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Analytic structure of dynamical systems*

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Abstract. The study of the analytic structure of nonlinear ordinary and partial differential equations is shown to provide a unified approach to determining their properties and finding their solutions.

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1. Introduction and overview

In these lectures we discuss how studying the complex space/time singularity structure of both ordinary and partial differential equations can provide a unified and wide ranging approach to understanding their properties as well as a variety of useful analytical solution techniques. We begin by giving a brief overview of results and the scope of this approach.

1.1 *Ordinary differential equations*

Firstly, consider systems of nonlinear ordinary differential equations. In some cases they can be completely *integrable* which results in the motion being multiply periodic for all initial conditions. Even today the precise meaning of "integrability" is proving to be a vexed issue. For Hamiltonian systems the notion of integrability is well understood and arises when there are as many "integrals of the motion" as there are degrees of freedom (N). This results in the stratification of the phase space by N-dimensional tori on which the solutions live. For non-Hamiltonian systems the notion of integrability is far less clear cut but, nonetheless, requires the existence of sufficient "integrals" (which themselves can take on unusual forms) to result in multiply periodic behavior.

The ability to identify integrable systems is very important since, in a certain sense, they are "exactly soluble" and their solutions can be represented analytically. These solutions can then be used as "building blocks" about which more complicated phenomena can be studied. Typically, however, systems of o.d.e's, be they Hamiltonian or dissipative, are nonintegrable and will exhibit chaotic behavior (Swinney and Gollub 1978; Eckmann 1981; Lichtenberg and Lieberman 1983). To date, most of the

*A course of lectures delivered at the School on Chaos and Nonlinear Dynamics held at the Indian Institute of Science, Bangalore, India (June 24th–July 18th 1987) analytical techniques that are available for studying chaos are only able to investigate local phase space structures such as fixed points, overlapping resonances, separatrix splitting etc. (Guckenheimer and Holmes 1983).

There is clearly a major need for analytical techniques of a more "global" nature that can determine such fundamental issues as to whether or not a given system is integrable and if so what its integrals are and, ultimately, to provide an algorithm for actually integrating the equations of motion. Furthermore, if a system is nonintegrable one would still like to provide, if possible, some type of analytical description of the solutions as they evolve in time.

We believe that many of these analytical insights can be obtained by examining the singularities that these systems exhibit in the complex domain. Thus a valuable test of integrability is proving to be the so called Painlevé test which has its origins in the classic work of Kovaleskaya on the rigid body problem (a semi-historical review is given in Tabor 1984). Here the basic idea is that for integrable systems the only movable singularities (i.e. singularities whose positions are initial condition dependent) exhibitied by the solutions in the complex (time) domain are ordinary poles. In practice this test is easily implemented by demonstrating that the dependent variables can be expanded in the neighborhood of some arbitrary singularity (say at t_0) in a Laurent series, i.e.

$$x(t) = \frac{1}{(t-t_0)^{\alpha}} \sum_{j=0}^{\infty} a_j (t-t_0)^j. \tag{1}$$

Here α is some "leading order" exponent—which must be integer for a single-valued expansion—and the a_j are a set of expansion coefficients of which the requisite number must be arbitrary in order that (1) be a local representation of the general solution. All sorts of interesting questions (discussed in §2) concerning this idea can arise—such as what happens if α is rational ("weak Painlevé") or if there are not enough arbitrary coefficients in the expansion ("singular solutions") and, of course, why should such a test work?

As it stands the expansion (1) is purely "local" and would appear to contain no further information. However, as we shall describe, recent results which use the techniques developed for partial differential equations (see below and §3) enable us to determine, from expansions like (1), the solutions to the problem in hand by determing the associated "Lax pair" from which we can then deduce the integrals of motion and the so called algebraic curve. This procedure gives us some deep geometrical insights into the properties of integrable systems and why the Painlevé test works.

The next question to ask is what happens to the singularities of nonintegrable systems? Here one finds that they become multivalued in all sorts of interesting ways. Past work (Chang *et al* 1983) has shown that when the singularities have complex exponents they cluster recursively in the complain domain forming self-similar natural boundaries. Our most recent work (Fournier *et al* 1988) has been concerned with commonly occurring logarithmic branch points. (These arise in physically important systems such as the Lorenz and Duffing equations.) Locally the solutions can be represented as so called psi-series of the form

$$x(t) = \frac{1}{(t-t_0)^{n}} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} a_{jk} (t-t_0)^j [(t-t_0)^m \log(t-t_0)]^k, \tag{2}$$

where n and m are certain integers determined directly from the differential equations. As it stands such a series might be regarded as being uninformative if not down right unpleasant! However, we have developed a novel resummation technique that enables one to represent (2) in terms of known analytical functions. This enables us to achieve a type of "local integrability", i.e. an explicit analytical representation of the solution in the neighborhood of a singularity. By piecing together such local solutions we believe that it is possible to provide a piecewise analytical representation of even chaotic trajectories.

1.2 *Partial differential equations*

The range of phenomena modelled by nonlinear partial differential equations is immense. As typical examples we cite: shallow water waves modelled by the $K\mathrm{d}V$ equation, wave propagation in nonlinear media modelled by the nonlinear Schrödinger equations (NLS equation), superconducting phenomena modelled by the Landau-Ginzburg equation, reaction-diffusion equations described by the Kuramoto-Sivashinsky equation and, perhaps most important of all, fluid dynamical processes described by the Euler and Navier-Stokes equations.

As with ordinary differential equations it is also possible to classify nonlinear p.d.e's according to whether or not they are integrable. Equations such as the $K\mathrm{d}V$ and NLS equations are found to have an integrable Hamiltonian structure possessing an infinite number of conservation laws (i.e. integrals). A remarkable property of the two, above mentioned, equations is their well known ability to exhibit *solitons*. These come about through the balancing of the equation's nonlinearities (which "sharpen up" a travelling wave form) with their dispersive terms (which spread out the wave). These integrable systems are also, in a certain sense, exactly soluble by means of the famous Inverse Scattering Transform (IST) method (Ablowitz and Segur 1981). Despite their special status integrable p.d.e's arise quite frequently in a variety of one dimensional physical problems.

However, as might be expected, most nonliear p.d.e's are nonintegrable and although they can often exhibit travelling wave-like solutions these do not have the stability and structure of solitons. In general the behavior of nonintegrable p.d.e's is very diverse ranging from the appearance of coherent like structures to solutions that can exhibit chaos in both space and time. There are many analytical techniques—primarily perturbation methods and stability analyses—that have been devised to obtain special solutions and their properties. More powerful computational facilities are now making large scale numerical studies feasible as well.

Again, however, there is a great need for more "global" analytical techniques which can, on the one hand, provide simple tests for integrability and a constructive algorithm for finding the associated IST and yet, on the other hand, still be capable of saying something useful about the properties of nonintegrable systems. Our contention is, again, that the singularity structure of nonlinear p.d.e's can provide many of the desired analytical insights. A first step in this direction was the observation by Ablowitz *et al.* (1980) that when some of the well known integrable p.d.e's were reduced to o.d.e's, via similarity or travelling wave reductions, the resulting equations had the Painlevé property. Although useful, this type of test, for p.d.e integrability, has a number of drawbacks since one needs to know all the possible reductions to o.d.e's (not easy in practice) and furthermore the method does not provide information about the actual solutions to the problem.

A more general approach has been developed by Weiss *et al* (1983) (henceforth referred to as WTC) which is based on the idea of expressing the solution of a given system as a local, Laurent like, expansion about a "*singular manifold*". Thus for some general nonlinear evolution equation of the form

$$q_t(x,t) = K(q, q_x, q_{xx}, \ldots), \tag{3}$$

where K is a nonlinear function of q and its derivatives, we attempt to expand the solution as

$$q(x,t) = \frac{1}{\varphi^{\alpha}(x,t)} \sum_{j=0}^{\infty} q_j(x,t)\varphi^j(x,t), \tag{4}$$

where α is some "leading order" exponent and the $q_j(x,t)$ are a set of coefficients. The equation

$$\varphi(x,t) = 0 \tag{5}$$

defines the singular manifold on which $q(x,t)$ is singular. The proposition of WTC is that if the above expansion is, in a certain sense, single-valued then the system (3) is integrable. This assertion is reinforced by the remarkable observation that if the expansion (4) is then truncated at order φ^0 it yields a system of equations from which the complete IST formalism, namely the Lax pair and auto-Bäcklund transformations, can be deduced. As well as being useful for finding the solutions to a specific equation this approach yields a particularly rich structure when applied to hierarcheis of evolution equations such as the $K\,dV$ hierarchy and NLS hierarchy. Here, in recent work with Newell *et al* (1987) we find much valuable information (and probably more that is not yet understood) in the "singular branches" of the WTC expansions, namely expansions of the form (4) which coexist with the general solutions but have less than its number of arbitrary functions. For nonintegrable systems our understanding of their singularity structure is far less advanced than that for nonintegrable o.d.e's. Nonetheless it seems that WTC expansions can still provide a useful means of constructing nontrivial special solutions.

The flow diagram (figure 1) attempts to show the scope of our singularity structure approach. In the following sections we will summarize these ideas in more detail and map out the large number of extensions, applications and possible connections with other methods that now need to be investigated. The overall aim will be to provide a unified family of solution techniques that can be used to solve the equations that arise in such physically important processes as nonlinear optics, fluid mechanics, plasma physics, nonlinear dynamics etc. as well as providing insights into mathematically important questions such as the meaning of integrability, the nature of chaos and a unified picture of soliton mathematics, etc.

2. Analytical structure of ordinary differential equations

2.1 *Integrable systems and Lax pairs*

As outlined in the introduction, the Painlevé test for integrability is implemented by determining whether the dependent variable(s), say $q(x)$, can be expanded locally in

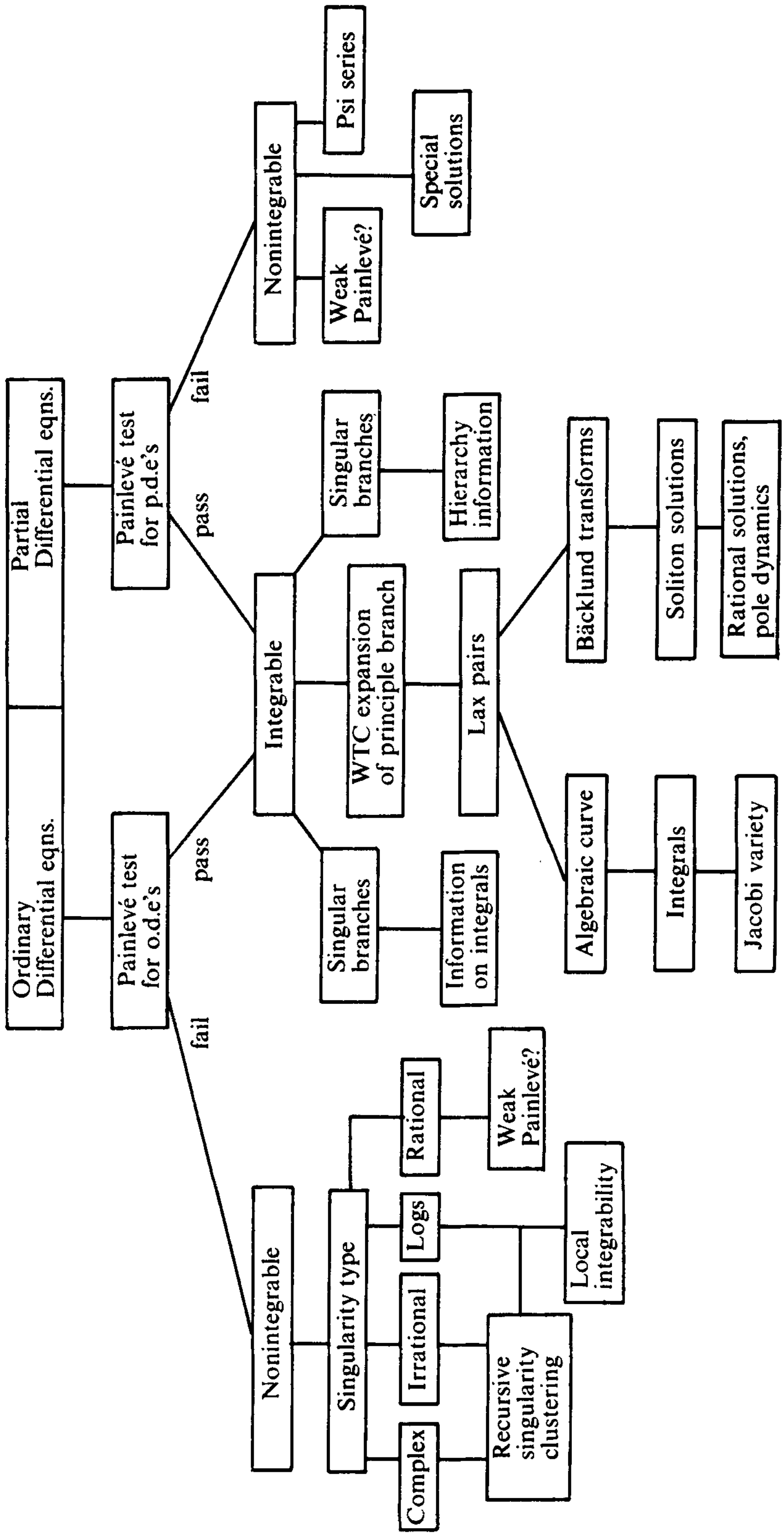
Ordinary Differential eqns.
Partial Differential eqns.
Painlevé test for o.d.e's
Painlevé test for p.d.e's
pass
fail
pass
fail
Integrable
Nonintegrable
Nonintegrable
Weak Painlevé?
Special solutions
Psi series
Singular branches
Hierarchy information
Singular branches
Information on integrals
WTC expansion of principle branch
Lax pairs
Bäcklund transforms
Soliton solutions
Rational solutions, pole dynamics
Algebraic curve
Integrals
Jacobi variety
Singularity type
Complex
Irrational
Logs
Rational
Weak Painlevé?
Local integrability
Recursive singularity clustering

Figure 1.

the neighborhood of a movable singularity x_0 as a Laurent series, i.e.

$$q(x) = \frac{1}{(x - x_0)^n} \sum_{j=0}^{\infty} a_j (x - x_0)^j, \tag{6}$$

where n is a (integer) leading order and a_j the expansion coefficients. The general solution will have as many arbitrary pieces of data (x_0 and various a_j) as the order of the system. Depending on the nonlinearities of the o.d.e other singularities can coexist with those of the general solution and may also have expansions of the form (6) but with less than the full complement of arbitrary coefficients. As will be discussed these "singular branches" may contain much valuable information.

A solution (of a differential equation) is said to have the "*Laurent property*" if it can be represented in the form (6). A system (of differential equations) is said to have the "*Painlevé property* if all the solution branches have Laurent property and is then deemed to be integrable. (The role of movable essential singularities is still not understood.) For a while it seemed that this was as far as one could go, i.e. the series (6) contained no further information about the actual solutions of the problem. However, it was recognized that Laurent-like expansions of the WTC type for partial differential equations (see §3) could yield more information. Thus one works with expansions of the form

$$q(x) = \frac{1}{\varphi^n(x)} \sum_{j=0}^{\infty} q_j(x) \varphi^j(x), \tag{7}$$

where $\varphi(x) = 0$ defines a singular manifold with the special choice $\varphi(x) = x - x_0$ recovering the standard Laurent expansion (6). We illustrate the approach with the simple example of the stationary solution to the $K\mathrm{d}V$ equation, i.e.

$$q_{xx} + 3q^2 = 0. \tag{8}$$

Substitution of the expansion (7), truncated at $O(\varphi^0)$, i.e. up to the coefficient q_2, yields

$$\begin{aligned} q_{xx} + 3q^2 = {} & \frac{1}{\varphi^2}(-8\varphi_x\varphi_{xxx} - 12\varphi_x^2 q_2 + 6\varphi_{xx}) \\ & + \frac{1}{\varphi}(2\varphi_{xxxx} + 12\varphi_{xx}q_2) + q_{2xx} + 3q_2^2, \end{aligned} \tag{9}$$

which on making the squared eigenfunctions substitution (discussed in more detail in §3)

$$\varphi_x = \psi^2 \tag{10}$$

becomes

$$\begin{aligned} q_{xx} + 3q^2 = {} & -\frac{4\psi^2}{\varphi^2}\left\{\int^x \psi(4\psi_{xxx} + 6q_2\psi_x + 3q_{2x}\psi)\,\mathrm{d}x'\right\} \\ & + \frac{4\psi}{\varphi}\left\{(4\psi_{xxx} + 6q_2\psi_x + 3q_{2x}\psi)\bigg| - 3\psi\left(\frac{\psi_{xx}}{\psi} + q_2\right)_x\right\} \\ & + q_{2xx} + 3q_2^2. \end{aligned} \tag{11}$$

The usual WTC prescription is to set each order of φ equal to zero which would here give the system of equations

$$\psi_{xx} + q_2\psi = \lambda\psi, \tag{12a}$$

$$4\psi_{xxx} + 6q_2\psi_x + 3q_{2x}\psi = 0, \tag{12b}$$

$$q_{2_{xx}} + 3q_2^2 = 0. \tag{12c}$$

Although this system of equations is self-consistent it is not general enough to integrate (12c) and can only provide special solutions. A crucial new observation, made in recent work with Newell *et al* (1987), is that the original WTC prescription is too restrictive and instead one should think of each order of φ being zero *modulo some function of* φ. Thus in this case if we

(i) add an amount $4\varphi_x\varphi y$ at $O(\varphi^{-2})$ and

(ii) subtract the identical amount $4\varphi_x y$ at $O(\varphi^{-1})$, where y is an additional free parameter, we obtain instead of (12b)

$$4\psi_{xxx} + 6q_2\psi_x + 3q_{2x}\psi = y\psi, \tag{13}$$

which together with (12a) is the correct Lax pair for (12c).

A less trivial example is an integrable case of the Henon-Heiles system, namely

$$q_{xx} = -3q^2 - \tfrac{1}{2}p^2, \tag{14a}$$

$$p_{xx} = -qp. \tag{14b}$$

WTC expansions of q and p, truncated at $O(\varphi^0)$ take the form

$$q = 2\frac{\partial^2}{\partial x^2}\log\varphi + q_2, \quad p = \frac{p_0}{\varphi} + p_1 \tag{15}$$

and when these are substituted in (14) one is able to obtain the Lax pair (Newell *et al* 1987)

$$\psi_{xx} + (q_2 + \lambda)\psi = 0, \tag{16a}$$

$$y\psi + 4\psi_{xxx} + 6q_2\psi_x + 3q_{2x}\psi - \tfrac{1}{4}p_1\int^x p_1\psi\,\mathrm{d}x' = 0, \tag{16b}$$

where again we use the relation $\varphi_x = \psi^2$. We believe that this result is the first example of the WTC method yielding a completely new Lax pair (apparently associated with the Lie algebra D_4).

It is convenient to rewrite (16), in system form as

$$W_t = PW, \quad y\lambda W = QW \tag{17}$$

with

$$W = \begin{pmatrix}\psi \\ \psi_t\end{pmatrix}, \quad P = \begin{pmatrix}0 & 1 \\ -\lambda - u & 0\end{pmatrix}, \tag{18}$$

$$Q = \begin{pmatrix}\lambda u_t + \frac{1}{4}vv_{t'} & 4\lambda^2 - 2\lambda u - \frac{1}{4}v^2 \\ -4\lambda^3 - 2\lambda^2 u - \lambda(u^2 + \frac{1}{4}v^2) + \frac{1}{4}v_t^2, & -\lambda u_t - \frac{1}{4}vv_t\end{pmatrix},$$

where we have set $u \equiv q_2$ and $v \equiv p_1$. The solvability condition of (17) is

$$Q_t = [P, Q] \tag{19}$$

which, when written in component form, gives (14) (with q and p replaced by u and v respectively). Further, if W is a fundamental solution matrix of (17), then the solution of (19) can be conveniently written as

$$Q = WQ_0W^{-1}, \tag{20}$$

with Q_0 independent of t. As a consequence of (20), the characteristic polynomial of Q

$$\det(Q - y\lambda I) = 0, \tag{21}$$

which is the condition that (17) has a nontrivial solution, is independent of t. In terms of y, λ, v and u, (21) is the algebraic curve,

$$\lambda^2 y^2 = -16\lambda^5 + 2H\lambda^2 + G\lambda, \tag{22}$$

where

$$H = \tfrac{1}{2}(U^2 + V^2) + \tfrac{1}{2}uv^2 + u^3 \tag{23}$$

is the Hamiltonian of the system (14) written in the canonically conjugate coordinate pairs u, $U = u_t$ and v, $V = v_t$ and

$$G = \tfrac{1}{4}v^2(u^2 + \tfrac{1}{4}v^2) + \tfrac{1}{2}V(vU - uV). \tag{24}$$

Because (22) is independent of time, we can identify G as the second constant of the motion for the flow (14) generated by the Hamiltonian H. G and H are in involution under the canonical Poisson bracket

$$\{G, H\} = \frac{\partial G}{\partial V}\frac{\partial H}{\partial v} + \frac{\partial G}{\partial U}\frac{\partial H}{\partial u} - \frac{\partial G}{\partial v}\frac{\partial H}{\partial V} - \frac{\partial G}{\partial u}\frac{\partial H}{\partial U}. \tag{25}$$

The choice of auxiliary variables which leads to the identification of the angle variables corresponding to the actions H and G is most conveniently made using the matrix from the Lax equation (17). The new variables are μ_1, μ_2 given by

$$\mu_1 + \mu_2 = \tfrac{1}{2}u$$

$$\mu_1\mu_2 = -\tfrac{1}{16}v^2 \tag{26}$$

which are the zeros of the (1, 2) element, $4\lambda^2 - 2\lambda u - \frac{1}{4}v^2$, of Q (see Newell *et al* 1987) The reader can also verify that this choice of variables separates the associated Hamilton-Jacobi equation. (I recommend strongly that the reader consult the works of Ercolani and Siggia (1986) on these ideas). Standard techniques of algebraic geometry can then yield an explicit linearization of the equations of motion on the associated Jacobi variety. A careful analysis of the rational solutions of (14) shows that the singular solution (lower branch) is simply a reexpansion of the general solution (principle branch) about a location on the singular manifold $\varphi = 0$ at which $\varphi_x = 0$. (A detailed discussion of this is given in Newell *et al* 1987).

Despite these successes important issues remain to be resolved. One vexed issue concerns the properties of systems with rational branch points. In the case of the

Henon-Heiles system one of the integrable cases was identified (Chang *et al* 1982) by relaxing the Painlevé test to include such singularities. However, in that case, the branch point could be transformed away by a simple change of dependent variable. These results led Ramani *et al.* (1982) to suggest that a "weak Painlevé" property, i.e. nontransformable branch points, could still identify integrable systems and a number of integrable Hamiltonians were constructed with this behavior. However, subsequent work (Ankiewicz and Park 1983) showed that some of these cases were separable thereby making the branch points transformable. In addition it is also possible to find (Weiss 1982) counter examples to the weak Painlevé property. We believe that one way of imposing some order onto this unsatisfactory state of affairs is to determine, at least for algebraically integrable Hamiltonians (i.e. those with integrals that are algebraic functions of the canonical variables), the order of the integrals of motion (if they exist). Interestingly enough this information does not require a complete solution to the problem but may be contained in the singular branches of the solutions. Indeed, this ties in with the other major issue—namely the information contained in these branches.

It would appear, at least for certain classes of Hamiltonian, that systems of N degrees-of-freedom exhibit N solution branches. One of these, as we have described, corresponds to the general solution and the associated Laurent series exhibits $2N-1$ arbitrary constants. The other branches have less free constants with the most singular branch exhibiting N-arbitrary parameters. Recent work of Ercolani and Siggia (1986) (as well as our own on the stationary $K\mathrm{d}V$ hierarchy (Newell *et al* 1987)—see §3) indictes that much information on the integrals is contained in this most singular branch. For systems with polynomial integrals it would appear that the order of the polynomial is easily determined by the so called "resonance" structure, namely the powers of the independent variable at which the free constants enter the Laurent expansions. This result could be rather useful since it suggests an alternative route (to our determination of the algebraic curve) for finding the integrals. Furthermore this may help with resolving the "weak Painlevé" issue. That is, if the structure of the singular branch indicates the possibility of polynomial integrals (so far all the weak Painlevé systems are of this type), we can allow the general solution to exhibit the weak property. (At this stage, however, we know little about the analytic structure of systems with transcendental integrals.)

2.2 *Nonintegrable systems and singularity clustering*

Systems with multivalued movable singularities fail the Painlevé test (as it currently stands) and are presumably nonintegrable. As indicated in §1 the local solution must now be represented in the form of a psi-series. The main types of multivaluedness are logarithmic, irrational or complex with the case of rational branch points ("weak Painlevé") having been discussed in the previous section.

Naturally one would like to know what information can be extracted from these psi-series. An example is provided by the work of Chang *et al* (1983) on certain nonintegrable regimes of the Henon-Heiles system. Here the singularities are complex and the psi-series are of the form

$$q(x) = \frac{1}{x^2}\sum_j \sum_k a_{jk} x^j (x^\beta)^k + \text{comp. conj.} \tag{27}$$

where β is a complex number. These workers observed that a substitution of the form

$$q(x) = \frac{1}{x^2}\theta(z), \tag{28}$$

where $\theta(z)$ is some function of the variable $z = x^\beta$, into the original equations of motion gives, in the limit $x \to 0$, a differential equation for θ that has the same analytic structure as the original equations. This is, in effect, a "renormalization" of the equations of motion in the neighborhood of a given singularity (here assumed to be at $x_0 = 0$). By mapping back the singularities from the z plane to the x plane, as determined by the transformation

$$z = x^\beta, \tag{29}$$

it is fairly easy to demonstrate (in confirmation with the numerical results) that the singularities in the x domain cluster in self-similar spirals.

Although the psi-series for logarithmic singularities had been studied (in the context of the Lorenz equations) by Tabor and Weiss (1981) sometime ago it was not understood how the singularities might cluster for this type of multivaluedness. Recent work with Fournier *et al* (1988) has demonstrated that for psi-series of the form

$$q(x) = \frac{1}{x^n}\sum_j \sum_k a_{jk} x^j (x^m \ln x)^k, \tag{30}$$

the singularities in the x-domain cluster recursively in the form of m-armed stars. As will be discussed in the next subsection the above group were able to push the earlier analysis of (Chang *et al* 1983) much further and obtain some remarkable asymptotic expansions that provide an effective "local integration" of the equations of motion.

This new understanding of the nature of logarithmic singularities enables us to probe a number of important questions. For example, in the case of the Lorenz equations there are certain parameter values for which the equations, although nonintegrable, possess one integral of the motion. The indications are that in these cases the existence of one integral, rather than none, coincides with a change in singularity clustering. This is important since this may help us understand a conjecture of Kruskal that a criterion for integrability is not so much meromorphicity (Painlevé property) but lack of singularity clustering.

2.3 *Locally integrable systems*

A remarkable feature of "renormalizing" substitutions of the form (28) is that in some cases, such as certain nonintegrable regimes of the Henon-Heiles system, they give back a rescaled version of the same equation whereas in other cases they yield *different* equations, for example in the case of the Duffing oscillator (here$'$ denotes d/dx)

$$y'' + \lambda y' + \tfrac{1}{2} y^3 = \varepsilon g(x), \tag{31}$$

the substitution

$$y(x) = \frac{1}{x}\theta(z) \tag{32}$$

where

$$z = x^4 \ln x, \tag{33}$$

gives an equation for θ that can be integrated in terms of elliptic functions—in fact the resulting equation is the integrable part of (31), namely of the form $f'' + \frac{1}{2} f^3 = 0$. In a way this is rather extraordinary since it means that in the neighborhood of a given singularity x_0 (here we have set $x_0 = 0$) we have an explicit analytical representation of the solution to (31) which is traditionally regarded as nonintegrable. A similar result was obtained for the Lorenz equations (Tabor and Weiss 1981) where, depending on whether the psi-series involves terms of the form $x^2 \ln x$ or $x^4 \ln x$ (this depends on the system parameters) the corresponding equation for θ could be integrated in terms of different types of elliptic functions. These observations suggest nonintegrable systems can be classified according to whether or not they have this "local integrability" property. This seems to be rather a deep concept but we have, as yet, little understanding of its significance.

Additional, important results have recently been obtained for the Duffing equation (31) (Fournier *et al* 1988). Here we have found that the substitution (32) is, in fact, just the first term in an asymptotic expansion of the form

$$y(x) = \sum_{j=0}^{\infty} \theta_j(z) x^{j-1}, \tag{34}$$

where the whole set of $\theta_j(z)$ can be determined analytically. The leading term θ_0 captures the essential nonlinearities of the system and the subsequent $\theta_j (j \geq 1)$ satisfy linear equations that can all be integrated in terms of Lamé functions. If the above expansion is compared with the original psi-series

$$y(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} a_{jk} x^{j-1} (x^4 \ln x)^k, \tag{35}$$

we realize that each θ_j is just

$$\theta_j(z) = \sum_k a_{jk} z^k, \tag{36}$$

($z = x^4 \ln x$) which are the generating functions for the coefficient sets a_{jk} ($k = 1, \ldots, \infty$)— Thus the series (34) constitutes a systematic resummation of the psi-series (35). It would appear that these new type of series have a very rich content whose properties have yet to be fully investigated.

3. Analytic structure of partial differential equations

3.1 *Integrable systems, Lax pairs and Bäcklund transformations*

The Painlevé test for p.d.e's developed by WTC (Weiss *et al* 1983) is proving to be a useful test of integrability for nonlinear evolution equations and, in some cases, a means of finding Lax pairs and Bäcklund transformations. In addition, the method can provide a convenient route for constructing rational solutions (Weiss 1982) and determining the pole dynamics (Chudnovsky *et al* 1983). Various extensions and applications of the method have been described in papers by Weiss (1983, 1985a, b), by Tabor and coworkers (Chudnovsky *et al* 1983; Gibbon *et al* 1985; Gibbon and Tabor 1985) and others. Lacking, however, has been a more serious attempt to gain a deeper insight into how and why the method actually works. The aim is to demonstrate that the WTC expansions can not only show that a system is integrable but that the expansions can be used to provide an algorithm which successfully captures all its properties; namely the Lax pairs, Bäcklund transformations, the motion invariants, symmetries and commuting flows and ultimately the algebraic properties.

As described in the introduction, the Painlevé test for p.d.e's involves the demonstration that the solutions can be expressed as local, single-valued expansions (4) about the singular manifold (5). Direct substitution of the anzatz (4) into a given evolution equation (3) leads to a set of recursion relations for the u_j of the form

$$P(j)u_j = F_j(\varphi_t, \varphi_x, \ldots, u_k; k < j) \tag{37}$$

where $P(j)$ is a polynomial in j whose roots (termed "resonances") determine which of the u_j should be arbitrary. As with the case of o.d.e's the solution can possess different branches. The structure and significance of the pattern of resonances for all branches is only just starting to be understood in recent work (Newell *et al* 1987) on the $K\,\mathrm{d}V$ hierarchy.

A standard example of the WTC method is provided by the $K\,\mathrm{d}V$ equation

$$q_t + 6qq_x + q_{xxx} = 0, \tag{38}$$

which is easily found to have the WTC expansion

$$q(x,t) = 2\frac{\partial^2}{\partial x^2}\log\varphi(x,t) + \sum_{j=2}^{\infty} q_j(x,t)\varphi^{j-2}(x,t), \tag{39}$$

where $\varphi(x,t)=0$ defines the singular manifold, and the resonances occur at $j=-1$ (corresponding to the arbitrariness of φ itself) and $j=4$ and 6. By setting the arbitrary functions $u_4 = u_6 = 0$ and requiring in addition that $u_3 = 0$, the expansion (39) can be self-consistently truncated at $0(\varphi^0)$ yielding (cf §2.1)

$$\begin{aligned} q_t + 6qq_x + q_{xxx} = {} & \frac{1}{\varphi^2}(6\varphi_{xx}^2 - 8\varphi_x\varphi_{xxx} - 12q_2\varphi_{xx}^2 - 2\varphi_x\varphi_t) \\ & + \frac{1}{\varphi}\{2\varphi_{xxxx} + 12q_2\varphi_{xx} + 2\varphi_{xt}\} \\ & + q_{2t} + 6q_2q_{2x} + q_{2xxx}, \end{aligned} \tag{40}$$

the squared eigenfunction substitution

$$\varphi(x,t) = \psi^2(x,t), \tag{41}$$

yields, on setting each order of φ equal to zero, the standard Lax pair

$$\psi_{xx} + (\lambda + q_2(x,t))\psi = 0, \tag{42a}$$

$$\psi_t + 4\psi_{xxx} + 6q_2\psi_x + 3q_{2x}\psi = 0, \tag{42b}$$

where

$$q_{2t} + 6q_2 q_{2t} + q_{2xxx} = 0. \tag{42c}$$

The truncated expansion

$$q(x,t) = 2\frac{\partial^2}{\partial x^2}\log\varphi(x,t) + q_2(x,t), \tag{42d}$$

then constitutes an auto-Bäcklund transformation for the $K\mathrm{d}V$ equation (38). We comment that until recently the substitution (41) was only identified for the $K\mathrm{d}V$ equation and that an alternative route to the Lax pairs was developed by Weiss (1983) using the properties of the Schwarzian derivative of φ.

The time is now ripe to address a series of questions (many of which complement those addressed for o.d.e's) including:

(i) How general is the WTC procedure?; (ii) What is the nature of the function $\varphi(x,t)$?; (iii) Can the WTC expansions always be self-consistently truncated if they pass the Painlevé test and can one thereby always find the associated Lax pair and Bäcklund transformation?; (iv) What information do the Painlevé expansions for the other branches contain?; (v) What is the structure and significance of the pattern of resonances for hierarchies of integrable systems?; (vi) Is the squared eigenfunction relation (41) special to the $K\mathrm{d}V$ equation or is it more general?; (vii) What is the connection between the WTC approach and (a) Hirota's method (b) Hamiltonian structures (c) the Estabrook-Wahlquist method of differential forms (d) Kac-Moody algebras?.

Recent work (Newell *et al* 1987) has started to answer some of these questions whereas others, such as (vii) b, c and d, have yet to be tackled. In the next section we summarize some of our new results for the $K\mathrm{d}V$ hierarchy which, in addition, provide a convenient framework for some of the questions such as (ii), (iii), (iv) and (v).

3.2 *The WTC method and integrable hierarchies*

When using the WTC method to find the Bäcklund transformation (42d) for the $K\mathrm{d}V$ equation (38) we should recall that this equation to just the first member of the infinite hierarchy

$$q_{t_{2n+1}} = \frac{\partial}{\partial x}(L^n q), \tag{43}$$

where t_{2n+1} is the 'time' associated with the $(2n+1)$th flow, and

$$L = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2}\int^x \mathrm{d}x' q_{x'}. \tag{44}$$

The first three members of this hierarchy are:

$$4q_{t_3} = -(q_{xx} + 3q^2)_x \tag{45}$$

$$16q_{t_5} = +(q_{xxxx} + 5q_x^2 + 10q^3)_x \tag{46}$$

$$64q_{t_7} = -(q_{6x} + 14qq_{4x} + 28q_xq_{xxx} + 21q_{xx}^2 + 70q^2q_{xx} + 70q_x^2 + 35q^4)_x. \tag{47}$$

Thus we should think of φ as a function of all the time coordinates $t_1 = x$, t_3, $t_5, \ldots, t_{2n+1} \ldots$ and therefore write (42d) as

$$q(x, t_3, t_5 \ldots) = 2\frac{\partial^2}{\partial x^2} \log \varphi(x, t_3, t_5, \ldots) + q_2(x, t_3, t_5, \ldots), \tag{48}$$

which, as we can demonstrate, is indeed the Bäcklund transformation (BT) for all members of the hierarchy (43). Equation (48) also provides the direct link between the WTC method and Hirota's method, i.e. it can be shown that φ is just the ratio of Hirota τ-functions (Newell *et al* 1987; Gibbon *et al* 1985).

For all members of the family it is easy enough to show that the leading order is -2 and hence one makes the anzatz

$$q = \sum_{j=0}^{\infty} q_j(x, t_3, \ldots)\varphi^{j-2}(x, t_3, t_5 \ldots), \tag{49}$$

At $O(\varphi^{-2n-2})$ one finds that there are n possible solutions

$$q_0 = -m(m+1)\varphi_x^2 \quad m = 1, 2, \ldots, n \tag{50}$$

and at $O(\varphi^{-2n-1})$ that

$$q_1 = m(m+1)\varphi_{xx} \quad m = 1, 2, \ldots, n. \tag{51}$$

Thus for the $(2n+1)$th flow there are n branches with the expansions

$$q = m(m+1)\frac{\partial^2}{\partial x^2}\log \varphi(x, t_3, t_5 \ldots) + \sum_{j=2}^{\infty} q_j\varphi^{j-2}. \tag{52}$$

For each branch there are $2n$ resonances and for the principal branch, $m = 1$, $2n-1$ of them occur for $j \geqslant 0$ and therefore this branch gives rise to the full complement (which includes φ itself) of $2n$ arbitrary functions. The lower branches ($m > 1$) have only $2n - m$ resonances for $j \geqslant 0$ and the corresponding expansion (52) has only $2n + 1 - m$ arbitrary functions.

For now we shall concentrate on the principal branch. Rather than attempt to demonstrate that this branch (or the others, for that matter) explicitly possess the Laurent property we have proved (Newell *et al* 1987) that the truncated expansion (48) leads to a self-consistent system of equations (the natural generalization of the set (42)) which yield the B.T. and Lax pair for each member of the $K\,dV$ hierarchy. An important feature of our analysis is that the squared eigenfunction substitution (41) may still be used for all members of the hierarchy. That this is so should not be too surprising since the equation for q_1, occurring at $O(\varphi^{-1})$ in the truncated expansion, always satisfies the p.d.e. linearized about the solution q_2. For example, for the t_3 flow this linearization is just (here $u \equiv q_2$)

$$4q_{1t} + q_{1xxx} + 6uq_{1x} + 6q_1u_x = 0. \tag{53}$$

Thus q_1 is a symmetry of the KdV equation and from previous studies it is known that the associated infinitesimal B.T.'s are generated by the derivative of the squared eigenfunction. Hence q_1, which is $2\varphi_{xx}$, should be identified with $(\psi^2)_x$ which immediately leads to the identification $\varphi_x = \psi^2$. Similar arguments apply to other types of integrable p.d.e such as the NLS hierarchy. This important result enables us to find the Lax pairs directly without recourse to the Schwarzian derivative approach of Weiss (1983).

In recent work (Newell *et al* 1987) we have been able to obtain the polynomials which determine the pattern of resonances for all branches of any member of the KdV hierarchy. Of vital importance is to understand further the information contained in the singular branches. It would appear that, on the basis of the properties of the rational solutions to the KdV hierarchy, these branches represent a *coalescence of poles* in the WTC expansions. Furthermore there is reason to believe that the resonances are associated with the fluxes of the system—the lowest branch containing information about the fluxes (i.e. integrals of motion) of the particular system studied (c.f. results for o.d.e's). The other branches appear to contain information about the other flows in the hierarchy in a way that is yet not fully understood.

There are clearly a variety of important tasks ahead including those listed at the end of §3.1. In particular we must understand more fully the role of the singular branches for both the time dependent and stationary flows associated with a given hierarchy. With some results already available for the KdV and NLS (not described here) hierarchies we must also look at the associated properties of hierarchies associated with different algebras (e.g. Boussinesq hierarchy) and multidimensional systems such as the K-P family of equations.

3.3 *The WTC method and nonintegrable systems*

It is interesting to consider what sort of information can be obtained from WTC expansions for nonintegrable p.d.e's. The use of the singular manifold function $\varphi = \varphi(x, t)$ gives us an extra flexibility not available in the usual Laurent or psi-series expansions used for o.d.e's. This arises in the following way. Consider a system with integer leading order and resonances. Typically, at a resonance, a certain "compatability condition" must be satisfied to ensure that the corresponding expansion coefficient is arbitrary. Sometimes this condition will only be satisfied for special values of the adjustable system parameters. Otherwise, failure to satisfy a compatability condition will result in the need for logarithmic terms in the local expansion. However, with WTC expansions for p.d.e's the compatability conditions can take the form of an (auxiliary) p.d.e that φ itself must satisfy. By forcing φ to satisfy such a constraint the expansions can be made to maintain their "single-valuedness". In such cases the system no longer possesses the "Painlevé property" for p.d.e's, but rather a "conditional Laurent" (or "conditional Painlevé") property which identifies a special sub-class of meromorphic solutions. An early application of this idea is due to Weiss (1984) in his study of (1 + N)-dimensional sine-Gordon equations. Only the 1 + 1 system passes the Painlevé test proper but the higher dimensional ones can be conditionally Painlevé if the Gaussian curvature of the associated φ is zero. This constraint identifies certain classes of travelling wave solutions.

An important class of nonintegrable equations that is currently being investigated

(Cariello and Tabor 1989) is the Newell-Whitehead equation

$$u_t = u_{xx} + u(a - u)(1 - u) \tag{54}$$

which is the "real" version of the $1 - D$ Landau-Ginzburg equation

$$u_t = u_{xx} + \mu u + \lambda u|u|^2 \tag{55}$$

where μ and λ are complex coefficients. Both of these systems fail the Painlevé test and require local psi- expansions of the form

$$u(x, t) = \frac{1}{\varphi} \sum_{jk} u_{jk} \varphi^j (\varphi^4 \ln \varphi)^k. \tag{56}$$

Little is know about the properties of such expansions for p.d.e's (in contrast to the results for o.d.e's) and for now we just concentrate on the class of solutions which have "conditional" Laurent expansions. Indeed, it can be shown (Cariello and Tabor 1989) that a subset of these solutions can be obtained by just computing the properties of the expansions truncated at $O(\varphi^{-1})$ which gives, as discussed in (Newell *et al* 1987), the correct form of the associated Hirota substitution. So, for example, in the case of (55) with both p and q complex, the solutions are easily shown to take the form

$$u(x, t) = \frac{u_0(x, t)}{\varphi^{1 + i\alpha}} \tag{57}$$

where α is a function of p and q and $u_0(x, t)$ is a determined function of ϕ. In this way we are able to obtain all of the special solutions obtained by Nozaki and Bekki (1984) who use a cumbersome, complex version of Hirota's bilinear calculus. Another, and rather nice illustrations of the use of truncated WTC expansions for nonintegrable systems is due to Fournier and Spiegel (1987) in their study of the Kuramoto-Sivashinsky equation, namely

$$u_t + uu_x + \sigma u_{xx} + \mu u_{xxxx} = 0. \tag{58}$$

Here the truncated expansion takes the form

$$u = 60\mu \frac{\partial^3}{\partial x^3} \ln u + \frac{60}{19} \sigma \frac{\partial}{\partial x} \ln \varphi \tag{59}$$

which, on substitution in (58) yields a homogeneous equation of the third degree! This can then be used to calculate the (correct) form of certain stationary solutions. It should be noted that the general solution of (58) requires a psi-series of the form

$$u(x, t) = \frac{1}{\varphi^3} \sum_{jk} \varphi^j \psi^k + \text{c.c.} \tag{60}$$

where $\psi = \varphi^{(13 + i\sqrt{71})/2}$, i.e. the system has complex resonances.

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