves to determine C and is consistent with (15).

Note that if η is a constant, then C will also be a constant and ξ would be independent of t. Obviously there cannot be any static solution of the type we are considering and this is indicated by the vanishing of e^ν for $\dot{\xi} = 0$.

The vanishing of pressure at the two boundaries r_1 and r_2 is automatically ensured if the continuity conditions with the vacuum fields at these two surfaces are satisfied. However the requirement that the pressure is positive inside requires, in view of equation (17), that ν' is negative at r_1 and positive at r_2. From (16), (13a, c) we get

$$\frac{\nu'}{2} = \frac{\xi^6}{8r\dot{\xi}\xi_x}\left(\dot{g}^2\frac{\dot{\xi}}{\xi} + \frac{\dot{g}\ddot{g}}{2}\right) \quad [\text{for } r \leqslant r_1] \tag{20a}$$

$$= \frac{\xi^6}{8r\dot{\xi}\xi_x}\left(\dot{g}^2\frac{\dot{\xi}}{\xi} + \frac{\dot{g}\ddot{g}}{2} - \frac{8K_2\dot{\xi}}{\xi^7}\right) \quad [\text{for } r \geqslant r_2] \tag{20b}$$

ξ_x will be positive at r_1 and negative at r_2, hence the condition on ν' requires that the expression within the brackets in (20a) and (20b) should be negative at r_1 and r_2 respectively (we are taking $\dot{\xi}$ to be positive – this does not mean any loss of generality owing to the time reversal symmetry). Due to (18), if $K_2 \geqslant 0$, (20b) will be automatically positive if (20a) is negative, while if $K_2 \leqslant 0$, (20a) would be negative if (20b) is positive.

Specifically, the conditions are

$$\left(\frac{\dot{\xi}}{\xi}\right)_b + \frac{\ddot{g}}{2\dot{g}} < 0 \quad \text{if } \xi_1 \geqslant \xi_2$$

$$\left(\frac{\dot{\xi}}{\xi}\right)_b + \frac{\ddot{g}}{2\dot{g}} - \frac{16\pi\rho}{3}\frac{(\xi_1^6 - \xi_2^6)}{\dot{g}^2\xi^7} < 0 \quad \text{if } \xi_2 \geqslant \xi_1$$

where the subscript b indicates that the values at the boundaries are to be taken.

3. The behaviour of ξ within the fluid region

In general an integration of (13b) will bring in elliptic functions which will degenerate into trigonometric functions only if the rhs of (13b) has two identical roots. However such identical roots can only occur if η is a constant; but as already noted, in that case we do not have any solution. Without writing out the integral in terms of elliptic functions, we attempt to study the qualitative behaviour of ξ in different cases, depending on whether η is positive or negative.

Case I: $\eta \equiv \frac{1}{16}\left(\dot{g}^2 - \frac{32\pi\rho}{3}\right) \geqslant 0$

In Friedmann metric, the above condition corresponds to negative or zero spatial curvature and an ever-expanding space and the inequality, if true at one epoch, is maintained at all times. In the present case, however, there does not seem to be any guarantee that the inequality will be preserved. However at the epoch when this inequality holds, we find that ξ_x starting from a positive value at r_1 will have a monotonic increase with x (see (12a)) and hence $\xi_2 > \xi_1$ and K_2 will be negative. We shall see that this corresponds to a positive Schwarzschild mass for the shell.

Case II: $\eta \equiv \frac{1}{16}\left(\dot{g}^2 - \frac{32\pi\rho}{3}\right) < 0$

In ξ_x starting from a positive value will lead to an increasing ξ which will finally make ξ_x vanish and ξ having a maximum. With further increase of x, ξ will show a monotone decrease [see (12b) and (13b)]. Again we may consider three subcases. Case IIa: Let r_2 be such that $\xi_2 > \xi_1$; K_2 is negative in this case and the situation is not significantly different from case I considered above. Case IIb: $\xi_2 < \xi_1$, so that $K_2 > 0$. In this case ξ_x would not have any zero in the region $r > r_2$. Hence it will continue to be negative and as $|\xi_x| > (K_2)^{1/2}$ will vanish at a finite value of x in the vacuous region extending beyond r_2. Case IIc: $\xi_2 = \xi_1$ so that $K_2 = 0$: In this case also ξ shows a monotone decrease for $r > r_2$ but approaches zero only asymptotically for $r \to \infty$.

To have positive pressure in the fluid region, we have insisted that ν' must be negative at r_1 and positive at r_2. In between, ν' will have a zero where the pressure will have the maximum value and $\exp(\nu)$ its minimum.

From (15), the condition for the vanishing of ν' is

$$\frac{\dot{\xi}_x}{\dot{\xi}} - \frac{\xi_x}{\xi} = 0 \tag{21}$$

or,

$$\dot{\xi}_x \xi \xi_x - \xi_x{}^2 \dot{\xi} = 0 \tag{22}$$

(Note that (22) does not lead to (21) if $\xi_x = 0$; hence (22) is a necessary but not a sufficient condition for the vanishing of v'). Substituting from (13b) we get from (22)

$$-\left(\frac{\pi\rho\xi^6}{3}\right) + \tfrac{1}{8}\dot{g}^2\xi^5\dot{\xi} + \tfrac{1}{16}\dot{g}\ddot{g}\xi^6 + \frac{2\pi\rho}{3}\xi^5\dot{\xi} - \frac{K_1\dot{\xi}}{\xi} = 0.$$

Using the divergence relation $(\rho\xi^6) = -6p\xi^6\dot{\xi}$, we get the expression for the maximum pressure (see (15)).

$$8\pi p = -\left[\frac{\dot{g}^2}{2} + \frac{\dot{g}\ddot{g}}{4}\frac{\xi}{\dot{\xi}} + \frac{8\pi\rho}{3}\left(1 - \frac{\xi_1{}^6}{\xi^6}\right)\right].$$

4. The fields in the vacuous regions

Equation (13a) may be readily integrated to give

$$\xi^4 = \frac{16a^2}{\dot{g}^2} \frac{u^2}{(a^2u^2 - 1)^2} \tag{23}$$

where

$$\begin{aligned} u &= r^{-1} \quad \text{for } \xi_x > 0 \\ &= r \quad \text{for } \xi_x < 0 \end{aligned}$$

and 'a' is a function of t alone.

We shall take $\xi_x > 0$ in the region (1) so that ξ starting from zero at the origin will be positive. The function 'a' of t is fixed by the condition of continuity of ξ at $r = r_1$. We have already seen that the field in region $r_1 \leqslant r \leqslant r_2$ has been completely determined, the function C being determined by (19). Hence the continuity of ξ at r_1 will now determine 'a'. Once ξ is made continuous, the choice of K_1 by (15) ensures the continuity of ξ_x.

In the region $r > r_2$, the nature of the integral will depend on K_2. In case $K_2 \neq 0$, we shall have an elliptic function of the form, $\xi = f(x + b, \eta, K_2)$ and the arbitrary function b of t will be chosen so as to obtain the continuity of ξ at r_2. Again the choice of K_2 by (15) will ensure the continuity of ξ_x.

As is to be expected, in view of Birkhoff's theorem, the metric in this region may be transformed to the Schwarzschild form. The requisite transformations are (Raychaudhuri 1953)

$$\left(1 + \frac{m}{2\hat{r}}\right)^4 \hat{r}^2 = \xi^4 \tag{24}$$

$$d\hat{t} = \left|\frac{8\dot{\xi}\xi_x}{\dot{g}(\xi^2 - 2m)}\right| dt + \left|\frac{\xi^6 \dot{g}}{2r(\xi^2 - 2m)}\right| dr. \tag{25}$$

with $m = -2k_2$. The metric then assumes the form (1). Thus the Schwarzschild mass is related with the constant K_2. It is useful to note also that the so-called curvature coordinate $\bar{r}$ which appears in the metric

$$ds^2 = \left(1 - \frac{2m}{\bar{r}}\right) dt^2 - \left(1 - \frac{2m}{\bar{r}}\right)^{-1} d\bar{r}^2 - \bar{r}^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{26}$$

is related with ξ by

$$\bar{r}^2 = \xi^4.$$

Consider now the case $\xi_2 > \xi_1$. In view of (15) and (25), the Schwarzschild mass is positive. Then at r_2, one may have $\xi_x > 0$, with either sign of η (see figures 1 and 2). In any case, an increasing ξ will increase the value of ξ_x and for large values of ξ,

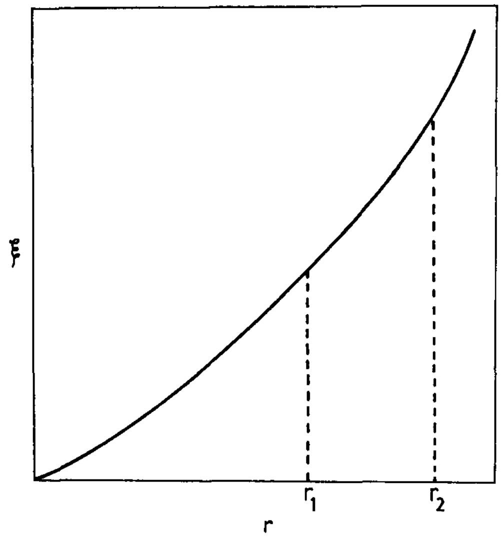

Figure 1. (not to scale) Case $\eta > 0$ $\xi_2 > \xi_1$.

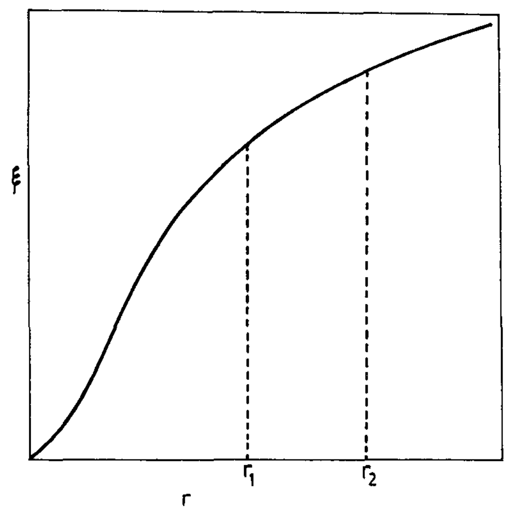

Figure 2. (not to scale) Case $\eta < 0$, $\xi_2 > \xi_1$, $(\delta\xi/\delta x)_2 > 0$

(13c) will have the asymptotic form

$$\xi_x^2 = \tfrac{1}{16}\dot{g}^2\xi^6.$$

The above equation integrates to give

$$\frac{2}{\dot{g}\xi^2} = q(t) - x.$$

Thus at a finite value of $x = q$, ξ^2 will blow up. This apparent singularity is however easily understood, when one notes that in view of (27) the corresponding $\bar{r} \to \infty$. Thus the singularity appears only because the sphere at infinity is mapped out at a finite value of ξ.

If however $\xi_x < 0$ at ξ_2 as may be the case with $\eta < 0$, ξ will have a minimum at ξ_0 where

$$\tfrac{1}{4}\xi_0{}^2 + \tfrac{1}{16}\dot{g}^2\xi_0{}^6 = \frac{m}{2}. \tag{27}$$

or,

$$\xi_0{}^2 < 2m.$$

Beyond this minimum, ξ will have a monotone increase, tending to ∞ as $r \to \infty$ (see figure 3). As $\xi_0{}^6 < 2m$ corresponds to $\bar{r} < 2m$, one has the interesting situation that there is a Schwarzchild horizon but no Schwarzschild singularity corresponding to $(\xi^2 = \bar{r} = 0)$.

Next let us suppose $\xi_2 < \xi_1$. In this case, we must have $\eta < 0$ and $\xi_x < 0$ at r_2. K_2 is now positive and the Schwarzschild mass negative. Equation (13c) shows that ξ_x cannot have any zero in the region $r > r_2$ and as $|\xi_x| > (K_2)^{1/2}$, ξ will vanish at a finite value of r (figure 5). The vanishing of ξ corresponds to $r = 0$ where there is a physical singularity as evident from the blowing up of the Riemann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. Thus one has to interpret the situation in the following manner—at $r = 0$, there is a negative mass particle, it is covered by a shell of positive mass exactly equal in magnitude to the negative mass of the particle. The resulting system has a vanishing

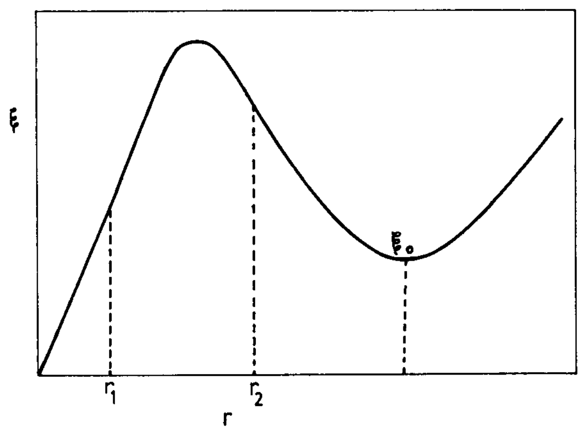

Figure 3. Case $\eta < 0$, $\xi_2 > \xi_1$, $(\partial\xi/\partial x)_2 < 0$. Note minimum at ξ_0 $[(\xi_0{}^2 < 2m)]$.

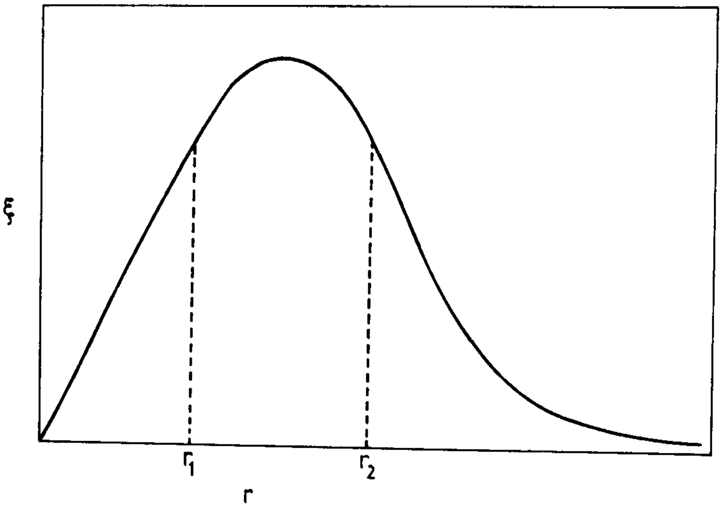

Figure 4. Case $\eta < 0$, $\xi_2 = \xi_1$, $K_2 = 0$ (zero mass case) $\xi \to 0$ as $r \to \infty$.

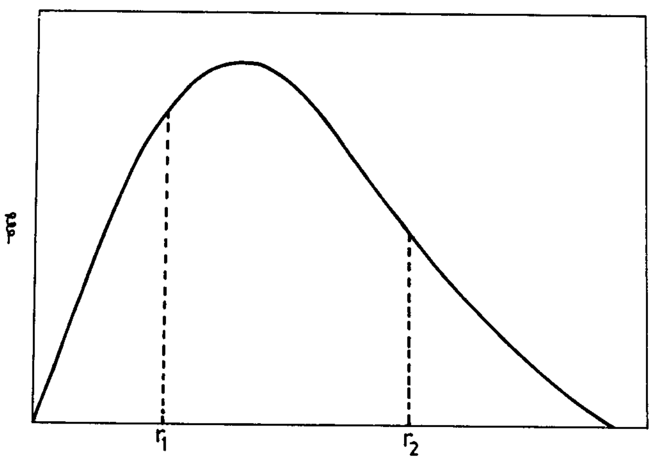

Figure 5. Case $\eta < 0$, $\xi_2 < \xi_1$, $K_2 > 0$ (negative mass case) ξ vanishes at finite r.

mass and thus one has a flat space-time beyond. In this picture the negative mass particle is at the centre of symmetry and the region which we have termed region 1 is euclidean but closed. In any case this situation does not allow us to assign a negative mass for a shell of positive density.

If however $\xi_2 = \xi_1$, $(\xi_x)_{r_2} < 0$, $K_2 = 0$; the Schwarzschild mass vanishes and (13c) reduces to (13a) (see figure 4). ξ will then be given by

$$\xi^4 = \frac{16S^2}{\dot{g}^2}\frac{\dot{r}^2}{(S^2r^2-1)^2} \quad (r \geqslant r_2) \tag{28}$$

where the function $S(t)$ is to be determined by the continuity of ξ at r_2. In fact the relation between S and a can be easily written out

$$\frac{S^2 r_2{}^2}{(S^2 r_2{}^2 - 1)^2} = \frac{a^2}{r_1{}^2\left(\frac{a^2}{r_1{}^2} - 1\right)^2}.$$

This leads to a quadratic equation for S which may be solved to give S in terms of a. It is clear from (28) that ξ will vanish as $r \to \infty$; however, the space-time in this region is euclidean as follows from the transformation to the Schwarzschild form with $m=0$. Only the space is now closed. Thus the shell although has a positive density and pressure everywhere does not produce any departure from euclidean geometry in the adjoining space except bringing about a closure ($e^{\mu} \to 0$ as $r \to \infty$).

One may be tempted to give an alternative interpretation as suggested by the transformation to the Schwarzschild coordinate $\bar{r}$ ($\bar{r}^2 = \xi^4$). The two regions which we have considered to be on opposite sides of the shell then get mapped in the same domain of $\bar{r}$—the sphere at infinity ($r \to \infty$, $\xi \to 0$) goes to the origin and the surface r_2 (ξ_2) to the surface r_1 (ξ_1). One may then suggest that the situation pictured above is merely of a vacuous region with the shell appearing as a singular boundary of the space.

However, the transformation to Schwarzschild coordinate bringing about a merging of two apparently distinct regions may be criticized as suppressing some aspects of the situation. One may cite the Schwarzschild black hole as an analogous case. The black hole and the white hole which appear separated in (say) the Kruskal–Szekeres coordinates are fused into a single region in the Schwarzschild coordinates and altogether disappear in the isotropic coordinates $\hat{r}$. There one prefers the Kruskal–Szekeres picture as giving the maximal extension. Here also we argue that the picture given by ξ is a maximal extension and the present discussion has indeed revealed an intriguing situation where a system of positive mass density and pressure appear to have vanishing mass to outside observers.

5. Newtonian analogue

Although it is quite conceivable that general relativity situations where the field is not weak may not have any newtonian analogue, it is interesting to explore the possibility of such analogues. In the newtonian case one may consider the energy to consist of three distinct parts—the kinetic energy (the random part being associated with pressure) which is positive definite, the negative gravitational potential energy and the rest mass energy which again is assumed to be positive. Thus the possibility of a non-positive total energy arises from the gravitational potential energy which varies as the fifth power of the radius for a given density (recall the expression for potential energy $\sim \mathrm{G}M^2 r^{-1} \approx \mathrm{G}\rho^2 r^5$). The rest mass energy $M_C{}^2 \sim \rho r^3 C^2$. Hence with r sufficiently large one sees the possibility of the total energy being negative.

However with the physically realistic equations of state, one knows that the equilibrium condition sets upper bounds to the mass and radius and consequently one cannot have a non-positive energy for systems in equilibrium.

The situation may be different for dynamic systems as considered in the present investigation.

6. Concluding remarks—earlier models of zero-mass

The integration of the field equations have brought in a number of undetermined functions of time e.g. $\dot{g}, a, b, c$ and S. However the conditions of continuity have linked up a, b, c, S with $\dot{g}$ and ρ (cf eq. (19)). Thus finally there remain two free functions of time $\dot{g}$ and ρ and two free parameters r_1 and r_2. It is natural to pose the question as to whether there is any constraint on their choice and the nature of the dependence of the solution on these functions and parameters. While it is not possible to spell out clearly the constraints on these, it is fairly obvious that they are not quite free, e.g. in the Friedmann case one cannot take ρ to be a constant as that would lead to a negative pressure. Again it is not possible to get rid of the function $\dot{g}$ by a mere scaling of the time coordinate as may appear from (16). This is because (10) can be written as

$$\left(\frac{d\mu}{dS}\right)^2 = \dot{g}^2$$

where $ds\,(\equiv \exp(\nu/2)\,dt)$ is the proper time interval as measured by an observer moving with the fluid. Thus for a simple transformation $t \to f(t)$, $\dot{g}$ is an invariant.

Lake and Roeder (1978) have attempted to establish a distinction between interior and exterior across a surface Σ of discontinuity. Thus they argue "whereas an exterior field arises due to the matter distribution interior to Σ, an interior field does not arise due to the matter distribution exterior to Σ". However in our case, this distinction disappears as the metrics in the interior and exterior vacua are both reducible to the Minkowski form. The same situation holds regarding the other criteria put forward by Lake and Roeder. Indeed one may be tempted to conclude that Lake and Roeder's work suggests an identification of the two vacua but as we have already pointed out, this appears to be an incomplete description of the space-time.

We next consider the zero-mass situations that appear already in the literature. Zel'dovich (1962) considers a star passing through a momentarily static configuration such that each element is just beyond the horizon i.e. if $m(\bar{r})$ represents the mass of the system up to the (curvature) coordinate $\bar{r}$, $\bar{r} = 2m\,(\bar{r}) + 0$. Now the metric is of the form

$$ds^2 = \left(1 - \frac{2m}{\bar{r}}\right)\exp(2\psi)\,dt^2 - \left(1 - \frac{2m}{\bar{r}}\right)^{-1} d\bar{r}^2 - \bar{r}^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{29}$$

where m and ψ are functions of r and t with

$$m = 4\pi \int_0^r T^0_0 \bar{r}^2\, d\bar{r}. \tag{30}$$

The baryon number A can be expressed in terms of their number density n:

$$A = 4\pi \int n v^0 \exp(\psi)\bar{r}^2\, d\bar{r}. \tag{31}$$

For a perfect fluid, one has

$$T^\mu{}_\nu = (p + \rho)v^\mu v_\nu - p\delta^\mu{}_\nu. \tag{32}$$

Using this expression in (30) and changing the integration variable from r to A by (31) we get

$$m = \int \frac{\rho}{n} \exp(-\psi) v_0 \, \mathrm{d}A + 4\pi \int p \left(1 - \frac{2m}{\bar{r}}\right) v_1{}^2 \, \mathrm{d}\bar{r}. \tag{33}$$

It is argued that for a momentarily static situation $v_1 = 0$, so that the second integral vanishes. Also, then

$$\exp(-\psi) v_0 = \left[1 - \frac{2m}{\bar{r}}\right]^{\frac{1}{2}} \to 0 \tag{34}$$

as $\bar{r} \to 2m$. Thus from (33) $m = 0$. But with (34), $A \to \infty$ (see (31)). Thus the Zel'dovich configuration, admittedly artificial as pointed out by Leibovitz and Israel (1970), is not at all realisable for finite baryon number.

Indeed from (30) and (31) it follows that with p and ρ non-negative, m can vanish only if $\mathrm{d}\bar{r}$ changes sign within the domain of integration. This is the case in our solution as well as in the model of Harrison *et al* (1965) who change the variable from $\bar{r}$ to χ by the relation

$$\bar{r} = \left(\frac{8\pi\rho}{3}\right)^{-\frac{1}{2}} \sin\chi$$

the domain of χ being set as $0 \leqslant \chi \leqslant \pi$. Obviously $\mathrm{d}\bar{r}$ changes sign at $\chi = \pi/2$. Thus for a momentarily static sphere, of uniform density

$$m = 4\pi\rho \int \bar{r}^2 \, \mathrm{d}\bar{r} = 4\pi\rho \left(\frac{8\pi\rho}{3}\right)^{-3/2} \int_0^\pi \sin^2\chi \cos\chi \, \mathrm{d}\chi$$

now vanishes. However $\chi = \pi$ makes the space closed. Thus while in our solution, there exists an outside vacuum where the Schwarzschild mass vanishes, in the situation conceived by Harrison *et al*, there is no outside vacuum and so it is not meaningful to talk of a Schwarzschild mass. Leibovitz and Israel have argued that the Harrison *et al* distribution is in the left quadrant of the Kruskal diagram while we, the observers, are in the right quadrant; so no communication is possible. However, it seems more natural to say that the distribution covers all space and thus there is no outside rather than invoking oddities of the Kruskal diagram.

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Appendix

A special class of solutions

$$\xi_x^2 = \gamma(\xi^2 - \alpha)^2 (\beta - \xi^2).$$

The above form obtains if there are two equal roots α. Here

$$\gamma = \frac{1}{16}\left(\frac{32\pi\rho}{3} - \dot{g}^2\right)$$

$$\alpha^2\beta\gamma = K_1$$

$$2\alpha + \beta = 0 \rightarrow \beta = -2\alpha$$

$$-\gamma\alpha(2\beta + \alpha) = \tfrac{1}{4}.$$

One can solve these equations

$$\alpha = -6K_1, \quad \beta = 12K_1, \quad \gamma = \frac{1}{432K_1^2}.$$

Note that γ is independent of t. As noted, in this case no solution can occur. However the integration can be performed to give

$$\frac{\beta}{\xi^2} = 1 + 3\cot^2\left(\frac{x + C}{2}\right)$$

where C is apparently a function of time but if we impose (19), C must be a constant and the solution does not exist (for then $\dot{\xi} = 0$ or $e^{\nu} = 0$).

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