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On mass-independence of the minimal subtraction scheme in dimensional regularization II

SATISH D JOGLEKAR
Department of Physics, Indian Institute of Technology, Kanpur 208 016, India

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Abstract. A self-contained argument is given for the mass independence of the renormalization constants in the minimal subtraction scheme in dimensional regularization in a two mass theory (Yukawa theory). An extension to a theory containing more mass parameters seems straightforward.

Keywords. Dimensional regularization; renormalization.

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1. Introduction

It is well known from 't Hooft's work ('t Hooft 1973) that the renormalization constants in the minimal subtraction scheme (henceforth abbreviated as MS scheme) in dimensional regularization are independent of the mass parameter(s) of the theory. This fact has been well-exploited in writing down renormalization group equation (RGE) and *solving it* (formally) *exactly* without making any approximations in the process ('t Hooft 1973; Gross 1975). This is not possible for RGE's in other (mass dependent) schemes nor for Callan-Symanzik equations (Callan 1972; Symanzik 1971) where approximations have to be made before actually proceeding to the solution of the relevant equation. In addition mass-independent MS scheme leads to many powerful and elegant results and great simplification in formal proofs (See e.g. (Gross 1975). A similar approach has been suggested by Weinberg (1973).

However, a simple but *complete* and *self-contained* qualitative understanding of the mass independence has not been available. In 't Hooft's work ('t Hooft 1973; 't Hooft and Veltman 1972) one needs to go over lengthy mathematical steps before arriving at mass-independence and the proof is not transparent.

With this in view, we gave (Joglekar 1987) an *independent* simple and a *self-contained* argument for the mass-independence of the MS scheme for the case of $\lambda\phi^4$ theory, which contains a single mass parameter. The arguments given by 't Hooft (1973) and by Collins and Macfarlane (1974) heavily rely on 't Hooft's claim that the divergences in the dimensional regularization are polynomials in mass parameters of the theory. This fact which makes the proof a mass independence of work of 't Hooft (1973) and Collins and Macfarlane (1974) simple is by no means transparent from 't Hoofts (1973) work.

With this in view, we have generalized the work of Joglekar (1987) to a two mass theory in which we do not assume polynomial nature of divergences with respect to mass parameters. It should be noted that there is a great deal of qualitative difference between a single mass theory of Joglekar (1987) and a two-mass (or a multi-mass) theory in this context. In a single mass theory the dependence of a dimensionless renormalization constant on mass m can be easily pinned down to the form (Collins and Macfarlane 1974) $\sum_s a_s (\ln(m/\mu))^s$, making it easy to deal with it. On the other hand in a two mass theory a dimensionless renormalization constant could depend, a priori, on the mass parameters m, M (and μ) through an infinite variety of functions. [Examples: $((m^2 - M^2)/(m^2 + M^2))^p$, $(\ln(\sqrt{m^2 + M^2})/\mu)^n$ etc.]. In this work, we have succeeded in proving mass-independence of the renormalization constants by completely bypassing the question of the kind of possible dependence of renormalization constants on the two mass parameters.

The proof given here is certainly not as simple as that in Joglekar (1987) for a single mass theory, but we believe it to be as simple as it could be in the present context. Its generalization to a multi mass theory seems reasonably straightforward, though it is not attempted here.

2. Preliminaries

We shall deal with a concrete case of a two mass theory, viz. Yukawa theory. It is described by a Lagrange density

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m_0)\psi + \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}M_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4 + ig_0\bar{\psi}\gamma_5\psi\phi$$

$$S \equiv \int d^N x \mathscr{L} \tag{1}$$

Here ψ and ϕ are bare fields, m_0 and M_0 are bare masses and λ_0 and g_0 are bare coupling constants.

We shall be using the minimal subtraction scheme (MS scheme) in dimensional regularization ('t Hooft and Veltman 1972) to renormalize Green's functions of the theory and of operators. The renormalization transformations are ($\varepsilon \equiv 4 - n$):

$$\begin{aligned} &\psi = Z_\psi^{1/2}\psi^R \quad \phi = Z_\phi^{1/2}\phi^R \\ &g_0 = Z_g g\mu^{\varepsilon/2} \quad \lambda_0 = [\lambda Z_\lambda + \delta\lambda(g^2)]\mu^\varepsilon \\ &m_0 = m + \delta m_0 \quad M_0^2 = M^2 + \delta M_0^2 \end{aligned} \tag{2}$$

where μ is the arbitrary mass scale in dimensional regularization and $\delta\lambda$ is chosen to be independent of λ. We shall show that δm_0 and δM_0^2 in (2) are such that

$$m_0 = Z_m m \quad M_0^2 = ZM^2 + \tilde{Z}m^2. \tag{3}$$

The Lagrange density of (1) can be reexpressed in terms of renormalized quantities as

$$\begin{aligned} \mathscr{L}[\psi, \phi, m_0, M_0^2, g_0, \lambda_0] &\equiv \mathscr{L}[\psi^R, \phi^R, m, M, g\mu^{\varepsilon/2}, \lambda\mu^{\varepsilon/2}] + \mathscr{L}_{\text{c.t.}} \\ &\equiv \mathscr{L}^R + \mathscr{L}_{\text{c.t.}} \end{aligned} \tag{4}$$

The unrenormalized but dimensionally regularized Green's functions are generated by

$$W^{UR}[\eta,\bar{\eta},J;\lambda_0,g_0;m_0,M_0^2,\varepsilon] \equiv \frac{1}{N}\int D\psi\, D\bar{\psi}\, D\phi \exp i\int d^n x[\mathscr{L}+\bar{\psi}\eta+\bar{\eta}\psi+J\phi] \tag{5}$$

whereas the renormalized Green's functions are generated by

$$W^R[\eta^R,\bar{\eta}^R,J^R;\lambda,g;m,M^2,\mu,\varepsilon] \equiv \frac{1}{N'}\int D\psi^R\, D\bar{\psi}^R\, D\phi^R \exp i\int d^n x[\mathscr{L}^R+\mathscr{L}_{\text{c.t.}}+\bar{\psi}^R\eta^R+\bar{\eta}^R\psi^R+J^R\phi^R] \tag{6}$$

N' in (6) is a quantity depending on coupling constants, masses, μ,ε so adjusted that $W^R[\mathscr{J}^R=0]=1$. Here $\mathscr{J}$ denotes collectively the sources. Further one has $W^{UR}=W^R$.

The renormalized and unrenormalized connected Green's functions are respectively generated by $Z^R=-i\ln W^R$ and $Z^{UR}=-i\ln W^{UR}$, with $Z^R=Z^{UR}$.

For an operator $O[\psi,\bar{\psi},\phi]$ we define $\langle\langle O\rangle\rangle$ and $\langle O\rangle$ by

$$\langle\langle O\rangle\rangle = \frac{1}{N}\int D\psi\, D\bar{\psi}\, D\phi O[\psi,\psi,\phi] \exp i\int d^n x[\mathscr{L}+\bar{\psi}\eta+\bar{\eta}\psi+J\phi] = W\langle O\rangle \tag{7}$$

$\langle\langle O\rangle\rangle$ and $\langle O\rangle$ generate the Green's functions and the connected Green's functions with one insertion of O. We further define operator insertion at momentum q by

$$\langle O(q)\rangle = \left\langle \int d^n x \exp(iq.x)O\right\rangle \tag{8}$$

with $\langle O_0\rangle=\langle O(q=0)\rangle$. It should, further, be noted that $\langle O(q)\rangle$ regarded as a functional of field expectation values $\langle\phi\rangle=\delta Z/\delta J$ etc. generates unrenormalized proper vertices with one insertion of $O(q)$.

We note that equations of motion imply that

$$\left\langle\frac{\delta S}{\delta\phi}\phi\right\rangle = -J\langle\phi\rangle = -J\frac{\delta Z^{UR}}{\delta J} = -J^R\frac{\delta Z^R}{\delta J^R} = \text{finite} = \left\langle\frac{\delta S}{\delta\phi}\phi\right\rangle^R \tag{9}$$

and similar equations for fermion fields.

We introduce the notation $F=-i\,\bar{\psi}\psi$ and $S=(-i/2)\phi^2$. Then*

$$\frac{\partial W^{UR}}{\partial m_0} = \langle\langle F_0\rangle\rangle^{UR};\quad \frac{\partial W^{UR}}{\partial M_0^2} = \langle\langle S_0\rangle\rangle^{UR} \tag{10a}$$

* In differentiating W^{UR} or W^R with respect to m_0 (or m) both the functional integral and the normalization factor N' (or N) get differentiated. As $W^R[\mathscr{J}^R=0]=1$, $(\partial W^R/\partial m)|_{\mathscr{J}^R=0}=0$. Thus the vacuum diagrams arising from one insertion of $\bar{\psi}\psi$ are subtracted out by the term arising from the differentiation of N.

and*

$$\frac{\partial W^R}{\partial m} = \langle\langle F_0 \rangle\rangle^R; \quad \frac{\partial W^R}{\partial M^2} = \langle\langle S_0 \rangle\rangle^R. \tag{10b}$$

3. Certain observations

In this section, we make a number of observations in the form of lemmas which are helpful in later sections. Many of them are evident by an inspection of Feynman graphs.

Lemma A. The unrenormalized proper vertices of the theory exist at $m_0 = 0 = M_0$ provided the external momenta are off-shell (Proof: Off-shell moments avoid infrared divergences).

This ensures that the limits $m_0 \to 0$ and/or $M_0^2 \to 0$ exist for the off-shell proper vertices.

Lemma B. The unrenormalized off-shell proper vertices of $S = -\frac{1}{2}\phi^2$ and $F = -i\bar{\psi}\psi$ at nonzero momentum q exist at $m_0 = 0$ and $M_0^2 = 0$. (Proof: Off-shell momenta and $q \neq 0$ avoid infrared divergences).

This ensures that the limits $m_0 \to 0$ and/or $M_0^2 \to 0$ exist for these proper vertices at $q \neq 0$.

Lemma C. The unrenormalized off-shell proper vertices generated by two insertions of F viz $\langle F(q), F(q') \rangle$ exist at $m_0 = 0$ and $M_0^2 = 0$ if $q \neq 0$ and $q' \neq 0$. (Proof: off-shell momenta and $q \neq 0$ and $q' \neq 0$ avoid infrared divergences).

Lemma D. The operators F and S satisfy relations of the sort

$$\langle S \rangle^{UR} = Z' \langle S \rangle^R \tag{11a}$$

$$\langle F \rangle^{UR} = T' \langle F \rangle^R + Z_1' \langle S \rangle^R \tag{11b}$$

Relations (11a) and (11b) define the renormalization constants Z', T', Z_1' uniquely in the MS scheme as can be seen by an order by order expansion.

Proof. This follows from (i) power counting (ii) scalar nature of F and S (iii) pseudo-scalar nature of ϕ.

Z' and T' are dimensionless and Z_1' has dimensions of mass. These relations can be inverted to yield

$$\langle S \rangle^R = Z \langle S \rangle^{UR} \tag{12a}$$

$$\langle F \rangle^R = T \langle F \rangle^{UR} + Z_1 \langle S \rangle^{UR} \tag{12b}$$

with

$$Z' = Z^{-1}, \quad T' = T^{-1} \quad \text{and} \quad Z_1 = -T' Z_1' Z'^{-1} \tag{12c}$$

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*These relations become obvious in the context of MS scheme from relations such as (45) and (48).

From (12b) we define $F^R = TF + Z_1 S$. Now $\langle F^R(q)F^R(q)\rangle_c$ (subscript "c" for connected) generates connected Green's functions of two insertions of F at momenta q and q' respectively. These Green's functions are "partially renormalized" in the sense that each operator F is a renormalized operator, but they need additional subtractions as $x \to y$ in $\langle F^R(x)F^R(y)\rangle_c$. By power counting these subtractions are needed* only for the 2 scalar Green's function of $\langle F^R(q)F^R(q')\rangle_c$. Hence a relation of the sort given below exists:

$$\langle F^R(q)F^R(q')\rangle_c^R = \langle F^R(q)F^R(q')\rangle_c + K(\varepsilon)\langle S(q+q')\rangle. \tag{13}$$

Now noting that

$$\langle F^R(q)F^R(q')\rangle = \langle F^R(q)\rangle\langle F^R(q')\rangle + \langle F^R(q)F^R(q')\rangle_c \tag{14}$$

and that $\langle F^R(q)\rangle$ is always finite, (13) yields,

$$\begin{aligned}\langle F^R(q)F^R(q')\rangle^R &= \langle F^R(q)\rangle\langle F^R(q')\rangle + \langle F^R(q)F^R(q')\rangle_c^R \\ &= \langle F^R(q)\rangle\langle F^R(q')\rangle + \langle F^R(q)F^R(q')\rangle_c + K(\varepsilon)\langle S(q+q')\rangle \\ &= \langle F^R(q)F^R(q')\rangle + K(\varepsilon)\langle S(q+q')\rangle. \end{aligned} \tag{15}$$

Lemma E. Eq. (15) is valid and $K(\varepsilon)$ is dimensionless (as seen by power counting).

Lemma F. The renormalization constants Z', T', Z'_1, Z, T, Z_1 and K have a smooth limit as $m_0 \to 0$ and/or $M_0^2 \to 0$.

Proof. This follows from Lemmas A, B and C. For example, the renormalization constant Z' can be determined (successively in higher and higher loop approximation) by considering the divergent part of $\langle S(q)\rangle^{UR}$ with $q \neq 0$. This is so because by power counting Z' is independent of q. But by Lemma B, $\langle S(q)\rangle^{UR}$ exists at $m_0 = 0 = M_0^2$ and Z' must have a smooth limit as $m_0 \to 0$ and/or $M_0^2 \to 0$. In a similar fashion Lemma B implies a smooth limit for T' and Z'_1 (and hence via Eq. (12c) for Z, T and Z_1). Lemma C implies a smooth limit for K as $m_0 \to 0$ and/or $M_0^2 \to 0$.

Lemma G. The renormalization constants Z_φ, Z_ψ, Z_λ $\delta\lambda$ and Z_g; δm_0, δM_0^2 have a smooth limit as $m_0 \to 0$ and/or $M_0^2 \to 0$.

Proof. This follows from the Lemma A and the logic used in Lemma F.

Lemma H. If a dimensionless renormalization constant Z_0 depends on a *single* mass,

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*The case of no external lines will not need to be considered here. If one were to consider (13) with zero external lines, there would be additional counterterms of the form $f(q^2, m, M^2, \varepsilon)$, which are independent of J, on the right hand side of (13).

When one considers a quantity of the kind $\langle F^R(x)F^R(y)\rangle_c$, internal subtractions are already made for those proper subdiagrams which contain insertion of only *one* operator $F(x)$ or $F(y)$. But this object develops a singularity as $x \to y$ indicating a need for additional subtraction in primitively divergent proper subgraphs containing both $F(x)$ and $F(y)$. By power counting only such graphs with two external scalars can contain (momentum independent) divergences. Hence the counterterm needed will be of the form $K(\varepsilon)S = K(\varepsilon)(-(i/2))\phi^2$. This leads to relation (13) in the momentum space.

say m, then in the r loop approximation it must be expressible as

$$Z_0(m) = \sum_{s=0}^{r-1} b_s \left(\ln \frac{m}{\mu} \right)^s \tag{16}$$

Proof. This follows from the Lemma D and E of Joglekar (1987).

Lemma I. If a dimensionless renormalization constant Z_0 depends on a single mass, say m, and has a smooth limit as $m_0 \to 0$ then it is mass-independent.

Proof. The proof proceeds by induction. The result is trivially true in the zero loop approximation. Let us assume the result in the $(r-1)$ loop approximation. Then by Lemma H

$$Z_{0[r]} = Z_{0[r-1]} + a^r \sum_{s=0}^{r-1} b_s \left(\ln \frac{m}{\mu} \right)^s$$

$$= Z_{0[r-1]} + a^r \sum_{s=0}^{r-1} b_s \left(\ln \frac{m_0}{\mu} \right)^s + O(a^{r+1}).$$

Here $Z_{0[r]}$ denotes the renormalization constant Z_0 up to r loop approximation and a is the loop expansion parameter. $Z_{0[r]}$ has a smooth limit as $m_0 \to 0$ iff $b_s = 0$ $s = 1, 2, \ldots, r-1$. Hence $Z_{0[r]}$ is mass independent.

4. A useful theorem

In this section we shall prove a very useful theorem. Let $X = \sum_{n=0} a^n x_n$ and $Y = \sum_{n=0} a^n y_n$ have MS-structure [i.e. x_0, y_0 are finite and x_n's and y_n's$(n \neq 0)$ have only poles in ε, and a is the loop expansion parameter].

Theorem. A relation

$$X\langle F\rangle^{UR} + Y\langle S\rangle^{UR} = \text{finite} \tag{17}$$

implies

$$X = x_0 T \quad \text{and} \quad Y = x_0 Z_1 + y_0 Z. \tag{18}$$

Proof. We introduce the notation $X_{[n]} = \sum_{m=0}^{n} a^m x_m$ etc.

(i) First consider the case $x_0 = 0$. Consider the relation (17) in one loop approximation:

$$x_1 \langle F\rangle_0 + y_0 \langle S\rangle_1^{UR} + y_1 \langle S\rangle_0 = \text{finite}$$

i.e.

$$y_0 \langle S\rangle_1^{UR} = -x_1 \langle F\rangle_0 - y_1 \langle S\rangle_0 + \text{finite} \tag{19}$$

whereas (12b) leads to

$$y_0 \langle S\rangle_1^{UR} = -y_0 z_1 \langle S\rangle_0 + \text{finite}. \tag{20}$$

As $\langle F\rangle_0$ and $\langle S\rangle_0$ are linearly independent, comparison of (19) and (20) implies that

$$x_1 = 0 \quad y_1 = y_0 z_1 \tag{21}$$

which agrees with (18) to one loop order.

Now let us assume the result of (18) up to $(n-1)$ loop approximation: i.e.

$$X_{[n-1]}=0 \quad \text{and} \quad Y_{[n-1]}=y_0 Z_{[n-1]}. \tag{22}$$

Then

$$X_{[n]}=x_n a^n \quad \text{and} \quad Y_{[n]}=y_0 Z_{[n]}+(y_n-y_0 z_n)a^n. \tag{23}$$

Now in the n-loop approximation (17) reads

$$x_n\langle F\rangle_0+y_0 Z_{[n]}\langle S\rangle_{[n]}^{UR}+(y_n-y_0 z_n)\langle S\rangle_0=\text{finite}. \tag{24}$$

Now using the fact that (i) $Z_{[n]}\langle S\rangle_{[n]}^{UR}$ is finite up to n-loop approximation [see (12a)] (ii) x_n, y_n Z_n contain only poles (iii) $\langle F\rangle_0$ and $\langle S\rangle_0$ are linearly independent one obtain

$$x_n=0 \quad \text{and} \quad y_n=y_0 z_n \tag{25}$$

proving (18) in n-loop approximation, etc.

(ii) Now consider the case $x_0\neq 0$. We can rewrite (17) using (12a) as

$$X\langle F\rangle^{UR}+(Y-y_0 Z)\langle S\rangle^{UR}=\text{finite}. \tag{26}$$

Dividing throughout by x_0, we have

$$X'\langle F\rangle^{UR}+Y'\langle S\rangle^{UR}=\text{finite} \tag{27}$$

with

$$X=x_0 X' \quad \text{and} \quad Y'=(1/x_0)(Y-y_0 Z) \quad \text{with} \quad y_0'=0. \tag{28}$$

Equation (18) then requires that we prove

$$X'=T \quad \text{and} \quad Y'=Z_1. \tag{29}$$

Consider (27) in one loop approximation:

$$\langle F\rangle_1^{UR}+x_1'\langle F\rangle_0+y_1'\langle S\rangle_0=\text{finite}. \tag{30}$$

Comparing this with (12b) in the one loop approximation (Z_1 vanishes in the tree approximation)

$$\langle F\rangle_1^{UR}+t_1\langle F\rangle_0+z_{1(1)}\langle S\rangle_0=\text{finite} \tag{31}$$

one obtains

$$x_1'=t_1 \quad \text{and} \quad y_1'=z_{1(1)} \tag{32}$$

proving (29) to one loop order.

Next, we assume the result of (29) to $n-1$ loop approximation, viz:

$$X'_{[n-1]}=T_{[n-1]} \quad \text{and} \quad Y'_{[n-1]}=Z_{1[n-1]} \tag{33}$$

so that

$$X'_{[n]}=T_{[n]}+a^n(x_n'-t_n);$$
$$Y'_{[n]}=Z_{1[n]}+a^n(y_n'-z_{1n}). \tag{34}$$

Equation (27), in n loop approximation, reads

$$T_{[n]}\langle F\rangle_{[n]}^{UR}+(x'_n-t_n)\langle F\rangle_0+Z_{1[n]}\langle S\rangle_{[n]}^{UR}+(y'_n-z_{1n})\langle S\rangle_0=\text{finite}. \tag{35}$$

Using Eq. (12b), this reduces to

$$(x'_n-t_n)\langle F\rangle_0+(y'_n-z_{1n})\langle S\rangle_0=\text{finite}. \tag{36}$$

This, as in (24), yields

$$x'_n=t_n \quad \text{and} \quad y'_n=z_{1n} \tag{37}$$

proving (29) to n loop approximation, etc.

5. Proof of mass-independence

Consider first

$$\begin{aligned}\text{finite}=\frac{\partial W^R}{\partial M^2}\bigg|_{\mathscr{J}^R,m,\lambda,g,\mu}&=\frac{\partial W^{UR}[\mathscr{J};\lambda_0,g_0,m_0^2,M_0^2,\varepsilon]}{\partial M^2}\bigg|_{\mathscr{J}^R,m,\lambda,g,\mu}\\&=\frac{\partial m_0}{\partial m}\frac{\partial W^{UR}}{\partial m_0}+\frac{\partial M_0^2}{\partial M^2}\frac{\partial W^{UR}}{\partial M_0^2}+\frac{\partial\lambda_0}{\partial M^2}\frac{\partial W^{UR}}{\partial\lambda_0}+\frac{\partial g_0}{\partial M^2}\frac{\partial W^{UR}}{\partial g_0}\\&\quad+\sum_{\mathscr{J}}\int\frac{\partial\mathscr{J}^{UR}}{\partial M^2}\frac{\delta W^{UR}}{\delta\mathscr{J}^{UR}}\mathrm{d}^n n.\end{aligned} \tag{38}$$

Using (9), (10a) and similar results one obtains

$$\begin{aligned}\text{finite}=\frac{1}{W^R}\frac{\partial W^R}{\partial M^2}&=\frac{\partial m_0}{\partial M^2}\langle F_0\rangle+\frac{\partial M_0^2}{\partial M^2}\langle S_0\rangle+\mu^\varepsilon\left[\lambda\frac{\partial Z_\lambda}{\partial M^2}+\frac{\partial\delta\lambda}{\partial M^2}\right]\\&\times\left\langle-\frac{i}{4!}\int\phi^4\,\mathrm{d}^n x\right\rangle+\mu^{\varepsilon/2}g\frac{\partial Z_g}{\partial M^2}\left\langle i\int\bar\psi\gamma_5\psi\phi\,\mathrm{d}^n x\right\rangle\\&+\frac{1}{2}\frac{\partial\ln Z_\phi}{\partial M^2}\left\langle-\int\frac{\delta S}{\delta\phi}\phi\,\mathrm{d}^n x\right\rangle+\frac{1}{2}\frac{\partial\ln Z_\psi}{\partial M^2}\left\langle-\int\left(\bar\psi\frac{\delta S}{\delta\bar\psi}-\frac{\delta S}{\delta\psi}\psi\right)\mathrm{d}^n x\right\rangle.\end{aligned} \tag{39}$$

Consider (39) in the one loop approximation. We note that all factors of operators on the right hand side except $\partial M_0^2/\partial M^2$ vanish in the tree approximation. Hence, one has

$$\begin{aligned}\langle S_0\rangle_1=&-\left(\frac{\partial M_0^2}{\partial M^2}\right)_1\langle S_0\rangle_0-\left(\frac{\partial m_0}{\partial M^2}\right)_1\langle F_0\rangle_0\\&-\mu^\varepsilon\left[\lambda\frac{\partial Z_\lambda}{\partial M^2}+\frac{\partial\delta\lambda}{\partial M^2}\right]_1\left\langle\frac{-i}{4!}\int\phi^4\mathrm{d}^n x\right\rangle_0\\&-\mu^{\varepsilon/2}g\left[\frac{\partial Z_g}{\partial M^2}\right]_1\left\langle i\int\bar\psi\gamma_5\psi\phi\,\mathrm{d}^n x\right\rangle_0\end{aligned}$$

$$+\frac{1}{2}\left[\frac{\partial}{\partial M^2}\ln Z_\phi\right]_1\left\langle -\int\frac{\delta S}{\delta\phi}\phi\,\mathrm{d}^n x\right\rangle_0$$
$$-\frac{1}{2}\left[\frac{\partial}{\partial M^2}\ln Z_\psi\right]_1\left\langle -\int\left(\bar\psi\frac{\delta S}{\delta\bar\psi}-\frac{\delta S}{\delta\psi}\psi\right)\mathrm{d}^n x\right\rangle + \text{finite.} \tag{40}$$

But S is a dimension two (multiplicatively renormalizable) operator and hence $\langle S\rangle_1$ cannot contain counterterms of dimension higher than two. This immediately implies that

$$\frac{\partial m_0}{\partial M^2}=0=\frac{\partial Z_\lambda}{\partial M^2}=\frac{\partial\delta\lambda}{\partial M^2}=\frac{\partial Z_g}{\partial M^2}=\frac{\partial}{\partial M^2}Z_\psi=\frac{\partial}{\partial M^2}Z_\phi \tag{41}$$

and using (12a)

$$\frac{\partial M_0^2}{\partial M^2}=Z \tag{42}$$

in *one* loop approximation. Now, one assumes the results of (41) and (42) to $(n-1)$ loop approximation. Then

$$\left[\frac{\partial M_0^2}{\partial M^2}\right]_{[n]}=Z_{[n]}+\left\{\left[\frac{\partial M_0^2}{\partial M^2}\right]_n-z_n\right\}a^n. \tag{43}$$

When this is used in (39) and use is made of (12a), (39) in the n loop approximation yields, as before, (41) and (42) in the n loop approximation. Hence the proof is complete by induction.

From (41) it follows that m_0 is only a function of m and on dimensional grounds it can be expressed as

$$m_0=Z_m(m)m. \tag{44}$$

Equation (41) also implies that Z_λ, $\delta\lambda$, Z_g, Z_ψ, Z_ϕ are dimensionless constants depending only on m and by lemma G they have a smooth limit as $m_0\to 0$. Hence by lemma I, they are mass independent. Hence it remains to show the mass-independence of $Z_m(m)$ and Z and $\tilde{Z}$ of (3). Analogous to (39) one can obtain

$$\text{finite}=\frac{1}{W^R}\frac{\partial W^R}{\partial m}=\frac{\partial m_0}{\partial m}\langle F_0\rangle^{UR}+\frac{\partial M_0^2}{\partial m}\langle S_0\rangle^{UR} \tag{45}$$

where use has been made of the mass independence of Z_ϕ, Z_ψ, $\delta\lambda$, Z_λ and Z_g. Equation (45) when compared to (17) yields [from (18)]:

$$\frac{\partial m_0}{\partial m}=T \quad\text{and}\quad \frac{\partial M_0^2}{\partial m}=Z_1. \tag{46}$$

Now from (44), $\partial m_0/\partial m$ is a renormalization constant that is a function of m alone and by lemma F, it has a smooth limit as $m_0\to 0$. Hence by lemma I it is mass independent. Hence,

$$\frac{\partial m_0}{\partial m}=\text{constant}\equiv Z_m. \tag{47}$$

This proves first of (3) and mass independence of Z_m.

Finally we shall prove second of (3) and mass independence of Z' and $\tilde{Z}$. To this end, we note that (39) now becomes, (using 42)

$$\frac{\partial W^R}{\partial M^2} = Z\langle S_0\rangle W^R = Z\langle\langle S_0\rangle\rangle. \tag{48}$$

Differentiating with respect to M^2 once

$$\begin{aligned}\text{finite} = \frac{\partial^2 W^R}{\partial (M^2)^2} &= Z\frac{\partial M_0^2}{\partial M^2}\frac{\partial}{\partial M_0^2}\langle\langle S_0\rangle\rangle + \frac{\partial Z}{\partial M^2}\langle\langle S_0\rangle\rangle \\ &= Z^2\langle\langle S_0, S_0\rangle\rangle + \frac{\partial Z}{\partial M^2}\langle\langle S_0\rangle\rangle \\ &= \langle\langle S_0^R, S_0^R\rangle\rangle + \frac{\partial Z}{\partial M^2}\langle\langle S_0\rangle\rangle. \end{aligned}\tag{49}$$

By power counting and the logic of the kind used in Lemma E, $\langle\langle S_0^R, S_0^R\rangle\rangle$ is finite. Hence

$$\frac{1}{W^R}\frac{\partial Z}{\partial M^2}\langle\langle S_0\rangle\rangle = \text{finite} = \frac{\partial Z}{\partial M^2}\langle S_0\rangle. \tag{50}$$

Hence the theorem of §4 implies (Here $y_0 = 0$)

$$\frac{\partial Z}{\partial M^2} = 0. \tag{51}$$

Thus Z is only a function of m. But by lemma F, Z has a smooth limit as $m_0 \to 0$. Hence by lemma I, Z is mass independent. Equation (42), then implies

$$M_0^2 = ZM^2 + A(m). \tag{52}$$

Equation (45), with the help of (46) and (12b) can be cast in the form

$$\frac{\partial W^R}{\partial m} = \langle\langle F_0^R\rangle\rangle = \left\langle\!\!\left\langle TF_0 + \frac{\partial M_0^2}{\partial m}S_0\right\rangle\!\!\right\rangle. \tag{53}$$

Differentiating (53) with respect to m once more and noting the mass independence of T, we obtain*

$$\frac{1}{W^R}\frac{\partial^2 W^R}{\partial m^2} = \langle F_0^R, F_0^R\rangle + \frac{\partial^2 M_0^2}{\partial m^2}\langle S_0\rangle = \text{finite}. \tag{54}$$

*The equation below can be made valid even with zero external lines provided suitable counterterms are included in the definition of $\langle F_0^R, F_0^R\rangle$ with zero external lines. These counterterms can be generated out of the derivatives of N' of (6), much as in footnote in p. 93. We, however, need this equation only with *nonzero* external lines.

As M_0^2 as a function of m and M has the MS form, so does $\partial^2 M_0^2/\partial m^2$. Hence,

$$\frac{1}{W^R}\frac{\partial^2 W^R}{\partial m^2}=\langle F_0^R, F_0^R\rangle+\frac{\partial^2 M_0^2}{\partial m^2}\langle S_0\rangle=\langle F_0^R, F_0^R\rangle^R. \tag{55}$$

Comparison of (55) with (15) yields

$$\frac{\partial M_0^2}{\partial m^2}=K. \tag{56}$$

Using (52),

$$\frac{\mathrm{d}A(m)}{\mathrm{d}m^2}=K. \tag{57}$$

Thus K is a dimensionless renormalization constant depending only on m and by lemma F, it has smooth limit as $m_0\to 0$. Hence, by lemma I, K is mass independent. Hence (57) implies

$$A(m)=\tfrac{1}{2}Km^2\equiv\tilde{Z}m^2 \tag{58}$$

(Here use is made of the fact that $A(m)$ has dimension two). Equations (52) and (58) now imply that

$$M_0^2=ZM^2+\tilde{Z}m^2$$

with Z and $\tilde{Z}$ mass-independent. This completes the proof.

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