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Triad of homogeneous and inhomogeneous three particle Lippmann–Schwinger equations

SUPROKASH MUKHERJEE
Saha Institute of Nuclear Physics, Calcutta 700009, India

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Abstract. A general distribution theoretic treatment of the convergence of sequences involving wave functions show that the problem of non-uniqueness does not exist for the solutions of the Lippmann–Schwinger equation for multichannel scattering, in the eigenfunction space.

Keywords. Scattering theory; Lippmann–Schwinger equation; non-uniqueness problem in multi channel scattering.

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1. Introduction

It was shown (Mukherjee 1978, 1981 a, b, c; 1982) that the solution $\Psi_\alpha(E_\alpha + io)>$ of Lippmann–Schwinger (LS) equation for multichannel scattering, at an energy E_α, is unique. The analysis used, however, was not rigorous and a few authors (Adhikari 1980; Adhikari and Glockle 1980; Levin and Sandhas 1984; Gerjuoy and Adhikari 1984; 1985) tried to counter these arguments, but their criticisms being just as poor, not much was actually achieved either way.

In §2 we discuss a higher level of rigour, using either L_2-theory of scattering for wave packets or distribution theoretic treatments for energy eigenfunctions. An attempt is made to show that some of the results, which are actually valid only in L_2-theory of scattering are taken to be valid in spaces other than L_2 and that these have led to some of the popular misunderstanding of some of the results of scattering theory. In the end we show that most of the results derived by Mukherjee (1978, 1981a, b, c; 1982) heuristically earlier remains valid if interpreted properly, using the distribution theoretic analysis.

2. Results of distribution theoretic limits

We discuss here the sense in which the equations of multichannel scattering like (see eqs A(12)–A(20) of Appendix)

$$|\Psi_\alpha^\pm(E_\alpha)\rangle = |\phi_\alpha(E_\alpha)\rangle + G_\alpha(E_\alpha \pm io)\bar{V}_\alpha|\Psi_\alpha \pm (E_\alpha)\rangle \tag{1}$$

are valid, where the channel wave functions $|\phi_\alpha(E_\alpha)\rangle$ is an improper state and also the sense in which the limit $\varepsilon \to 0$ is performed on vector sequence like

$$|\psi_\alpha^\pm(E_\alpha)\rangle = \lim_{\varepsilon\to 0} |\psi_\alpha(E_\alpha \pm i\varepsilon)\rangle \tag{2}$$

where

$$|\Psi_\alpha(E_\alpha \pm i\varepsilon)\rangle = \pm i\varepsilon G(E_\alpha \pm i\varepsilon)|\phi_\alpha(E_\alpha)\rangle. \tag{3}$$

It must be noticed that strong or weak limits of vector sequences are useful and well defined concepts in normed spaces (e.g. Hilbert space); but these theories cannot be applied to evaluate limits of vector sequence as in (2), for $|\phi_\alpha(E_\alpha)\rangle$ of (3) is an improper state vector and hence not normalizable. Formal algebraic manipulation of (3) yields (eq. A(13)):

$$|\Psi_\alpha(E_\alpha \pm i\varepsilon)\rangle = |\phi_\alpha(E_\alpha)\rangle + G_\alpha(E_\alpha \pm i\varepsilon)\bar{V}_\alpha|\Psi_\alpha(E_\alpha \pm i\varepsilon)\rangle \tag{4}$$

but passing from (4) to (1) as limit $\varepsilon\to 0$ is not at all straight forward when the vector sequences (2) are not members of Hilbert space. The question, therefore, arises as to the meaning one can assign to the multichannel Lippmann–Schwinger eq. (1) where scattering states are defined through limits in (2) of vector sequence involving improper states, when it is noted that its counterpart in Hilbert space (i.e. LS-equation in H^1 of A(3)) exists in the sense of strong limit for a large class of potential (Reed and Simon 1979). An additional problem arises in (1) due to presence of Dirac delta functions in the momentum of spectator particles in the Green's function in (1), which necessitates the use of distribution theory for treatment of these equations. From treatment (Faddeev 1964) of three body scattering by Faddeev, one has (in the notation of Faddeev (1964))

$$\Psi_\alpha^\pm(k_\alpha, p_\alpha, p_\alpha^0) = \lim_{\varepsilon\to 0} \langle k_\alpha, p_\alpha | i\varepsilon G(E_\alpha + i\varepsilon)|\phi_\alpha(E_\alpha)\rangle \tag{5}$$

$$= \hat{\psi}_\alpha(k_\alpha)\delta(p_\alpha - p_\alpha^0) + \left(\frac{k_0^2}{2m_\alpha} + \frac{p_\alpha^2}{2n_\alpha} - E_\alpha\right)^{-1} \sum_\beta K_{\beta\alpha}(k_\alpha, p_\alpha, p_\alpha^0). \tag{6}$$

Here the channel state, in three body system, is $|\phi_\alpha(E_\alpha)\rangle = |\psi_\alpha\rangle|p_\alpha\rangle^0$; the Fourier component $\hat{\psi}_\alpha(k_\alpha) = \langle k_\alpha|\psi_\alpha\rangle$; k_α is the relative momentum of αth pair with reduced mass m_α binding energy-ε_α and wave function $|\psi_\alpha\rangle$, p_α^0 is the momentum of the third particle (with reduced mass n_α) from the centre of mass of the pair; $E_\alpha = -\varepsilon_\alpha + (p_\alpha^{0\,2}/2n_\alpha)$. The singularity structure of the matrix element of Green's functions was shown by Faddeev to be of the form (§§V, IX of Faddeev (1964)):

$$\langle k_\alpha, p_\alpha | G(z)|\phi_\alpha(E_\alpha)\rangle = \langle k_\alpha, p_\alpha | G_\alpha(z) + G_0(z)\sum_\beta M_{\beta\alpha}(z)G_0(z)|\phi_\alpha(E_\alpha)\rangle$$

$$+ \langle k_\alpha, p_\alpha | \{G_0(z) \sum_{\beta\neq\alpha} t_\alpha(z) + \Sigma M_\beta(z)G_0(z)\}|\phi_\alpha(E_\alpha)\rangle \tag{7}$$

$$= \frac{1}{i\varepsilon}F_1(z) + F_2(z) \tag{8}$$

with $z = E + i\varepsilon$, and

$$\langle k_\alpha, p_\alpha | M_{\beta\alpha}(z)G_0(z)|\phi_\alpha(E_\alpha)\rangle = \frac{K_{\beta\alpha}(k_\alpha, p_\alpha, p_\alpha^0)}{Z - E_\alpha}. \tag{9}$$

Thus the singularity structure of the above matrix element of Green's function of $1/i\varepsilon$ type with the numerical sequence $F_1(z)$ and $F_2(z)$ as well behaved as $\varepsilon \to 0$. From A(12) and A(13) we get

$$|\Psi_\alpha(E_\alpha + i\varepsilon)\rangle = |\phi_\alpha(E_\alpha)\rangle + G_\alpha(E_\alpha + i\varepsilon)\bar{V}_\alpha[i\varepsilon G_\alpha I E_\alpha + i\varepsilon)|\phi_\alpha(E_\alpha)\rangle]. \tag{10}$$

As mentioned earlier, the matrix element $\langle k_\alpha, p_\alpha | G_\alpha(E_\alpha + i\varepsilon)\bar{V}_\alpha | k_\alpha, p_\alpha \rangle$ has Dirac delta function, in momentum of spectator particle (see §§II to IV of Faddeev (1964)), which has no meaning in L_1 or L_2 space, so that (10) can be handled rigorously from the standpoint of distribution theory only. For a wide class of potentials v_α including singular ones, suitable test functions ω of compact support in C_0^∞ (R^6) or Schwartzian space $S(R^6)$, $\langle \omega | G_\alpha(E_\alpha + io)$. $\bar{V}_\alpha | k'_\alpha, p'_\alpha \rangle = \omega_1(k'_\alpha, p'_\alpha)$ can also be used as a test function in momentum space (see Gelfand *et al* (1966); Richtmyer (1978), Kanwal (1983)) which would be then sufficient to handle the Dirac delta function of (6) so that $\langle \omega_1(k'_\alpha, p'_\alpha) | \Psi_\alpha^\pm (k'_\alpha, p'_\alpha, p_\alpha^0 \rangle$ coming from the last square bracket of (10), by virtue of (5), remain well defined for the distribution theoretic treatment. Since the finite contribution to $\langle \omega_1 | i\varepsilon G(E_\alpha + i\varepsilon) | \phi_\alpha(E_\alpha)$ comes only from $F_1(7)$ of (8) as $\varepsilon \to 0$, we may rewrite the distribution theoretic limit of (2) through (10) as

$$|\Psi_\alpha(E_\alpha + io)\rangle = D\text{-}\lim_{\varepsilon \to 0} |\Psi_\alpha(E_\alpha + i\varepsilon)\rangle \tag{11}$$

$$= |\phi_\alpha(E_\alpha)\rangle + D\text{-}\lim_{\varepsilon \to 0} G_\alpha(E_\alpha + i\varepsilon)\bar{V}_\alpha \cdot D\text{-}\lim_{\varepsilon \to 0} |\Psi_\alpha(E_\alpha + io)\rangle \tag{12}$$

$$= |\phi_\alpha(E_\alpha)\rangle + G_\alpha(E_\alpha + io)\bar{V}_\alpha |\Psi_\alpha(E_\alpha + io)\rangle \tag{13}$$

where the distribution theoretic limit, denoted by D-lim, is defined in Appendix B. Clearly, the only surviving term in the numerical sequence $\langle \omega | \Psi_\alpha(E_\alpha + i\varepsilon)\rangle$ comes from $F_1(z)$ of (8), so that the limit of the product of terms on the right side of (12) reduces to the product of their respective limits, provided the limits are both taken in the distribution theoretic sense and this in turn leads to the multi channel Lippmann–Schwinger equation (13) whose validity is, therefore, established only in the distribution theoretic sense for a wide class of potentials. Further, by (3),

$$(E_\alpha - H)|\Psi_\alpha(E_\alpha \pm i\varepsilon)\rangle = \pm i\varepsilon |\phi_\alpha(E_\alpha)\rangle \tag{14}$$

since for test function, in C_0^∞ (R^6) $S(R^6)$, $\lim_{\varepsilon \to 0} \langle \omega | i\varepsilon | \phi_\alpha(E_0)\rangle = 0$, we have from (14)

$$(E_\alpha - H)|\Psi_\alpha^\pm(E_\alpha)\rangle = 0 \tag{15}$$

as distribution theoretic limit, so that the $|\Psi_\alpha^\pm(E_\alpha)\rangle$, which is the solution of the LS-equation (13) is also the distribution theoretic solution of Schrödinger equation (15) as is also seen by evaluating $\langle \omega | E_\alpha - H | \psi_\alpha^\pm (E)\rangle = 0$, as applied to eq. (13). It is in this sense the multichannel wave functions $|\Psi_\alpha^\pm(E_\alpha)\rangle$ are the eigenfunctions of the total Hamiltonian H. In single channel case, the LS-equation A(9) in eigenfunction space, is valid for Rollnick class of potentials or locally square integrable class of potentials, while in general A(9) is valid also in distribution theoretic sense for much wider class of potentials. In contrast, the multichannel LS-equation (13) is valid only in distribution theoretic sense, also for a wider class of potentials including singular ones with integrable singularities, in Schwartzian, compact or other suitable space of test functions (Kanwal 1983).

Since the multichannel wave function $|\Psi_\alpha^\pm(E_\alpha)\rangle$ is to be defined as limit $\varepsilon\to 0$ of (2) in the distribution theoretic sense we should make similar treatment to different terms of equation (A14):

$$|\Psi_\alpha(E_\alpha\pm i\varepsilon)\rangle = i\varepsilon G_\beta(E_\alpha\pm i_\varepsilon)|\phi_\alpha(E_\alpha)\rangle + G_\beta(E_\alpha\pm i\varepsilon)\bar{V}_\beta|\Psi_\alpha(E_\alpha\pm i\varepsilon)\rangle. \quad (16)$$

The distribution theoretic treatment of the first term on the right side of equation (A19) for $\varepsilon\to 0$ can be shown give a single layer surface distribution in general, instead of Kronecker delta function as appears in Lippmann's identity in A(19).

It is noted that $i\varepsilon\, G_\beta\,(E_\alpha+i\varepsilon)$, $\varepsilon\neq 0$, has ("in some sense") an infinite number of eigenstates $|\xi_\beta(E_\alpha,\, f)\rangle$ belonging to the eigenvalue unity and that the following holds, (in obvious notations):

$$i\varepsilon G_\beta(E+\ i\varepsilon)|\xi_\beta(E,f)\rangle = |\xi_\beta(E,f)\rangle \quad (17)$$

where (Baumgartel *et al* 1983)

$$|\xi_\beta(E,f)\rangle = p_\beta(E)|f\rangle \quad (18)$$

$$p_\beta(x) = S\text{-}\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\{G_\beta(x-i\varepsilon)-G_\beta(x+i\varepsilon)\} = \delta(x-E_\beta) \quad (19)$$

where $|f\rangle$ is any vector in the Hilbert space H^1. Expanding (see Prugovecki 1981) the Green's functions in terms of the eigenfunctions $|\phi_\beta(E_\beta)\rangle$ of H_β, the vectors $|\xi_\beta(E_\beta,f)$ can be expressed, by the Radon transform (Gelfand *et al* 1966) of $|f\rangle$ over the surface $S_\beta\,(E_\beta)$ of a sphere in momentum variable k_j, where one of the degrees of freedom is frozen by the constraint (see (A7))

$$E_{\beta'} = E_\beta = \sum\frac{h^2}{2\mu_j}k_j^2 - \sum\varepsilon_j^\beta \quad (20)$$

and we have

$$|\xi_\beta(E_\beta,f)\rangle = \int|\phi_\beta(E'_\beta)\rangle\, \mathrm{d}^3k_1 a^3k_2\cdots\delta(E'_\beta-E_\beta)\langle\phi_\beta(E'_\beta)|f\rangle \quad (21a)$$

$$= \delta(E_\beta - H_\beta)|f\rangle. \quad (21b)$$

As $|f\rangle$ runs over the Hilbert space H^1, the eigenvectors $|\xi_\beta(E_\beta,f)\rangle$ spans an infinite dimensional linear manifold $M_\beta(E_\beta)$ (of codimension $(n-1)$ in momentum space, being constrained by (23)) onto which H^1 is mapped by $p_\beta(E_\beta)$.

In distribution theory (Kanwal 1983), a single layer surface distribution $\delta(S)\,f$ is defined by the surface integral, over the surface S,

$$\langle\omega|f\rangle_S = \langle\omega|\delta(S)f\rangle = \int_S \omega f\,\mathrm{d}S \quad (22)$$

whenever the surface integral (22) is found to exist for all test functions ω in a suitable space, like C_0^∞ or Schwartzian. Now

$$\hat{A}_\beta(E'_\beta;\varepsilon,E_\beta,f) = \langle\phi_\beta(E'_\beta)|i\varepsilon G_\beta(E_\beta+i\varepsilon)|f\rangle$$

$$= \frac{i\varepsilon}{E_\beta+i\varepsilon-E'_\beta}\langle\phi_\beta(E'_\beta)|f\rangle \quad (23)$$

$$\underset{\varepsilon\to 0}{\to}\ {}^{\delta}E_\beta, E'_\beta\langle\phi_\beta(E'_\beta)|f\rangle. \quad (24)$$

In a straightforward way we have the following (using $\hat{f}=\langle\phi_\beta(E_\beta)|f\rangle$, $\hat{\omega}=\langle\phi_\beta(E_\beta)|\omega\rangle$):

$$I_\beta(E_\beta,\omega,f)=\lim_{\varepsilon\to 0}\langle\xi_\beta(E_\beta,\omega)|i\varepsilon G_\beta(E_\beta+i\varepsilon)|f\rangle \tag{25}$$

$$=\lim_{\varepsilon\to 0}\int\langle\omega|\phi_\beta(E'_\beta)d^3k_1^1a^3k_2^1\cdots\delta(E'_\beta-E_\beta)\hat{A}_\beta(E'_\beta,\varepsilon,E_\beta,f)\rangle \tag{26}$$

$$=\lim_{\varepsilon\to 0}\int_{S_\beta(E_\beta)}\langle\omega|\phi_\beta(E'_\beta)\rangle a^3k_1^1a^3k_2^1\cdots\hat{A}_\beta\,(E_\beta;\varepsilon,E_\beta,f) \tag{27}$$

$$=\langle\hat{\omega}|\hat{A}_\beta(\ \ ;\varepsilon,E_\beta,f)\rangle_{S_\beta(E_\beta)} \tag{28}$$

$$=\langle\hat{\omega}|\hat{f}\rangle_{S_\beta(E_\beta)}. \tag{29}$$

The surface integrals in (25)–(29) are over a surface $S_\beta\,(E)$ of a $(n-1)$ sphere in momentum variables, constrained by $\sum(k^2/(2m))=E_\beta+\sum\varepsilon^\alpha$, correspond to a restriction mapping of $L_2(R^n,n)$. Although $M(R^n,n-1)$ has a Lebesgue measure zero in $L_2(R^n,n)$, it does not imply that the above surface integrals in $M(R^n,n-1)$ are zero. In fact, for the non-trivial existence of the single layer surface distribution, it is enough that the surface integrals in (25)–(29) exist for all test functions ω or $\hat{\omega}$ in suitable test space. Further, the above discussion can clearly be repeated for $|f\rangle$ replaced by $|\phi_\alpha(E_\alpha)\rangle$, when ω or $\hat{\omega}$ is taken to belong to suitable class of test functions for which the above surface integrals in (25)–(29) exist; we get from (28)–(29), for

$$E_\beta=E_\alpha,\ \langle\hat{\omega}|\hat{A}_\beta(\ \ ;\varepsilon,E_\beta,\phi_\alpha(E_\alpha))_{S_\beta(E_\beta)}$$
$$=\langle\hat{\omega}|\phi_\alpha(E_\alpha)\rangle_{S_\beta(E)_\alpha} \tag{30}$$

so that we may write, (by B(1)–B(3) and (22))

$$D\text{-}\lim_{\varepsilon\to 0} i\varepsilon G_\beta(E_\alpha+i_\varepsilon)|\phi_\alpha(E_\alpha)\rangle=\delta(S_\beta(E_\alpha))|\phi_\alpha(E_\alpha)\rangle \tag{31}$$

From (16) we then have, in distribution theoretic limit,

$$|\Psi_\alpha^\pm(E_\alpha)\rangle=\delta(S_\beta(E_\alpha))|\phi_\alpha(E_\alpha)\rangle+G_\beta(E_\alpha\pm io)\bar{V}_\beta|\Psi_\alpha^\pm(E_\alpha)\rangle. \tag{32}$$

The single layer surface distribution of (31) on the surface $S_\beta(E_\alpha)$ correspond to the projection operator $P_{\beta\alpha}(E_\alpha)$ of (A20) obtained heuristically earlier and it shows also that the limit $\omega\to 0$ in (A20) is to be taken only in the distribution theoretic sense to arrive at the correct result. The result in (31) is distinctly different from the Kronecker delta function of Lippmann's identity. The point is that for the non-trivial existence of the surface integral and equation (30), it is enough to have $E_\beta=E_\alpha$, which will of course be true for $\beta=\alpha$, but E_β may be equal to E_α (see (A7)) even when $\beta\neq\alpha$. This in turn means that (30) or the surface distribution (31) does not yield a Kronecker delta (e.g. $^\delta\beta\alpha$ as in (A19)), but behaves more like $^\delta E_\beta$, E_α instead, as in (24) and this makes a lot of difference in the context of the non-uniqueness problem in multichannel scattering theory. The $P_{\beta\alpha}\,(E_\alpha)$ of the heuristic treatment in A(20) also point in the same direction. It implies, as emphasised earlier, that equation (A21) or (32) above is no longer a homogeneous equation for $\beta\neq\alpha$ as in (A18), and hence there is no problem of non-uniqueness in the solution of multichannel Lippmann–Schwinger equation in the eigenfunction space, for a three body scattering problem. The above result is easily extended to many body multichannel scattering and one can similarly conclude that the Lippmann–Schwinger equation in many particle multichannel scattering problem is valid in the sense of distribution theory, and there is no problem of non-uniqueness of its solution as envisaged earlier.

Appendix A

In this appendix we quote without proof a few results of non-relativistic scattering theory for ready reference for our discussion.

In the Hilbert space theory of multichannel scattering, the channel scattering states $|f^{\alpha}_{\pm}\rangle$ are given by (see Prugovecki 1981; Reed and Simon 1979) the following strong limits:

$$|f^{\alpha}_{\pm}\rangle = S\text{-lim}\,\Omega^{\alpha}_{\pm}(\varepsilon)E_{m^{\alpha}_{\pm}}|f\rangle = \underset{\varepsilon\to 0}{S\text{-lim}} \pm i\varepsilon \int \exp(\pm\varepsilon\tau)\exp(iHt)$$

$$\times \exp(iH_{\alpha}t)E_{M^{\alpha}_{\pm}}|f\rangle \,\mathrm{d}t \tag{A1}$$

$$= \underset{\varepsilon\to 0^{+}}{S\text{-lim}} \pm i\varepsilon \int_{-\infty}^{+\infty} G(\lambda + i\varepsilon)d_{\lambda}(\varepsilon_{H_{\alpha}}(\lambda))E_{M^{\alpha}_{\pm}}|f\rangle \tag{A2}$$

$$|f^{\alpha}_{\pm}\rangle = E_{M^{\alpha}_{\pm}}|f\rangle + S\text{-lim}\int_{-\infty}^{+\infty} G_{\alpha}(\lambda + i\varepsilon)(H - H_{\alpha})d_{\lambda}(\varepsilon_{H}(\lambda))|f^{\alpha}_{\pm}\rangle. \tag{A3}$$

In (A1)–(A3), $H = H_{\alpha} + \bar{V}_{\alpha} = H_0 + V$; $V = V_{\alpha} + \bar{V}_{\alpha} = V_{\beta} + \bar{V}_{\beta}$ where H_0 and V are total kinetic and potential energy operators, respectively; $H_{\alpha} = H_0 + V_{\alpha} =$ Hamiltonian for the α-channel with its potential V_{α}; the α-channel Green's function is $G = (Z - H_{\alpha})^{-1}$; $G(Z) = (Z - H)^{-1}$; $E_{M^{\alpha}_{\pm}}$ is the projection operator on to the subspace M of Hilbert space H^1 consisting only of those vectors for which the strong limits in (A1)–(A3) exist; $\varepsilon_H(\lambda)$ and $\varepsilon_{H_{\alpha}}(\lambda)$ are, respectively, the spectral projection operators for the Hamiltonian H and H_{α} giving rise to the Boçhner integrals in (A1)–(A3). The equation (A3) is the Lippmann–Schwinger (LS) equation in the Hilbert space H^1, and both the range and the domain space of the scattering operator $\Omega^{\alpha}_{\pm}(\varepsilon)$ are Hilbert spaces (Reed and Simon 1979). This implies that $|f\rangle$ cannot be replaced by the channel state $|\Phi_{\alpha}(E_{\alpha})\rangle$ the eigenstate of H_{α}, for the subspace $M^{\alpha}_{\pm}$ is a Hilbert space and $\phi_{\alpha}(E_{\alpha})$ is not normalizable as H_{α} contains the kinetic energy operators for the free propagations of the centre of mass motions of bound group of particles forming clusters constituting the channel α:

$$E_{\alpha}|\phi_{\alpha}(E_{\alpha})\rangle = H_{\alpha}|\phi_{\alpha}(E_{\alpha})\rangle = (H_0 + V_{\alpha})|\phi_{\alpha}(E_{\alpha})\rangle \tag{A4}$$

$$= \sum_{i=1}^{n_{\alpha}}\left\{T_i + \sum_{j=1}^{n_{\alpha}} h^{\alpha}_j\right\}|\Phi_{\alpha}(E_{\alpha})\rangle. \tag{A5}$$

Here α-channel is formed from a group of η_{α}-clusters, with internal Hamiltonian $h^{\alpha}_j(j = 1, z, \ldots, n_{\alpha})$ with its eigenfunction χ^{α}_j and internal energy $-\varepsilon^{\alpha}_j$, with T_i as the kinetic energy operator of centre of mass motion of the i th group of bound particles moving with momentum $\hbar\vec{k}_i$. In obvious notation, the following may be written:

$$|\Phi_{\alpha}(E_{\alpha})\rangle = \prod_{j=1}^{n_{\alpha}}|\chi^{\alpha}_j\rangle \prod_{j-1}^{n_{\alpha}}|k_j\rangle \tag{A6}$$

$$E_\alpha = -\sum \varepsilon_j^\alpha + \sum \frac{\hbar^2}{2\mu_j} k_j^2 \tag{A7}$$

$$T_i|\vec{k}_i\rangle = \frac{\hbar^2}{2\mu_i}|\vec{k}_i\rangle; \quad h_j^\alpha|\chi_j^\alpha\rangle = -\varepsilon_j^\alpha|\chi_j^\alpha\rangle. \tag{A8}$$

Since $|\phi_\alpha(E_\alpha)\rangle$ of (6) contains the non-normalizable improper states (plane wave, $|\vec{k}_i\rangle$), it cannot be used for $|f\rangle$ in (1)–(3), as strong limits are defined for normalizable states only. The LS-equation (3) in L_2-space and its single channel version, is known to exist for a wide class of potential (Reed and Simon 1979).

Side by side with the LS-equation in L_2-space, we have the old familiar (Schiff 1968) LS-equation for the scattering states $\Psi^\pm$ $(k, r; E)$, in single-channel case:

$$\Psi^\pm(k, r; E) = \phi(k, r; E) + \int G_0^\pm(r, r'; E)V(r')\Psi^\pm(k, r', E)\,\mathrm{d}^3r'. \tag{A9}$$

The space where (A9) is valid is not a L_2 space and here the unperturbed state $\phi(k, r; E)$ and the scattering state $\Psi^\pm$ $(k, r; E)$ are not L_2 objects and (in some sense) also the eigenstates of H_0 and H, respectively, for the energy eigenvalue $E = (h^2k^2/2m)$, written formally as

$$(H_0 - E)\phi(k, r; E) = 0 \tag{A10}$$

$$(H - E)\Psi^\pm(k, r; E) = 0 \tag{A11}$$

corresponding to the absolutely continuous part of the spectrum of the Hamiltonians. Such functions $\Psi_\pm$ $(k, r; E)$ from eigenfunction space which is not a L_2 space. The solution $\Psi_\pm$ $(k, r; E)$ of (A9) is also a solution of the Schrödinger equation (A11) in classical function theoretic sense, only for a very restricted class of potentials. For wide class of potentials, the solution of the LS-equation (A9) is also a solution of the Schrödinger equation (A11) in the distribution theoretic sense only (see Reed and Simon 1979). In the single channel version of (A3) in two body L_2 theory of scattering, the LS-equation is

$$|f_\pm\rangle = E_m|f\rangle + \underset{\varepsilon\to 0^+}{S\text{-lim}}\left\{\int_{-\infty}^{+\infty} G_0(\lambda + i\varepsilon)(H - H_0)d_\lambda(\varepsilon_H(\lambda)\right\}|f_\pm\rangle. \tag{A3$'$}$$

It is shown by Prugovecki (1981) that only for a restricted class of potential the LS-equation (A3′) in Hilbert space for single channel case leads to the existence of LS-equation (A9) in the eigenfunction space of the Hamiltonian H for the absolutely continuous part of its spectrum. But it may be emphasised that, in single channel case the LS-equation (A3′) and (A9) exist independently of each other, under a very general condition and that (A9) does not have to exist if and when the single channel version (A3′) in L_2 space does.

The situation in multichannel case is, however, more complicated, because the analysis of single channel case cannot be carried over verbatim in multichannel case and hence the equations corresponding to (A9) for multichannel case is not obtainable from (A3), for condition on potential only.

However, in the heuristic treatment of three body scattering theory (Sandhas 1976; Levin 1984), one writes for scattering wave function ($\alpha \neq \beta \neq \gamma \neq 0$):

$$|\Psi_\alpha(E_\alpha \pm i\varepsilon)\rangle = \pm i\varepsilon G(E_\alpha \pm i\varepsilon)|\phi_\alpha(E_\alpha)\rangle \tag{A12}$$

$$|\Psi_\alpha(E_\alpha \pm i\varepsilon)\rangle = |\phi_\alpha(E_\alpha)\rangle + G_\alpha(E_\alpha \pm i\varepsilon)\bar{V}_\alpha|\Psi_\alpha(E_\alpha \pm i\varepsilon)\rangle \tag{A13}$$

$$|\Psi_\alpha(E_\alpha \pm i\varepsilon)\rangle = \pm i\varepsilon G_\beta(E_\alpha \pm i\varepsilon)|\phi_\alpha(E_\alpha)\rangle + G_\beta(E_\alpha \pm i\varepsilon)\bar{V}_\beta|\Psi_\alpha(E_\alpha \pm i\varepsilon)\rangle \tag{A14}$$

where the total wave function, defined by

$$|\Psi_\alpha^\pm(E_\alpha)\rangle = \lim_{\varepsilon \to 0} |\Psi_\alpha(E_\alpha \pm i\varepsilon)\rangle \tag{A15}$$

is considered (in some sense) to be the eigenfunction of the total Hamiltonian H:

$$(E_\alpha - H)|\Psi_\alpha^\pm(E_\alpha)\rangle = 0 \tag{A16}$$

with outgoing $(+)$ or incoming $(-)$ wave boundary condition. The limits performed formaly in A(12)–(A14) one writes (Sandhas 1976; Levin 1984):

$$|\Psi_\alpha^\pm(E_\alpha)\rangle = |\phi_\alpha(E_\alpha)\rangle + G_\alpha(E_\alpha \pm io)\bar{V}_\alpha|\Psi_\alpha^\pm(E_\alpha)\rangle \tag{A17}$$

$$|\Psi_\alpha^\pm(E_\alpha)\rangle = \delta_{\alpha\beta}\,|\phi_\alpha(E_\alpha)\rangle + G_\beta(E_\alpha \pm io)\bar{V}_\beta|\Psi_\alpha^\pm(E_\alpha)\rangle \tag{A18}$$

where one uses the "Lippmann's Identity" given by

$$\lim_{\eta \to 0} |A_\beta(\eta, E_\alpha, \phi_\alpha(E_\alpha)) = \lim_{\eta \to 0} i\eta G_\beta(E_\alpha + i\eta)|\phi_\alpha(E_\alpha)\rangle$$
$$= \delta_{\alpha\beta}|\phi_\alpha(E_\alpha)\rangle. \tag{A19}$$

The simultaneous existence of the triad of inhomogeneous equation (A17), for α, β, $\gamma \neq 0$, along with the homogeneous equation (A18) (for $\alpha \neq \beta$), precipitate the solution of LS-equation (A17). This is because, from any solution of the inhomogeneous equation one can obtain infinity of other solution by adding an arbitrary linear combination of the solutions of homogeneous equation. It was shown (Mukherjee 1981a) that the root cause of this non-uniqueness problem is the use of Lippmann's identity (A19) and that the correct evaluation of the limit in (A19) gives, instead of the crucial Kronecker delta function $\delta_{\alpha\beta}$ of (A19), a projection operator $P_{\beta\alpha}(E)$ defined by

$$\lim_{\eta \to 0} i\eta G_\beta(E_\alpha + i\eta)|\phi_\alpha(E_\alpha)\rangle = P_{\beta\alpha}(E_\alpha)|\phi_\alpha(E_\alpha)\rangle$$
$$= \lim_{\eta \to 0} \int_{E_\alpha - \eta}^{E_\alpha + i\eta} |\phi_\beta(E)\rangle\, d\rho(E) \langle \phi_\beta(E)|\phi_\alpha(E_\alpha)\rangle. \tag{A20}$$

It can be shown that $P_{\beta\alpha}(E_\alpha) = 1$ when $\alpha = \beta$, and for $\alpha \neq \beta$, $P_{\beta\alpha}(E_\alpha)$ is non zero in a suitable hyperspace so that instead of (A18) one gets

$$|\Psi_\alpha^\pm(E_\alpha)\rangle = P_{\beta\alpha}(E_\alpha)|\phi_\alpha(E_\alpha)\rangle + G_\beta(E_\alpha + io)\bar{V}_\beta|\Psi_\alpha^\pm(E_\alpha)\rangle. \tag{A21}$$

Since $P_{\beta\alpha}(E_\alpha)$ is a projection operator, and does not behave like Kronecker delta function $\delta_{\alpha\beta}$, (A21) does not reduce to a homogeneous equation for $\alpha \neq \beta$, so that we no longer have simultaneous existence of homogeneous and inhomogeneous equation and, with that, have no longer the problem of non-uniqueness of the solution of LS-equation (A17). It should further be noted that there is no problem of non-uniqueness in the complex energy plane, for (A12) to (A14) are trivially equal to each other for $\varepsilon \neq 0$, and they do not even form a set of independent equations, being one equation written in three different forms only. It can be shown that our equation (21) merely assures that the same is true for (A12)–(A14) in the limit $\varepsilon \to 0+$. But if one uses the Lippmann's identity in the limit of $\varepsilon \to 0$ it changes the situation completely by giving rise to independent triad (A17) and (A18) giving rise to the above mentioned non-uniqueness problem. One can give other examples of problems created by use of Lippmann's identity, like appearance of spurious solutions of Weinberg's equation of scattering and similar other equations (Glockle 1973), and it can be shown in a similar way that these spurious solutions do not appear when the projection operator $P_{\beta\alpha}(E_\alpha)$ is used instead of the Kronecker delta $\delta_{\alpha\beta}$ in Lippmann's identity.

Appendix B

The distribution theoretic limit of a vector sequence, defined as

$$D\text{-}\lim_{\varepsilon \to 0} |f(\varepsilon)\rangle = |f\rangle \tag{B1}$$

is said to exist (see Gelfand *et al* 1966; Kanwal 1983) when one or both of the two following equalities exist

$$\langle \omega | D\text{-}\lim_{\varepsilon \to 0} | f(\varepsilon)\rangle = \lim_{\varepsilon \to 0} \langle \omega | f(\varepsilon)\rangle = \langle \omega | f\rangle \tag{B2}$$

or

$$\lim_{\varepsilon \to 0} \langle \hat{\omega} | f(\varepsilon)\rangle = \lim_{\varepsilon \to 0} \langle \hat{\omega} | f(\varepsilon)\rangle = \langle \omega | \hat{f}\rangle = \langle \hat{\omega} | f\rangle \tag{B3}$$

and equation (B2) and/or (B3) exist for all test functions or their Fourier transform $\hat{\omega}$, in a suitable space of test functions like $S(R^n)$ or $C_0^\infty(R^n)$ or on some of their submanifolds The distribution theoretic $|A\rangle = |B\rangle$ means $\langle \omega | A\rangle = \langle \omega | B\rangle$ for all such ω in a test space.

Let $f(z)$, with $z = \mathrm{d}/\mathrm{d}x$, is a function of differential operator giving rise to the differential equation

$$\left\{E - f\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\right\} | Y\rangle = 0 \tag{B4}$$

subject to some boundary condition. Then $|\psi\rangle$ is said to be the distributional solution of the differential equation (B4), when

$$\left\langle \omega \middle| \left(E - f\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)\right) \middle| \psi \right\rangle = 0 \tag{B5}$$

for all test function ω in a suitable space like compact space $C_0^\infty(R)$ or Schwartzian space $S(R^2)$ etc. There are interesting situations when the distributional solution, of equation (B5) exist when the classical solution, of equation (B4) does not exist.

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