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Phase-function method for velocity-dependent potentials

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Abstract. We have adapted the phase-function method for studying on- and off-shell properties of velocity-dependent potentials. The main result presented in this paper is an ansatz for the interpolating T-matrix function (on- or off- the energy-shell as the case may). Based on this ansatz we have presented an efficient method for computing the off-shell extension function which plays a role in the theories of three particle system. We have demonstrated this by means of a model calculation.

Keywords. Nuclear reactions; scattering theory; phase function method; velocity dependent potential; off-shell amplitudes.

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1. Introduction

The phase-function method (PFM) (Calogero 1967) provides an alternative approach to traditional quantum theory of scattering and is particularly useful for dealing with scattering on a class of potentials for which the Schrödinger equation does not admit straightforward analytical solutions. The velocity-dependent potential (VDP) (Green *et al* 1968) serves as a typical example of this class. Although the efficiency of the PFM and the usefulness of the VDP were realised almost simultaneously (Calogero 1967; Green *et al* 1968), very little attention has been paid to treat the VDP by the PFM. Therefore, it is of considerable interest to adapt the PFM for studying scattering on velocity-dependent potentials. The present paper is an effort in this direction.

The mathematical foundation of the PFM is well known in the theory of differential equations. For example, a linear homogeneous equation of the second order like the Schrödinger equation can be reduced to a first order nonlinear equation of the Riccati type. The function which satisfies the Riccati equation is called the phase function and has at each point the meaning of the phase shift for scattering by the potential cut off at that point. Accordingly, the problem reduces to direct determination of the desired scattering phase shift.

One can also introduce interpolating functions corresponding to other physical quantities like the partial wave scattering amplitudes. The intutive meaning of the phase function and of other related interpolating functions enables one to see the effect of the action of different regions of the potential in producing the corresponding observables. The importance of such physical intuition can hardly be overestimated; it is certainly very helpful in most problems of physical interest. By treating the velocity-dependent potential within the framework of PFM one can therefore expect

to attain added realism and sophistication which are not obvious in the traditional methods for computing phase shifts and scattering amplitudes.

We begin by considering the potential of the form (Herndon *et al* 1963)

$$U(\mathbf{r},\mathbf{p}) = -K_1 \exp(-r^2/a^2) - K_2[\nabla^2 \exp(-r^2/b^2) + \exp(-r^2/b^2)\nabla^2], \quad (1)$$

with a, b, K_1 and K_2 as energy independent parameters. More than twenty years ago McKellar and May (1965) employed the PFM to deal with the elastic or on-energy-shell scattering by this potential. In a many particle system various pairs of particles do not scatter elastically from each other and therefore knowledge of two particle on-shell scattering amplitudes is not sufficient to determine the properties of the many particle system. We need the so called off-shell amplitudes (Watson and Nuttall 1967). In view of this we develop a generalization of the PFM for the VDP which, on the one hand, reproduces the result of McKellar and May and, on the other hand, allows one to calculate the off-shell transition amplitudes.

In §2 we present the algorithms of the traditional PFM by introducing an ansatz for the interpolating on-energy-shell T-matrix function which arises naturally within the framework of our approach. We seek an off-energy-shell extension of the PFM in §3 and finally, we deal with a case study in §4 and examine how the algorithms of our combined variable phase off-shell scattering theory for the VDP describe the process of accumulation of the phase shift and transition matrices due to the potential. Here we also present some results which are of interest to few-body physicists.

2. Phase function method (PFM)

For clarity of presentation we shall treat only the s-wave scattering by the potential in (1) and omit the subscript $l=0$ for the radial wave function $u_l(k,r)$. In this case the radial Schrödinger equation is given by

$$\left(\frac{d^2}{dr^2} + k^2\right) u(k,r) = \left(V^{(A)}(k,r) + V^{(B)}(k,r)\frac{d}{dr}\right) u(k,r). \quad (2)$$

Here k represents the two body centre of mass momentum or wave number, and the quantities $V^{(A)}(k,r)$ and $V^{(B)}(k,r)$ are written as

$$V^{(A)}(k,r) = \frac{1}{[1+g_1(r)]}\left[k^2 g_1(r) - g_2(r) + \frac{1}{b^2} g_1(r) - \frac{2r^2}{b^4} g_1(r)\right], \quad (3a)$$

and

$$V^{(B)}(k,r) = \frac{1}{[1+g_1(r)]b^2}\, 2r g_1(r), \quad (3b)$$

with

$$g_1(r) = 2K_2 \exp(-r^2/b^2), \quad (4a)$$

and

$$g_2(r) = K_1 \exp(-r^2/a^2). \quad (4b)$$

The PFM is based on the separation of the radial wave function $u(k,r)$ into an amplitude part $\alpha(k,r)$ and an oscillating part with a variable phase $\delta(k,r)$. Physically, this amounts to factorizing out the two effects of the potential which manifest

themselves by deforming the wave function and by producing the scattering phases. In the PFM the phase shift $\delta(k)$ is obtained from the phase function $\delta(k,r)$ using the limiting condition

$$\delta(k) = \lim_{r\to\infty} \delta(k,r). \tag{5}$$

Thus, the phase function $\delta(k,r)$ at a distance r from the origin is just the scattering phase shift produced by the potential $V(r)$, if all parts extending beyond r were removed. The amplitude function $\alpha(k,r)$ is a measure of focussing or defocussing of the projectile by the potential field (Fano *et al* 1976) and is related to the modulous of the Jost function produced by the potential truncated at r. The amplitude function $\alpha(k,r)$ should not be confused with the scattering amplitude or the T-matrix function which will be introduced presently.

The PFM proceeds by an ansatz for the wave function $u(k,r)$ accompanied by a constraint imposed through its derivative. These ansatz and constraint are given by (Fano *et al* 1976)

$$u(k,r) = \alpha(k,r)\sin(kr + \delta(k,r)), \tag{6a}$$

and

$$u'(k,r) = k\alpha(k,r)\cos(kr + \delta(k,r)). \tag{6b}$$

Here prime denotes differentiation with respect to r.

Let us define an interpolating on-shell T-matrix function by (Calogero 1967)

$$T(k,r) = -\frac{2}{\pi k}\sin\delta(k,r)\exp[i\delta(k,r)], \tag{7}$$

with $T(k,0) = 0$ and $T(k,\infty) = T(k)$, the elastic or on-shell T-matrix. As with the regular wave function, the outgoing or physical wave function for a potential truncated at r is given by (Talukdar *et al* 1981)

$$\psi^{(+)}(k,r) = \sin kr - \tfrac{1}{2}\pi k T(k,r)\exp(ikr). \tag{8}$$

We now venture to suggest that the relation

$$T'(k,r) = \frac{2}{\pi k^2}\alpha^{-1}(k,r)\exp(i\delta(k,r))\psi^{(+)}(k,r)V_{\text{eff}}(k,r)u(k,r), \tag{9}$$

with

$$V_{\text{eff}}(k,r) = V^{(A)}(k,r) + V^{(B)}(k,r)\frac{\mathrm{d}}{\mathrm{d}r}, \tag{10}$$

leads to a first order non-linear differential equation for $T(k,r)$. From (6a), (6b), (8), (9) and (10) we have

$$\begin{aligned} T'(k,r) = \frac{2}{\pi k^2}&\{\sin kr - \tfrac{1}{2}\pi k T(k,r)\exp(ikr)\} \\ &\times [V^{(A)}(k,r)\{\sin kr - \tfrac{1}{2}\pi k T(k,r)\exp(ikr)\} \\ &+ kV^{(B)}(k,r)\{\cos kr - \tfrac{1}{2}i\pi k T(k,r)\exp(ikr)\}]. \end{aligned} \tag{11}$$

Using (7) we obtain a differential equation for $\delta(k,r)$ in the form

$$\delta'(k,r) = -k^{-1}\{V^{(A)}(k,r)\sin(kr+\delta(k,r)) + kV^{(B)}(k,r) \times \cos(kr+\delta(k,r))\}\sin(kr+\delta(k,r)), \tag{12}$$

with $\delta(k,0)=0$. Equation (12), which represents the so called phase equation for the VDP, was derived by McKellar and May (1965) by using a Green's function approach to the problem. Thus we see that our ansatz for the T-matrix function introduced through (9) yields in a rather natural way the well known phase equation for the VDP.

For scattering on short-range potentials, the phase shift $\delta(k)$ and consequently also $\tan\delta(k)$ are odd functions which are regular at the point $k=0$. In this case $k^{-1}\tan\delta(k)$ can be expanded in power series in k (Green *et al* 1968), whose first coefficients completely determine scattering at low energies. A PFM for calculating these coefficients has been developed by Levy and Keller (1963). Here one represents $\tan\delta(k,r)$ as

$$\tan\delta(k,r) = -k^{-1}[c(r)+k^2\,\mathrm{d}(r)+k^4e(r)+\cdots]. \tag{13}$$

The quantities $c(r)$ and $d(r)$ are related to the scattering length a_s and the effective range r_e by the formulas

$$a_s = \lim_{r\to\infty} c(r) \tag{14a}$$

and

$$r_e = \lim_{r\to\infty} \frac{2d(r)}{c^2(r)} \tag{14b}$$

and satisfy the initial conditions $c(0)=0$ and $d(0)=0$. From (12) and (13) we have found the non-linear differential equations for $c(r)$ and $d(r)$ in the following forms

$$c'(r) = \frac{1}{[1+g_1(r)]}\left[\left\{-g_2(r)+\frac{1}{b^2}g_1(r)-\frac{2r^2}{b^4}g_1(r)\right\} \times (r-c(r))^2+\frac{2r}{b^2}(r-c(r))g_1(r)\right] \tag{15}$$

and

$$d'(r) = \frac{1}{[1+g_1(r)]}\left[\{r^2-2rc(r)+c^2(r)\}g_1(r) + \left\{-g_2(r)+\frac{1}{b^2}g_1(r)-\frac{2r^2}{b^4}g_1(r)\right\} \times \left\{-\frac{r^4}{3}+\frac{4r^3}{3}c(r)-r^2c^2(r)-2rd(r)+2c(r)d(r)\right\} + \frac{2r}{b^2}g_1(r)\left\{-\tfrac{2}{3}r^3+2r^2c(r)-rc^2(r)-d(r)\right\}\right]. \tag{16}$$

The solution of (15) as $r\to\infty$ determines the value of the scattering length. As opposed to this, (15) and (16) should be solved simultaneously to get the value of r_e.

3. Off-shell extension of the PFM

Another important element of (9) is that it can easily be continued off-the energy shell. To see this we proceed as follows.

The interpolating half-shell T-matrix function $T(k,q,k^2,r)$ satisfies the boundary conditions (Talukdar *et al* 1981) $T(k,q,k^2,0)=0$ and $T(k,q,k^2,\infty)=T(k,q,k^2)$, the half-off-shell T-matrix. Here q is an off-shell momentum. With (9) we now introduce

$$T'(k,q,k^2,r)=\frac{2}{\pi kq}\alpha^{-1}(k,r)\exp[i\delta(k,r)]\psi^{(+)}(k,q,r)\times V_{\text{eff}}(k,r)u(k,r). \quad (17)$$

In (17) $\psi^{(+)}(k,q,r)$ stands for the outgoing off-shell wave function for the potential truncated at the point r and is given by (Talukdar *et al* 1981)

$$\psi(k,q,r)=\sin qr-\tfrac{1}{2}\pi qT(k,q,k^2,r)\exp(ikr). \quad (18)$$

Equations (6a), (6b), (10), (17) and (18) can be combined to get

$$\begin{aligned}T'(k,q,k^2,r)=\frac{2}{\pi kq}&\{\sin qr-\tfrac{1}{2}\pi qT(k,q,k^2,r)\exp(ikr)\}\\&\times\Big[V^{(A)}(k,r)\{\sin kr-\tfrac{1}{2}\pi kT(k,r)\exp(ikr)\}\\&+kV^{(B)}(k,r)\Big\{\cos kr-\frac{i}{2}\pi kT(k,r)\exp(ikr)\Big\}\Big].\end{aligned} \quad (19)$$

It is of interest to note that (19) represents a linear differential equation, but it is coupled to the non-linear equation (11).

In the context of combined variable phase off-shell scattering theory there appears a quasi-phase function (Sobel 1968) $\Delta(k,q,r)$, which is related to the interpolating half-shell T-matrix function by

$$T(k,q,k^2,r)=-\frac{2}{\pi k}\Delta(k,q,r)\exp[i\delta(k,r)], \quad (20)$$

and satisfies the boundary conditions $\Delta(k,q,0)=0$ and $\Delta(k,q,\infty)=\Delta(k,q)$, the quasi phase. Using (7) and (20) in (19) and separating the real and imaginary parts of the resulting equations we get

$$\begin{aligned}\Delta'(k,q,r)=&-[q^{-1}\sin qr+k^{-1}\Delta(k,q,r)\cos(kr+\delta(k,r))]\\&\times[V^{(A)}(k,r)\sin(kr+\delta(k,r))+kV^{(B)}(k,r)\cos(kr+\delta(k,r))],\end{aligned} \quad (21)$$

and the phase equation. Interestingly,

$$\Delta(k,q,r)\underset{q\to k}{\sim}\sin\delta(k,r),\quad \Delta(k,q)\underset{q\to k}{\sim}\sin\delta(k)$$

and $T(k,q,k^2)$ in terms of $\Delta(k,q)$ is given by

$$T(k,q,k^2)=-\frac{2}{\pi k}\Delta(k,q)\exp[i\delta(k)]. \quad (22)$$

The off-shell extension function (Noyes 1965) $F(k, q)$ which plays a role in the theories of three-particle scattering is related to the quasi phase by (Talukdar *et al* 1983)

$$F(k, q) = \Delta(k, q)/\sin \delta(k). \tag{23}$$

At a given on-shell energy the off-shell properties of a potential are often studied (Haidenbauer *et al* 1984) in terms of the behaviour of $F(k, q)$ as a function of q.

4. Results and discussion

Herndon *et al* (1963) and, Derewych and Green (1967) claim that the potential in (1) leads not only to good $^1S_0$ and $^3S_1$ $n-p$ scattering phases but also serves quite well in three-and four-body problems. In our case study we have chosen to work with $a^{-2} = 0{\cdot}35$ fm^{-2}, $b^{-2} = 2{\cdot}2$ fm^{-2}, $K_1 = 0{\cdot}8392$ fm^{-2} and $K_2 = 0.9$.

We have solved (12), (15) and (16) by the Runge Kutta method with an appropriate stability check. Based on the constant asymptotic values of $c(r)$ and $d(r)$ we have calculated the values of scattering length and effective range. For a_s and r_e we found $a_s = -16{\cdot}7242$ fm and $r_e = 2{\cdot}7684$ fm. These results are in exact agreement with those of Herndon *et al* (1963).

The simple way in which the potential enters the phase equation (12) provides an opportunity to learn how phases are accumulated by the potential to produce the scattering phase shift. To see this we have plotted in figure 1 the quantities $V^{(A)}(k, r)$, $V^{(B)}(k, r)$ and the phase function $\delta(k, r)$ as a function of r at $E_{lab} = 20$ MeV. The curves

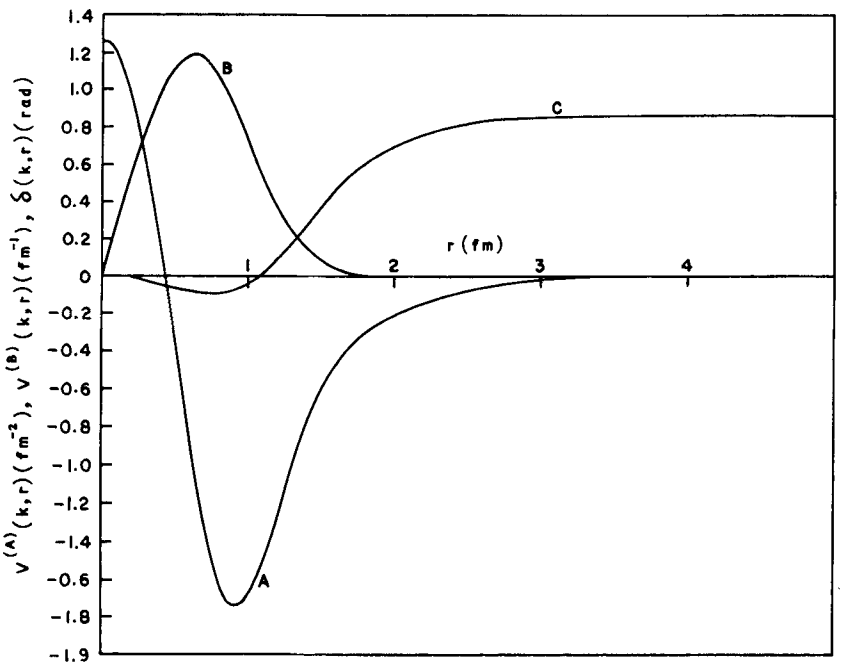

Figure 1. Quantities $V^{(A)}(k, r)$, $V^{(B)}(k, r)$ of (3a) and (3b), and the phase function $\delta(k, r)$ as a function of r at $E_{lab} = 20$ MeV.

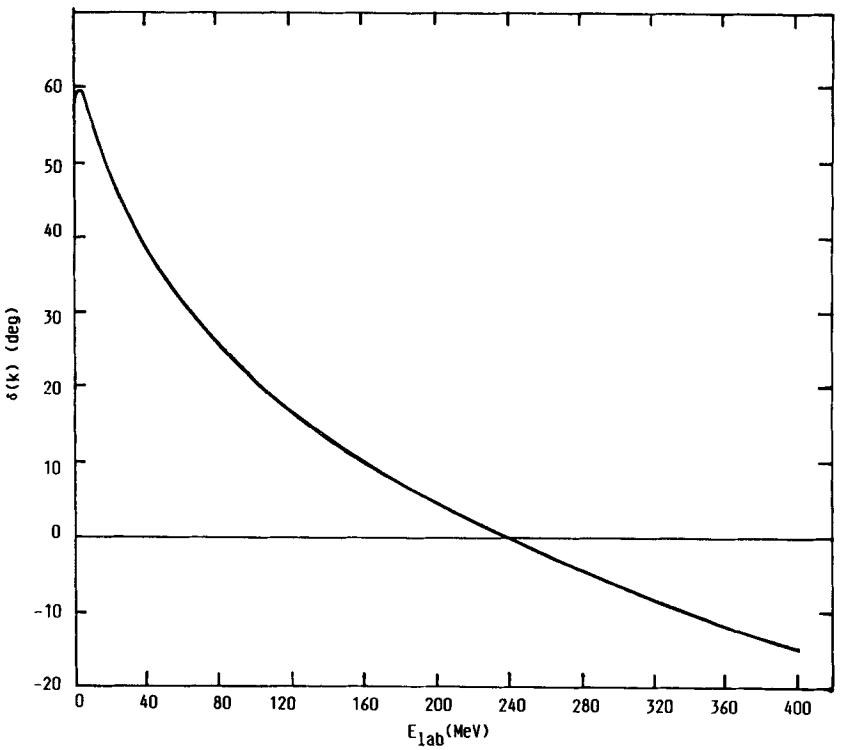

Figure 2. Phase shift $\delta(k)$ as a function of E_{lab}.

A, B and C represent the spatial variation of $V^{(A)}(k,r)$, $V^{(B)}(k,r)$ and $\delta(k,r)$ respectively. The quantity $V^{(A)}(k,r)$ has a repulsive soft core for $r < 0{\cdot}43$ fm and is attractive elsewhere. In contrast to this $V^{(B)}(k,r)$ is altogether repulsive. Looking at curve C we see that $\delta(k,r)$ starts from the origin, goes through a minimum, and then increases smoothly to take up a positive saturation value $\delta(k)$ beyond $r = 3$ fm. This behaviour of $\delta(k,r)$ may be attributed to the following.

As we move away from the origin the repulsive cores of $V^{(A)}(k,r)$ and $V^{(B)}(k,r)$ give negative contributions to the phase function. The attractive part of $V^{(A)}(k,r)$ then operates and completely absorbs the negative phase around $r = 1{\cdot}1$ fm. Since we are not yet out of the range of the potential, the residual part of $V^{(A)}(k,r)$ goes on giving positive contribution to $\delta(k,r)$ until it reaches its asymptotic value $\delta(k) = 0{\cdot}857$ rad.

Solving (12) we have computed $\delta(k)$ for $E_{lab} = 1 - 400$ MeV in steps of 1 MeV. In figure 2 we have plotted $\delta(k)$ as a function of E_{lab}. As expected $\delta(k)$ exhibits a maximum at $E_{lab} = 4$ MeV and decreases as one moves to higher laboratory energies. The phase shift changes sign at $E_{lab} = 240$ MeV. It is of interest to note that our calculated values of $\delta(k)$ are in excellant agreement with those of Haidenbour and Plessas (1984) calculated from a separable representation of the Paris potential.

To examine the off-shell properties of the potential in (1) we have calculated the half-shell T-matrix $T(k,q,k^2)$ and off-shell extension function $F(k,q)$ by using the algorithms in (19)–(23). Figure 3 displays how the potential contributes to the interpolating function $T(k,q,k^2,r)$ at $E_{lab} = 20$ MeV to build up $T(k,q,k^2)$. Curves E and F represent the variation of real and imaginary parts of $T(k,q,k^2,r)$ for

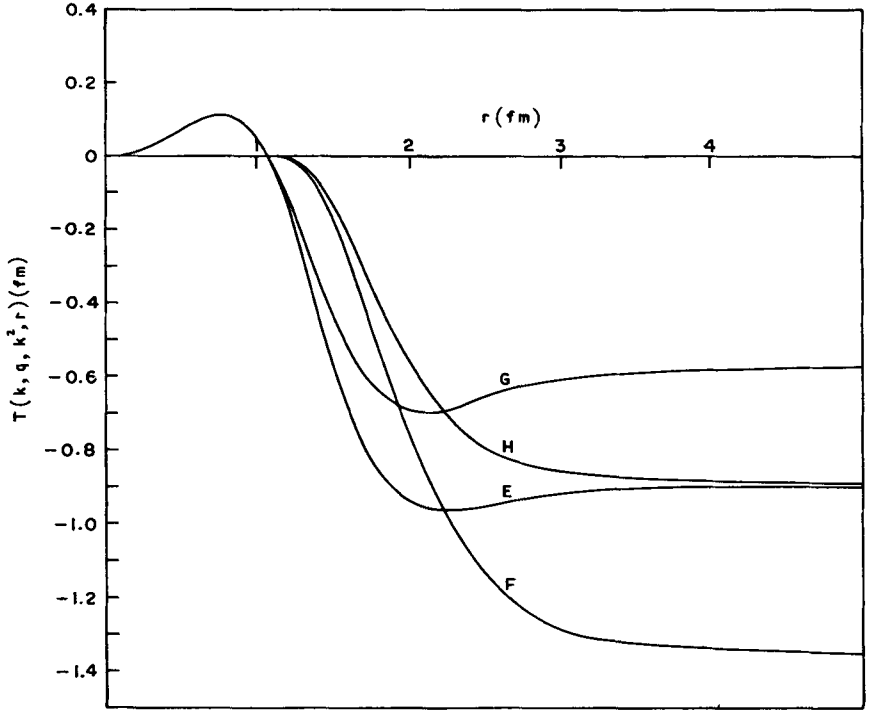

Figure 3. Half-shell T-matrix function $T(k, q, k^2, r)$ as a function of r.

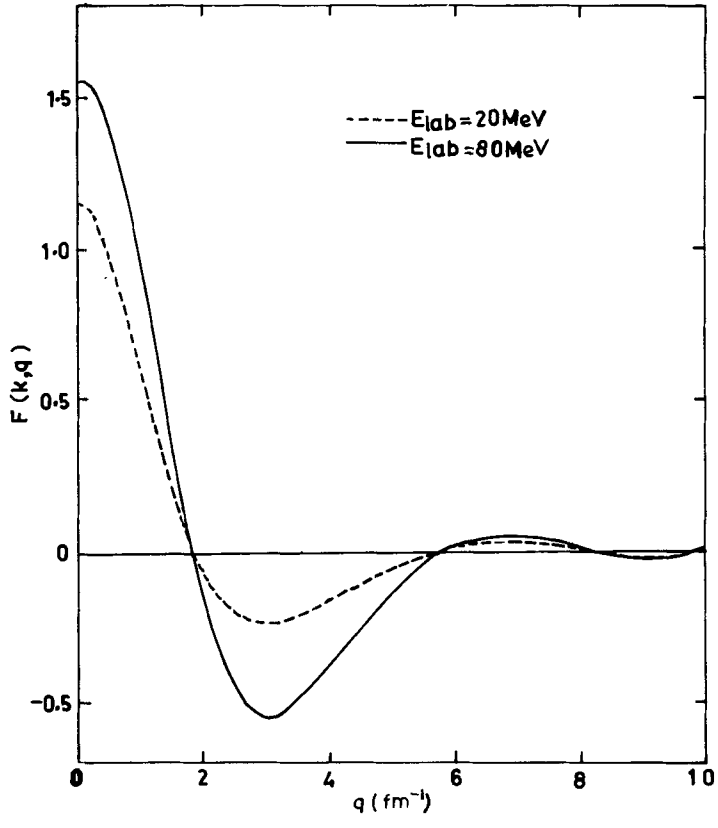

Figure 4. Off-shell extension function $F(k, q)$ at two different laboratory energies as a function of the off-shell momentum q.

$q = k/2\,\text{fm}^{-1}$. We have shown similar results for $q = 2k\,\text{fm}^{-1}$ by the curves G and H respectively. From these curves it is clear that for small values of r the real parts of various T matrices are not discernable. The corresponding imaginary parts are of the order of 10^{-6} for $r \leqslant 1{\cdot}1$ fm. As we move further along the r axis the values of real and imaginary parts of $T(k, q, k^2, r)$ decrease in their characteristic way to take up appropriate saturation values. On a very general ground one knows that the phase of the half-shell T matrix is the scattering phase shift. We have verified that for the results of T matrices shown in figure 3. Figure 4 portrays $F(k, q)$ as a function of q for two different laboratory energies, namely, $E_{\text{lab}} = 20$ MeV and $E_{\text{lab}} = 80$ MeV. Looking at these curves we see that $F(k, q)$ is an oscillatory function of q. The oscillations however, rapidly damp out as q increases. Interestingly, at the lower laboratory energy the potential exhibits a large off-shell effect. This is in agreement with the observation of Haidenbauer and Plessas (1984).

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