Pramāṇa - J. Phys., Vol. 33, No. 5, November 1989, pp. 581–586. © Printed in India.

On the non-radiating motion of a charged particle

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MS received 6 April 1989

Abstract. The most general class of the electric and magnetic fields such that a charged particle moving in this field will not radiate is obtained. Apart from suitably orientated constant fields, it includes some special varying (but steady) fields.

Keywords. Radiation reaction; Lorentz-Dirac equation; radiation-fields; runaway solution; initial condition.

PACS No. 41·70

1. Introduction

Born (1909) first showed that two charged particles, equal in all respect but with charges of opposite sign, moving in the same straight line with equal but opposite velocities and acceleration, do not radiate. Since then it has attracted the attention of many authors. Milner (1921, 1922) also investigated the same problem. It was shown by Sommerfeld (1910), Schott (1912), Pauli (1921), that Born's case is only a special case of hyperbolic motion in one dimension, (hyperbola in $x-t$ plane). In these investigations, it is shown that no radiation is emitted by the charge, when the motion is hyperbolic, in $x-t$ plane. More precisely, the magnetic field due to such a moving charge is zero, Pauli (1921). They do not discuss about the dynamics of the problem, i.e., what should be the nature of the external electro-magnetic field which results in the hyperbolic motion. Schott (1912) solved the relativistic equation of motion of a charged particle, in the absence of radiation reaction in the field of a uniform electric field and observed that the motion is hyperbolic if the motion is along the lines of force, hence it does not radiate. Later Schott (1915), showed that for linear hyperbolic motion, the radiation reaction is zero, hence it is non-radiating. Further, in a later paper Schott (1923) posed the direct problem, to find the nature of the external field under which an accelerated electron will not radiate, i.e., there is no radiation reaction and found the electric field **E** and magnetic field **H** are constant, such that $\mathbf{E}\cdot\mathbf{H} = 0$ and $|\mathbf{E}| > |\mathbf{H}|$. The motion is in the plane perpendicular to **H**, with initial velocity perpendicular to **E** and **H**, along which it remains constant and the initial acceleration is along **E**. The locus is a hyperbola in the plane of the motion.

In this paper we show that the nature of the electromagnetic field for which the accelerating charge does not radiate is much wider. Schott arrived at the nature of electromagnetic field after solving the motion for which radiation reaction is zero and then suggested the nature of the field. But we use only the qualitative nature of motion in the absence of radiation reaction and find out from the equation of motion the most general nature of the electro-magnetic field.

After a brief discussion on the nature of the equation of motion with radiation reaction in the next section, the main problem is investigated in §§3 and 4. The last section contains a few comments on the results obtained.

2. The equation of motion

The Lorentz-Dirac equation of motion for a charged particle with radiation reaction is

$$\dot{v}^{\mu} - \varepsilon(\ddot{v}^{\mu} + \dot{v}_{\nu}\dot{v}^{\nu}v^{\mu}) = (e/mc)f^{\mu}_{\nu}v^{\nu} \tag{1}$$

where $v^{\nu}v_{\nu} = 1$, and $\varepsilon = \frac{2}{3}(e^2/mc^3)$. Dots denote differentiation with respect of the proper time of the particle τ and $v^{\mu} = (1/c)\dot{x}^{\mu}$. f^{μ}_{ν} is the external electromagnetic field tensor. The radiation reaction is the second term on the left hand side with the coefficient ε. Hence for the non-radiating motion,

$$\ddot{v}^{\mu} + \dot{v}_{\nu}\dot{v}^{\nu}v^{\mu} = 0, \tag{2}$$

in addition to the usual equation of motion

$$\dot{v}^{\mu} = \frac{e}{mc}f^{\mu}_{\nu}v^{\nu}. \tag{3}$$

The problem is to obtain $f^{\mu}_{\nu}(x)$ so that the above two equations are simultaneously satisfied. This is basically the problem considered by Schott (1923) and solved by him, in restricted cases with particular initial condition.

It is important to point out that the equation of motion Schott (1915) considered is exactly the same as the Lorentz-Dirac equation (1). But Schott wrote the space component in the usual three vector notation and the time component separately. He obtained them from the Abraham-Lorentz equation, when terms related to the structure of the particle are not considered. The following interesting observation was made just below the two equations, Schott (1915), 'The curious similarity of form of the last two equations is worthy of remark'. Of course, Dirac arrived at the same equation from a different standpoint [Dirac (1938)].

The equation of motion with radiation reaction (1), is a third order differential equation, hence to obtain its solution it is not sufficient to know the initial position and velocity only. But one must know the initial acceleration also. This was observed by Schott (1915). This is the source of so-called 'runaway solutions', in later period after Dirac's paper. Because a third order equation admits of a wider class of solutions due to the arbitrary choice of the initial acceleration. Schott (1915) heuristically chose 'the initial acceleration to be the external force per unit mass when the velocity of the particle is zero'. The reaction due to radiation, the second term in (1), is small for all practical problems. The equation of motion (1) has two distinct (non-overlapping) classes of solution, namely the solutions which are analytic in ε, i.e., bounded for all finite time as $\varepsilon \to 0$ and the other class of solutions are singular as $\varepsilon \to 0$. Because, $\varepsilon \ll 10^{-23}$ s for electrons and $\varepsilon \ll 10^{-26}$ s for protons, it is but evident that we are interested in solutions which are analytic as $\varepsilon \to 0$. Hence this should be the criteria for choosing the initial acceleration, Sen Gupta (1970, 1971, 1987). The initial acceleration is uniquely determined with this choice, Sen Gupta (1972a). Observing this difficulty with third order equation, Bonnor (1974) in an interesting paper proposed a new equation of motion, which is still second order, but with varying proper mass. However, our present interest is on the Lorentz-Dirac equation, (more appropriately Lorentz-Schott-Dirac equation).

3. The nature of the non-radiating motion

Equation (2) is the condition for absence of radiation reaction. Rather to be more precise following Schott (1915), the absence of total radiation reaction. Because according to Schott, in (2) the first term represents the reversible reaction, (rate of change of acceleration energy), while the second term represents the irreversible reaction of radiation and the condition of no radiation is a subtle balance of the two, (note $\dot{v}^\nu \dot{v}_\nu < 0$).

From (2)

$$\dot{v}^\nu \dot{v}_\nu = -\omega^2 \tag{4}$$

and

$$\dot{v}^\mu v_\nu - v^\mu \dot{v}_\nu = \text{constant (tensor)}. \tag{5}$$

ω is a real positive constant as $\dot{v}$ is space-like, since $\dot{v}^\mu v_\mu = 0$, from (1).

Excluding the trivial case of uniform velocity motion, $\omega \neq 0$, (5) leads to

$$\dot{\mathbf{v}} \times \mathbf{v} = \mathbf{j} \times \text{constant} \tag{6}$$

$$\dot{\mathbf{v}} v_0 - \mathbf{v} \dot{v}_0 = \mathbf{k} K \tag{7}$$

and

$$\omega^2 \mathbf{v} v_0 - \dot{\mathbf{v}} \dot{v}_0 = \mathbf{i}(K^2 - \omega^2)^{1/2} \tag{8}$$

where

$$\mathbf{i} = \mathbf{j} \times \mathbf{k}, \tag{9}$$

i, **j** and **k** are three mutually orthogonal unit vectors. K and ω depend on the initial velocity and acceleration. Equation (8) follows from (6) and (7).

Let us introduce a Lorentz transformation with velocity $\mathbf{i}(K^2 - \omega^2)^{1/2}/K$. In this transformed frame of reference,

$$|K| = \omega \tag{10}$$

and

$$\dot{\mathbf{v}} = \mathbf{k} \omega v_0 \tag{11}$$

$$\mathbf{v} = \mathbf{k} \dot{v}_0 / \omega. \tag{12}$$

Thus in the new frame of reference velocity and acceleration are along **k**, hence the motion is linear. The equation of motion (3), is

$$\dot{\mathbf{v}} = \frac{e}{mc}(v_0 \mathbf{E} + \mathbf{v} \times \mathbf{H}). \tag{13}$$

4. The nature of the electro-magnetic field

From (11–13),

$$\mathbf{E}' + f(z)\mathbf{k} \times \mathbf{H} = 0 \tag{14}$$

where

$$\mathbf{E}' = \mathbf{E} - \frac{mc}{e}\omega - \frac{mc}{e}\omega\mathbf{k} \tag{15}$$

and

$$f(z) = \frac{\dot{v}_0}{v_0\omega}. \tag{16}$$

Since the motion is along $\mathbf{k}$, $\dot{v}_0/v_0$ is a function of $z = \mathbf{k}\cdot\mathbf{r}$ only. The very nature of the problem restricts us to steady fields. So that in addition to the relation (14), $\mathbf{E}$ and $\mathbf{H}$ must also satisfy the Maxwell equations,

$$\nabla \times \mathbf{E} = 0, \quad \nabla\cdot\mathbf{E} = 0 \tag{17}$$

$$\nabla \times \mathbf{H} = \mathbf{J}(r), \quad \nabla\cdot\mathbf{H} = 0 \tag{18}$$

(i) the simplest solution of the problem is constant $\mathbf{E}$ and $\mathbf{H}$ along $\mathbf{k}$, so that $\omega = (e/mc)$ $|\mathbf{E}|\cdot|\mathbf{H}|$ apparently has no effect in the frame in which the initial velocity and acceleration are along the same direction, namely $\mathbf{k}$. The general case with arbitrary orientation of initial velocity and acceleration may be obtained by introducing the inverse Lorentz transformation with velocity along $\mathbf{i}$. The motion is then in the plane of $\mathbf{i} - \mathbf{k}$, i.e., that of the initial velocity and acceleration. The locus in this plane is a hyperbola. The special case when the initial velocity and acceleration are perpendicular was reported by Schott (1923). But the electric and magnetic fields are not necessarily mutually orthogonal for linear motion. Our analysis shows, in general, $\mathbf{E}\cdot\mathbf{H} \neq 0$, excluding the special case for which $\mathbf{H} = 0$. When $|\mathbf{E}| > |\mathbf{H}|$ and both are constants, as in the example of Schott, it can be transformed to electric field $\mathbf{E}$ only and in the transformed frame the motion is linear.

(ii) In the system in which the motion is along the direction $\mathbf{k}$, the general nature of the field which satisfies (15)–(18) may be easily obtained.

$$\mathbf{E}' = -\nabla\phi(x, y), \quad \nabla\cdot\nabla\phi = 0 \tag{19}$$

$$\mathbf{H} = \mathbf{k}g(x, y) + (1/f(z))\nabla\phi \times \mathbf{k} \tag{20}$$

where x, y are co-ordinates in the plane perpendicular $\mathbf{k}$, $g(x, y)$ is any arbitrary function of x and y. Hence, the current which is the source of magnetic field is

$$\mathbf{J} = \nabla g \times \mathbf{k} - \frac{1}{f^2}\frac{\mathrm{d}f}{\mathrm{d}z}\nabla\phi. \tag{21}$$

It has no component along $\mathbf{k}$, but it depends on z. Because of the dependence on z, the current cannot be zero for finite z, in general. However from (3), (11) and (12), it follows that the nature of the motion is such that $f(z) \to 1$, as $\tau \to \infty$, i.e., as $t \to \infty$, so that $g(x, y)$ should be constant if the region along z is not bounded. In which case,

$$\mathbf{J}' = \frac{1}{f^2}\frac{\mathrm{d}f}{\mathrm{d}z}\mathbf{E}' \tag{21'}$$

(iii) The uniform velocity motion appears to be an obvious solution of the problem. Nonetheless, it is interesting in this context, (2) is trivially satisfied and (3) is

$$\mathbf{E} + V\mathbf{k} \times \mathbf{H} = 0, \tag{22}$$

where $\mathbf{k}Vc$ is the constant velocity. It may be pointed out that apart from constant $\mathbf{E}$ and $\mathbf{H}$ satisfying (22), so that $\mathbf{E}$, $\mathbf{H}$ and the direction of velocity are mutually orthogonal and $|\mathbf{E}| = V|\mathbf{H}|$, it has other interesting physically admissible solutions. Let $\phi(x, y)$ and $\psi(x, y)$ be the real and imaginary parts respectively of an analytic function $F(x + iy)$ of $x + iy$. It can be easily checked that (22) is satisfied with

$$\mathbf{E} = -\nabla\phi \tag{23}$$

and

$$\mathbf{H} = (-1/V)\nabla\psi. \tag{24}$$

The field is source free, i.e., satisfies homogeneous Maxwell's equation, and current is zero in finite region. In this case, though the field is non-vanishing but the resultant mechanical force on the charge is zero, due to a subtle balance between the mechanical force on the charged particle due to electric and magnetic fields. It may be mentioned that for the asymptotic motion mentioned in the previous subsection (ii) the field configuration belongs to this category, but with $V = 1$.

5. Comments

In the conclusion, we first want to point out that (1) with constant $\mathbf{E}$ can be exactly solved when the initial velocity is along $\mathbf{E}$. The initial acceleration is to be determined by the criteria that the kinematical quantities are bounded for all finite time as $\varepsilon \to 0$. The resultant motion is such that (2) and (3) are separately satisfied. Hence for this motion the radiation reaction vanishes.

Next, according to Schott's hypothesis the initial acceleration is the mechanical force per unit mass when velocity is zero. By this choice the role of the magnetic field on the choice of the initial acceleration is not taken into consideration. On the other hand if one restricts to bounded solutions for $\varepsilon \to 0$, the initial acceleration is to be found from (13), such that it is the Lorentz force per unit mass, Sen Gupta (1972a). Further, Schott posed the problem to find the electric and magnetic field such that the motion is non-radiating. But according to his hypothesis to obtain the initial acceleration one should know at least the electric field. However intuitively he chose the initial acceleration such that the hypothesis is satisfied posteriorly.

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