Pramāṇa - J. Phys., Vol. 32, No. 4, October 1989, pp. 449–454. © Printed in India.

Spin precession of a charged particle in a uniform magnetic field on a static space-time

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MS received 7 February 1989; revised 24 April 1989

Abstract. We investigate the ratio of spin precession frequency to orbital frequency for a spinning charged particle confined to circular orbit in the equatorial plane of a compact object, with a uniform magnetic field, as described by the Wald and the Ernst potentials. In order to see the difference in behaviours for particles with different g values we consider the cases of electron and proton separately.

Keywords. Spin precession frequency; charged particle; uniform magnetic field; curved space-time.

PACS Nos 04·40; 41·70; 97·60

1. Introduction

One of the significant aspects of astrophysical phenomena is the role of magnetic fields in the dynamics of charged particles which govern the electrodynamical features leading to different kinds of emission mechanisms. It is hardly necessary to emphasize the usefulness of a detailed understanding of the dynamical features of charged particles in the magnetic fields surrounding compact objects particularly when the particles also possess spin. The orbit theory of single charged particles without spin, in magnetic fields on a curved space time was considered, a comprehensive review of which is presented in Prasanna (1980). It is in fact known that almost all charged particles possess spin and thus it is desirable to understand spin precession (due to interaction with magnetic fields) in a curved space-time particularly since this could have a bearing on certain emission features. Though at this stage it is not completely clear as to the role of variation of the ratio of spin frequency to orbital frequency from the usual flat space values, one does find that on curved space time, the dynamical role of the gravitational field influencing the magnetic field has an effect on this ratio.

Prasanna and Kumar (1973) attempted a very limited analysis of the spin precession of a charged particle in Melvin's magnetic universe, wherein due to the complete negligence of the spin-orbit coupling terms no quantitative effect was seen. Recently, Prasanna and Virbhadra (1988) considered this problem and found that the ratio of spin frequency to orbital frequency of a charged particle in a dipole magnetic field, superposed on Schwarzschild geometry does get altered from the classical value, when the particle is very close to the compact object.

In the following, these discussions are extended to the case of a spinning charged particle in a uniform magnetic field as given by the Wald solution (painted field on

Schwarzschild background) and by the Ernst solution (wherein the space time metric is changed due to the magnetic field).

2. Formalism

The general equations governing the motion of a spinning charged particle in an electromagnetic field in curved space time are obtained following the treatments of Anderson (1967) and Papapetrou (1951), and are given by, for orbit

$$\frac{D}{D\tau}\left(m_0 u^i + \frac{DS^{ik}}{D\tau}u_k\right) + \tfrac{1}{2}S^{ab}u^c R^i{}_{cab} = eF_k{}^i u^k \tag{1}$$

and for spin

$$\frac{DS^{ik}}{D\tau} + u^i u_l \frac{DS^{kl}}{D\tau} - u^k u_l \frac{DS^{il}}{D\tau} = g\frac{e}{2m_0}(S^{kl}F^i{}_l - S^{il}F^k{}_l), \tag{2}$$

which satisfy the Pirani constraint

$$S^{ik}u_k = 0, \tag{3}$$

g being the gyromagnetic ratio also known as Lande factor. It is useful to re-express the spin equation in terms of the spin four vector

$$S^i = \tfrac{1}{2}\varepsilon^{ijkl}S_{jk}u_l \tag{4}$$

as given by

$$\frac{DS^i}{D\tau} = -S^k u^i \frac{DU_k}{D\tau} + g\frac{e}{2m_0}(F_j{}^i - F_j{}^k u^i u_k)S^j. \tag{5}$$

In order to see the different contributions to spin precession, one could regroup the terms and also use (1) in (5) and rewrite the orbit and spin equations in the form:

$$\frac{du^i}{d\tau} = -\Gamma^i_{jk}u^j u^k - \frac{e}{m_0}F^i_k u^k + \Sigma^i \tag{6}$$

$$\frac{ds^i}{d\tau} = -(\Gamma^i_{jk}u^k + u^i\Sigma_j)S^j + \frac{e}{2m_0}[gF_j{}^i - (g-2)F_{jk}u^i u^k]S^j, \tag{7}$$

with Σ^i representing the major spin-orbit curvature coupling

$$m_0\Sigma^i = -\frac{D}{D\tau}\left(\frac{DS^{ik}}{D\tau}u_k\right) - \tfrac{1}{2}S^{ab}u^c R^i{}_{cab}, \tag{8}$$

wherein one can recognize the geodetic precession term and the generalized Thomas precession term apart from new spin-curvature interaction terms.

As we are looking for the effects of spin precession in a uniform magnetic field around compact objects, it is necessary to consider both the possibilities that the field is weak and thus does not affect the background space-time governed by the gravitational field of the central object and the case when the field is strong enough to change the background space-time. In literature there already exist solutions covering both situations, one given by Wald (1974) for the test field and the other by Ernst (1976) for the case of the Einstein-Maxwell equations. We shall now consider the orbit and spin equations (under certain approximations) for both these situations and look for a comparison of the effects. However as we are presently looking for the effects of spin precession only, we will not consider the effects of spin terms in the orbit equations and further in the first approximation assume that the effects of spin-curvature interaction term is small enough to be negligible. With this the equations reduce to

$$\frac{\mathrm{d}u^i}{\mathrm{d}\tau} = -\Gamma^i_{jk}u^j u^k + \frac{e}{m_0}F_k{}^i u^k \tag{9}$$

and

$$\frac{\mathrm{d}s^i}{\mathrm{d}\tau} = -(\Gamma^i_{jk}s^j u^k) + \frac{e}{2m_0}[gF_j{}^i - (g-2)F_{jk}u^i u^k]s^j. \tag{10}$$

As said earlier, we consider the Schwarzschild spacetime

$$\mathrm{d}\tau^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{11}$$

alongwith the potential uniform magnetic field given by $A = (0, 0, \frac{1}{2}\mathrm{B}r^2, 0)$, the Wald potential and the Ernst spacetime

$$\mathrm{d}\tau^2 = \Lambda^2\left[\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2\right]$$

$$- \Lambda^{-2}r^2\sin^2\theta\,\mathrm{d}\phi^2, \tag{12}$$

with $\Lambda = 1 + Br^2\sin^2\theta$ and its corresponding vector potential $(0, 0, Br^2/(1 + r^2B^2\sin^2\theta), 0)$ for evaluating the ratio of the spin precession frequency to orbital frequency.

Presently we shall restrict our attention to the case of a particle confined to the equatorial plane ($\theta = \pi/2$, $u^\theta = 0$) and having a circular orbit ($u^r = 0$, $r = R$ constant). Thus we have the constraint equation for the orbital frequency u^ϕ

$$(\Gamma^r_{\phi\phi} - \Gamma^r_{tt}g^{tt}g_{\phi\phi})(u^\phi)^2 + \frac{e}{m_0}F^r_\phi u^\phi + \Gamma^r_{tt}g^{tt} = 0. \tag{13}$$

Rewriting (13) for the respective background geometries one has for the Wald solution

$$(R - 3m)(u^\phi)^2 + \frac{eBR}{m_0}\left(1 - \frac{2m}{R}\right)u^\phi - \frac{m}{R^2} = 0, \tag{14}$$

and for the Ernst solution

$$[(R - 3m) - (\Lambda - 1)(3R - 5m)](u^\phi)^2 + \frac{2\Lambda eB}{m_0}(R - 2m)u^\phi$$

$$- \Lambda^2\left[\frac{m}{R^2} + \frac{2(\Lambda - 1)}{R}\left(1 - \frac{3m}{2R}\right)\right] = 0. \tag{15}$$

The spin equations with the same restrictions, $\theta=\pi/2$, $u^\theta=0$, $u^r=0$, $r=R$ alongwith the Pirani constraint (3) yield

$$\frac{\mathrm{d}S^r}{\mathrm{d}\tau}=\left[\left(-\Gamma^r_{\phi\phi}+\frac{g_{\phi\phi}}{g_{tt}}\Gamma^r_{tt}\right)u^\phi-\frac{ge}{2m_0}g^{rr}\frac{\partial A_\phi}{\partial r}\right]_{r=R}S^\phi \tag{16}$$

$$\frac{\mathrm{d}S^\phi}{\mathrm{d}\tau}=\left[-\Gamma^\phi_{r\phi}u^\phi+\frac{ge}{2m_0}g^{\phi\phi}\frac{\partial A_\phi}{\partial r}-\frac{(g-2)e}{2m_0}\frac{\partial A_\phi}{\partial r}(u^\phi)^2\right]_{r=R}S^r \tag{17}$$

and

$$\frac{\mathrm{d}S^t}{\mathrm{d}\tau}=\left[-\Gamma^t{}_{rt}-\frac{e(g-2)}{2m_0}\frac{\partial A_\phi}{\partial r}u^\phi\right]_{r=R}u^tS^r. \tag{18}$$

Introducing cartesian coordinates x and y instead of r and ϕ for the representation of the spin vector, one gets

$$\begin{aligned}\frac{\mathrm{d}S^X}{\mathrm{d}\tau}=\frac{S^Y}{R}\Bigg[&u^\phi(g_{\phi\phi}g^{tt}\Gamma^r_{tt}-\Gamma^r_{\phi\phi}-R)-\frac{\partial A_\phi}{\partial r}g\frac{e}{2m_0}g^{rr}\\ &-\sin^2\theta\Bigg\{u^\phi(g_{\phi\phi}g^{tt}\Gamma^r_{tt}-\Gamma^r_{\phi\phi}-R^2\Gamma^\phi_{r\phi})\\ &-\frac{e}{2m_0}\frac{\partial A_\phi}{\partial r}[g(g^{rr}-R^2g^{\phi\phi})+(g-2)(Ru^\phi)^2]\Bigg\}\Bigg]\\ &+\frac{\cos\phi\sin\phi}{R}S^x[u^\phi(R^2\Gamma^\phi_{r\phi}-g_{\phi\phi}g^{tt}\Gamma^r_{tt}+\Gamma^r_{\phi\phi})\\ &+\frac{e}{2m_0}\frac{\partial A_\phi}{\partial r}\{g(g^{rr}-R^2g^{\phi\phi})+(Ru^\phi)^2(g-2)\}]\end{aligned} \tag{19}$$

and

$$\begin{aligned}\frac{\mathrm{d}S^Y}{\mathrm{d}\tau}=-\frac{\sin\phi\cos\phi}{R}S^Y\Bigg[&(R^2\Gamma^\phi_{r\phi}+\Gamma^r_{\phi\phi}-\Gamma^r_{tt}g_{\phi\phi}g^{tt})u^\phi\\ &+\frac{e}{2m_0}\frac{\partial A_\phi}{\partial r}\{g(g^{rr}-R^2g^{\phi\phi})+(g-2)(Ru^\phi)^2\}\Bigg]\\ &+\frac{S^x}{R}\Bigg[\sin^2\phi\Bigg\{\frac{\partial A_\phi}{\partial r}[g(g^{rr}-R^2g^{\phi\phi})+(g-2)(Ru^\phi)^2]\\ &+u^\phi\{R^2\Gamma^\phi_{r\phi}+\Gamma^r_{\phi\phi}-g_{\phi\phi}g^{tt}\Gamma^r_{tt}\}\Bigg\}\\ &+R^2\left\{\frac{\partial A_\phi}{\partial r}\frac{e}{2m_0}[g(g^{\phi\phi}-(g-2)(u^\phi)^2]-u^\phi\left(\Gamma^\phi_{r\phi}-\frac{1}{R}\right)\right\}\Bigg].\end{aligned} \tag{20}$$

3. Discussion

If we now consider the spin frequency to be ω_S by taking Fourier transforms one can write the expression for ω_S from (19) and (20), which for the Wald case is given by

$$\omega_S^2 = \frac{eB}{4m_0^2 R}[\{g + (g-2)(Ru^\phi)^2\}\{ge(R-2m)B - 6m_0 mu^\phi\}] \tag{21}$$

while for the Ernst case, one gets:

$$\omega_S^2 = \frac{1}{m_0^2 R\Lambda^7}[(geB)^2\Lambda^3(R-2m) - geB\Lambda^2 m_0 u^\phi\{R(\Lambda^S + 5\Lambda - 6) + m(12-9\Lambda)\} + (u^\phi)^2\Lambda(\Lambda-1)\{m_0^2 R(2\Lambda^5 + 6\Lambda - 8) + m_0^2 m(16-10\Lambda) + g(g-2)e^2(R-2m)\} - (g-2)m_0 eBR^2(u^\phi)^3\{m(8-5\Lambda) + R(\Lambda^S + 3\Lambda - 4)\}]. \tag{22}$$

Solving for u^ϕ from (14) for Wald potential and from (15) for the Ernst potential one can evaluate the ratio ω_S/u^ϕ for two cases.

Figures 1 and 2 give the plots for both the cases for different values of the magnetic

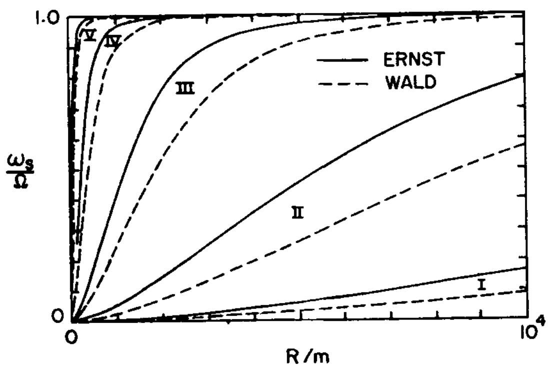

Figure 1. The ratio of the spin precession frequency to the orbital frequency vs. the radial distance of the test particle electron ($g = 2{\cdot}0023$) from the central object (in terms of mass of the central object) for magnetic fields $B = 10^{-9}$(I), 10^{-8}(II), 10^{-7}(III), 10^{-6} (IV), 10^{-5} (V)G for Ernst and Wald cases.

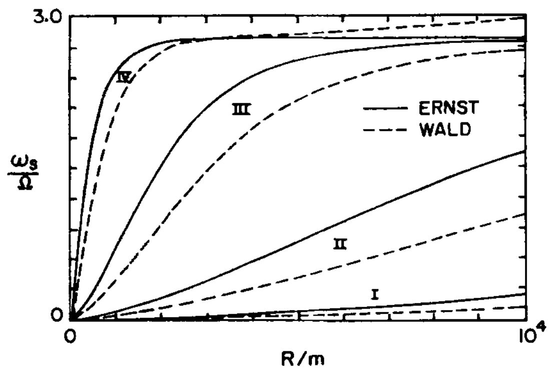

Figure 2. The ratio of the spin precession frequency to the orbital frequency vs. the radial distance of the test particle proton ($g = 5{\cdot}59$) from the central object (in terms of mass of the central object) for magnetic fields $B = 10^{-6}$ (I), 10^{-5} (II), 10^{-4} (III), 10^{-3} (IV) G for Ernst and Wald cases.

field for electron and proton separately. In the case of proton as $g = 5{\cdot}59$, the difference in the ratio from the point of view of asymptotic value is clearly brought out. Wald solution being for a painted magnetic field on Schwarzschild background, one can take the asymptotic limit $m/R \to 0$ for which the ratio of frequencies is given by

$$\frac{\omega_S}{u^\phi} = \frac{g}{2}\left[1 + \left(1 - \frac{2}{g}\right)(Ru^\phi)^2\right]^{1/2}. \tag{23}$$

Obviously for electron with $g = 2{\cdot}0023$ the ratio is approximately equal to $g/2$, whereas for the particles like proton where 'g' value differs from 2 considerably, the expression differs appreciably from $g/2$ by quantity proportional to $v^2/2c^2$. The origin of importance of $(g-2)$ value of a particle is purely relativistic as has been discussed by Bargmann *et al* (1959) and Anderson (1967), in the context of special relativity. On the other hand, in Ernst potential one cannot have a meaningful limit of $m/R \to 0$ as Ernst space-time is not asymptotically flat. However, it *appears* from the curves that the ratio ω_S/u^ϕ reaches a saturation value of approximately equal to $g/2$.

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