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Confinement model for quarks

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Abstract. A confinement model of hadron with its constituent quarks bound in a strong gravitational field is presented. The gravitational field plays the role of a medium having, as if, space dependent permeabilities from a fixed centre. The massless Dirac equation modified by the gravitational field is solved. The solution for the wavefunction of the quarks obtained shows the characteristic features of confinement, i.e., (i) wavefunction with higher energy states lying closer to the centre, (ii) equispaced energy levels without continuum, (iii) the quark orbits lying within a distance $\sim 10^{-14}$ cm, the characteristic radius of a typical hadron.

Keywords. Quark motion; strong gravity; confinement.

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1. Introduction

There have been attempts to understand the confinement in various ways, although quantum chromodynamics (QCD), and colour geometrodynamics (Mielke 1980; Mielke and Scherzer 1980) have been partially successful in finding a solution, the mechanism of quark confinement is not yet fully understood. Alternative approaches like bag models (Chodos *et al* 1974; Hasenfratz and Kuti 1978) and various other models try to describe some properties of hadrons at phenomenological levels but the way the confinement goes is not clear because of some otherwise phenomenological input leading to confinement. The colour dielectric models (Khare and Pradhan 1983; Jena and Pradhan 1981, 1984; Khadkikar and Vinod Kumar 1987) try to explain confinement mechanism solving Maxwell-like equations for quarks and gluons moving in a colour dielectric medium. In the models the dielectric constant, crucial to the understanding of confinement mechanism, is introduced with an assumed form. However the judicious choice, though phenomenological, is an indication to the way the confinement is achieved. Moreover the same form of dielectric constant used for the confinement of photon (in charge space), quark and gluons (in colour space) suggests that there is a mechanism through which the dielectric constant couples in the same way to all the fields. In this paper we investigate those two subtle aspects. We derive the form of the dielectric constant for confinement of photon, vector gluon and massless quarks. The confinement of photon and gluon has been discussed elsewhere; here we discuss only the confinement of massless quarks. To do so we write down the quark equation in Maxwell-like form and couple it with strong gravitational background in an analogous way the Maxwell equations of electrodynamics couple with the normal gravitational field (Landau and Lifshitz 1980).

To proceed with the confinement of massless quarks we make the following assumptions:

(i) The field equations of the confining particles are written in Maxwell-like form. The particles are taken to be massless.
(ii) There is a gravity-like field, called strong gravity, that satisfies Einstein-like equations with a coupling constant $G_f \simeq 10^{38} G_N$, characteristic of strong interaction.
(iii) This gravity field couples with the Maxwell-like equations in the same way a normal gravitational field modifies the Maxwell equations of classical electrodynamics in 3-metric space.

The idea that the quarks can be described by Maxwell-like equations was introduced by Lee (Lee 1979) and was further carried out by Jena and Pradhan (1981, 1984) to describe the motion of quarks in a colour dielectric medium. It is described elsewhere (Biswas and Kumar 1989) that the gravitational field simulates the effect of a dielectric medium and hence assumption (iii) may be taken as a model approach to couple the gravitational-like field with a spinor field in view of the fact that the solution of nonlinear spinor equation in curved space-time is extremely difficult. Assumptions (ii) and (iii) lead to a form of the metric of the strong gravity identified as the dielectric constant. The gravitational field simulates the effect of a medium having space-dependent dielectric permeabilities. We describe the strong interaction between quarks by a tensor field $f_{\mu\nu}$. As in colour geometrodynamics (Mielke 1980), the origin of $f_{\mu\nu}$-field is assumed to be due to the flavor and the colour charge of quarks.

2. Role of strong gravity

It is an old idea to treat the gravitational field as an electromagnetic effect (Dicke 1957). The gravitational bending of a light beam near the sun can be explained by considering the motion of photon in a polarized vacuum. The vacuum acts like a dielectric medium and the effects of vacuum polarization can be described by classical field quantities $\varepsilon(r)$ and $\mu(r)$. We adopt an analogous approach and assume that the metric of the strong gravity background is given by

$$\mathrm{d}s^2 = e^{2\lambda}[\mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)], \tag{1}$$

where $\lambda = \lambda(r)$ is determined from the solution of Einstein equation with the Lagrangian

$$L = \frac{(-f)^{1/2}}{2K_f}(R(f) - 2\Lambda_f) + L_\phi. \tag{2}$$

Here $R(f)$ is the curvature scalar for f-gravity field

$$f_{\mu\nu} = \exp(2\lambda)\eta_{\mu\nu}. \tag{3}$$

Λ_f is the cosmological constant and $K_f = 8\pi G_f/c^4$ with $G_f = 10^{38} G_N$, characteristic of strong interaction (Sivaram and Sinha 1979). L_ϕ is the Lagrangian for massless quarks in 'equivalent description'. By 'equivalent description' we mean that ϕ is a field that satisfies the squared version of the Heisenberg-Pauli-Weyl nonlinear spinor equation generalized to curved space-time (Deppert and Mielke 1979). The occurrence of cosmological constant Λ_f suggests that the space is nonempty. Since it is nongeometrical in origin it may be interpreted as a contribution coming from other fields (e.g., colour fields) acting like energy momentum tensor $\sim \Lambda_f f_{\mu\nu}$ in the field equations. The form of conformal solution (3) is dictated by the modification of coloumb's law in presence of a dielectric medium. The transformation $r^2 \to \varepsilon(r) r^2$ in dielectric medium allows us to consider $e^{2\lambda}$ as a dielectric constant. However, we will find later that the identification of $e^{2\lambda}$ as a dielectric constant is determined from the structure of Maxwell-like equations in the strong gravity field. The above disussion allows us to put forward a plausible explanation for the origin of the strong gravity field $f_{\mu\nu}$.

It is well known that the nonlinear terms of gluon field equations can be considered as supercurrent (in colour space) analogous to Ginzburg-Landau theory of superconductivity (Khadkikar and Kumar 1987). This current simulates the effect of a medium having space dependent dielectric constant. From the above discussion it is clear that such space-dependent permeabilities will make the space-time curved and the dielectric constants can be treated as tensor field $f_{\mu\nu}$. It is quite possible that the $f_{\mu\nu}$ field may also have some contribution from vacuum polarization. We assume that all the contributions have been taken into account in $f_{\mu\nu}$ via Einstein field equations corresponding to the Lagrangian (2). As this gravity field couples in the same way to photon, gluon and quarks (written in Maxwell-like form), the use of the same form of dielectric constant finds a natural explanation in our approach. The solution of Einstein field equations corresponding to (2) has already been obtained and is given by (Biswas and Kumar 1989)

$$f_{\mu\nu} = (1 - \Lambda_f r^2)\eta_{\mu\nu}. \tag{4}$$

It is worthwhile to mention that unless one considers the back coupling of the spinor ψ to generalized Einstein equation with L_ϕ replaced by a Lagrangian corresponding to HPW nonlinear equation, it is extremely difficult to solve the HPW equation exactly. In the 'equivalent description' the theory is founded on a dynamics which is similar to those derived for the spinor case. The background is then obtained with the assumption that around the centre of confinement the quarks are almost free. This 'equivalent description' then leads to the solution (4) above.

We write Maxwell equations in 3-dimensional form with the metric given by (1), (3) and (4). Consider a general strong gravity background written as

$$\begin{aligned} ds^2 &= f_{\mu\nu}\, dx^\mu\, dx^\nu \\ &= f_{00}(dx^0)^2 + 2 f_{0i}\, dx^0\, dx^i + f_{ij}\, dx^i\, dx^j. \end{aligned} \tag{5}$$

Now the spatial distance dl between two infinitesimally separated events occurring at one and the same time is given by

$$dl^2 = \gamma_{ij}\, dx^i\, dx^j, \tag{6}$$

where

$$\gamma_{ij} = \left(-f_{ij} + \frac{f_{0i} f_{0j}}{f_{00}} \right), \tag{7a}$$

$$-f = f_{00}\gamma, \tag{7b}$$

$$f = \det(f_{\mu\nu}), \quad \gamma = \det(\gamma_{ij}). \tag{7c}$$

Introducing the three vectors **E**, **D** and the antisymmetric 3-tensors B_{ij} and H_{ij} according to the definitions

$$E_i = F_{0i}, \qquad B_{ij} = F_{ij},$$
$$D^i = -(f_{00})^{1/2} F^{0i}, \quad H^{ij} = (f_{00})^{1/2} F^{ij},$$

one writes down (Landau and Lifshitz 1980) Maxwell equations in 3-dimensional form as

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{1}{(\gamma)^{1/2}}\frac{\partial}{\partial t}((\gamma)^{1/2}\mathbf{B}), \tag{8a}$$

$$\operatorname{div} \mathbf{D} = 0, \quad \operatorname{curl} \mathbf{H} = \frac{1}{c(\gamma)^{1/2}}\frac{\partial}{\partial t}((\gamma)^{1/2}\mathbf{D}), \tag{8b}$$

where

$$\mathbf{B} = \frac{\mathbf{H}}{(f_{00})^{1/2}} + \mathbf{f} \times \mathbf{E}$$

$$\mathbf{D} = \frac{\mathbf{E}}{(f_{00})^{1/2}} + \mathbf{H} \times \mathbf{f}$$

and $f_i = f_{0i}/f_{00}$. In our case $f_{0i} = 0$ and γ is independent of time so that converting the covariant divergence and curl with respect to the metric $\gamma_{\mu\nu}$ in terms of flat space quantities we get,

$$\nabla \cdot ((\gamma/f_{00})^{1/2}\mathbf{H}) = 0, \tag{9a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}((\gamma/f_{00})^{1/2}\mathbf{H}), \tag{9b}$$

$$\nabla \cdot ((\gamma/f_{00})^{1/2}\mathbf{E}) = 0, \tag{9c}$$

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}((\gamma/f_{00})^{1/2}\mathbf{E}). \tag{9d}$$

Thus we see that the gravity field plays the role of a medium with dielectric permeabilities

$$\varepsilon = \mu = \left(\frac{\gamma}{f_{00}}\right)^{1/2} = \exp(2\lambda) = (1 - \Lambda_f r^2) \tag{10}$$

so that $\mathbf{B} = \varepsilon\mathbf{H}$ and $\mathbf{D} = \varepsilon\mathbf{E}$. In the next section we consider the motion of quarks in the strong gravity field in an analogous way.

3. Quark motion

In solving the Einstein field equation we consider only massless quarks. The equation in a gravity field for a spinor is described by Heisenberg-Pauli-Weyl (HPW) non-linear equations (Weyl 1950; Heisenberg 1966). In colour geometrodynamics the quarks are treated as massless scalar objects and the effect of nonlinear terms is taken into consideration by introducing an interaction term (of course nonlinear) so that it resembles mostly the squared version of HPW equations. There is also a model (Khadkikar and Gupta 1985) that deals with massive quarks in which vector potentials are required to deal with confinement. In our model we consider the quarks as massless, coloured spin $\frac{1}{2}$ objects moving in the strong gravitational field $f_{\mu\nu}$. The quarks are described by massless Dirac equations

$$(\boldsymbol{\sigma}\cdot\mathbf{p})\varphi = \omega\chi, \quad (\boldsymbol{\sigma}\cdot\mathbf{p})\chi = \omega\varphi. \tag{11}$$

Equation (11) can be cast into Maxwell-like form (Akhiezer and Berestetskii 1965)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{12}$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \tag{13}$$

as follows. Let

$$(\mathbf{E}, \mathbf{B}) \sim \exp(-i\omega t)$$

(12) and (13) take the form

$$\left.\begin{aligned}(\mathbf{S}\cdot\mathbf{p})_{jk}E_k &= i\omega B_j\\ (\mathbf{S}\cdot\mathbf{p})_{jk}B_k &= -i\omega \mathrm{E}_k\end{aligned}\right\} \tag{14}$$

where $P_k = -i\partial_k$ and $(Si)_{jk} = -i\varepsilon_{ijk}$. Similarity of (11) with (14) suggest that we may identify $\varphi \equiv \mathbf{E}$, $\chi \equiv i\mathbf{B}$ and $\boldsymbol{\sigma} \equiv \mathbf{S}$ to get the Maxwell-like form. Now the manner in which (12) and (13) is modified by the gravity field is known from (9), so that (11) can be modified and cast into the form

$$(\boldsymbol{\sigma}\cdot\mathbf{p})\varphi = \omega\varepsilon\chi, \tag{15}$$

$$(\boldsymbol{\sigma}\cdot\mathbf{p})\chi = \omega\varepsilon\varphi, \tag{16}$$

in the gravitational background. Equations (15) and (16) now represent motion of massless quarks in strong gravity field that takes care of all the interactions among the quarks. An analogous approach was also taken by Jena and Pradhan (1984) to treat confinement of quarks in a colour dielectric medium.

With the choice $\varepsilon(r) = 1 - \Lambda_f r^2/3$, the solution of (15) and (16) can be put in the standard form

$$\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \begin{pmatrix} f(r)\Omega_{jlm}(\theta,\varphi)\\ ig(r)\Omega_{jl'm}(\theta,\varphi)\end{pmatrix}, \tag{17}$$

with $l = j + \frac{1}{2}$ and $l' = 2j - 1$. The Ω_{jlm} are the spinor spherical harmonics. Using the relation $\Omega_{jl'm} = (i)^{l-l'}\boldsymbol{\sigma}\cdot\hat{r}\Omega_{jlm}$ in (15) and (16) we get the radial equations

$$f' + \frac{1+k}{r}f - \omega\varepsilon g = 0, \tag{18a}$$

$$g' + \frac{1-k}{r}g - \omega\varepsilon f = 0, \tag{18b}$$

where

$$k=\begin{cases} j+\frac{1}{2}=l; & \text{for } j=l-\frac{1}{2} \\ -(j+\frac{1}{2})=-(l+1); & \text{for } j=l+\frac{1}{2}, \end{cases}$$

is either a (+ve) or (−ve) integer and can never take zero. With the form of $\varepsilon(r)=1-\Lambda_f r^2/3$ we get

$$f''+\frac{2}{r}f'+\left\{\omega^2(1-\Lambda_f r^2/3)^2-\frac{k(k+1)}{r^2}\right\}f=0. \tag{19}$$

We solve (19) in the region $(\Lambda_f)^{1/2}r \ll 1$ so that (19) reduces to the form

$$f''+\frac{2}{r}f+\left\{\omega^2-\tfrac{2}{3}\omega^2\Lambda_f r^2-\frac{k(k+1)}{r^2}\right\}f=0. \tag{20}$$

With

$$\left.\begin{aligned} f&=\frac{1}{r}\chi_k \\ \lambda^2&=\tfrac{2}{3}\omega^2\Lambda_f \\ \mu&=\omega^2/2\lambda \end{aligned}\right\} \tag{21}$$

(20) becomes

$$\frac{d^2\chi_k}{dr^2}+\left\{\omega^2-\lambda^2r^2-\frac{k(k+1)}{r^2}\right\}\chi_k=0. \tag{22}$$

Substituting

$$t=\lambda r^2,\quad \chi_k=r^{k+1}\exp(-\lambda r^2/2)v(r) \tag{23}$$

(22) reduces to

$$t\frac{d^2v}{dt^2}+\{(k+\tfrac{3}{2})-t\}\frac{dv}{dt}-\left\{\tfrac{1}{2}(k+\tfrac{3}{2})-\frac{\mu}{2}\right\}v=0. \tag{24}$$

This is a Kummer equation with the complete solution

$$\begin{aligned} v&=C_1\,{}_1F_1(\tfrac{1}{2}(k+\tfrac{3}{2}-\mu),k+\tfrac{3}{2};\lambda r^2) \\ &\quad+C_2r^{-(2k+1)}{}_1F_1(\tfrac{1}{2}(-k+\tfrac{1}{2}-\mu),-k+\tfrac{1}{2};\lambda r^2). \end{aligned} \tag{25}$$

Here C_1, C_2 are constants and ${}_1F_1(a,b;\rho)$ is a confluent hypergeometric function. There are two sets of solutions for the (+ve) and (−ve) values of K.

(i) For $K>0, j=l-\frac{1}{2}$ and $K=j+\frac{1}{2}=l=1,2,3$ etc. Here $l=0$ is excluded as $K=0$. So at $r=0$, the second part of (25) blows up at origin and hence we take $C_2=0$

$$\therefore v=C_1\,{}_1F_1(\tfrac{1}{2}(l+\tfrac{3}{2}-\mu),l+\tfrac{3}{2};\lambda r^2), \tag{26}$$

(ii) For $K<0; j=l+\frac{1}{2}$ and $K=-(l+1)=-1,-2,\ldots$ etc. with $l=0,1,2,\ldots$. Now $2K+1=-2l-1$ and the second part of (25) can be defined at origin. Further $K+\frac{3}{2}=-l+\frac{1}{2}$ and the first part of (25) cannot be defined for $l>1$. So we put $C_1=0$ and

$$v=C_2r^{-(2k+1)}{}_1F_1(\tfrac{1}{2}(-k+\tfrac{1}{2}-\mu),-k+\tfrac{1}{2};\lambda r^2). \tag{27}$$

Using (26) and (27) we write the solution for $f_l(r)$ as

$$f_l(r) = \begin{cases} C_1 r^l \exp(-\lambda r^2/2)\, {}_1F_1(a,b;\rho), & \text{for } j = l - \tfrac{1}{2} \text{ with } l = 1,2,3,\ldots. \\ C_2 r^l \exp(-\lambda r^2)\, {}_1F_1(a,b;\rho), & \text{for } j = l + \tfrac{1}{2} \text{ with } l = 0,1,2,\ldots. \end{cases} \tag{28}$$

with $a = \frac{1}{2}(l + \frac{3}{2} - \mu)$, $b = l + \frac{3}{2}$ and $\rho = \lambda r^2$. A confluent hypergeometric series behaves asymptotically for a large positive value of its argument as

$${}_1F_1(a,b;\rho) \to \frac{\Gamma(b)}{\Gamma(a)} \exp(\rho) \cdot \rho^{a-b}$$

This leads to an expression

$$\chi_l \alpha r^{l+1} \exp(-\lambda r^2/2) \exp(\lambda r^2) r^{-(1/2)(l+(3/2)-\mu)},$$

which is exponentially divergent. To ensure good asymptotic behaviour, the confluent hypergeometric series must terminate and reduce to a polynomial. This imposes a condition

$$a = -n$$

or

$$\tfrac{1}{2}(l + \tfrac{3}{2} - \mu) = -n$$

with $n = 0, 1, 2, \ldots$ etc. which restricts the ω values to

$$\omega_{nl} = \left(\frac{8\Lambda_f}{3}\right)^{1/2} (2n + l + \tfrac{3}{2}), \tag{29}$$

on substitution of the values of μ and λ from (21). Using the relation

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\, {}_1F_1(a,b;\rho) = \frac{a}{b}\, {}_1F_1(a+1, b+1; \rho)$$

we get from (18)

$$g_{nl} = \begin{cases} \dfrac{C_1}{\omega\varepsilon} r^l \exp(-\lambda r^2/2) \left\{ \left(\dfrac{l}{r} - \lambda r^2 + \dfrac{1+l}{r} \right) {}_1F_1(a,b;\rho) \right. \\ \qquad \left. + 2\lambda r \cdot \dfrac{a}{b}\, {}_1F_1(a+1,b+1;\rho) \right\}, \quad \text{for } l = 1,2,3,\ldots \text{etc.} \\ \dfrac{C_2}{\omega\varepsilon} r^{l+1} \exp(-\lambda r^2) \lambda \{ {}_1F_1(a+1,b+1;\rho) - 2\, {}_1F_1(a,b;\rho) \}, \\ \qquad \text{for } l = 0,1,2,\ldots \text{etc.} \end{cases} \tag{30}$$

Introducing

$$M = 2n + l$$

We get

$$E_M = \omega_M = \left(\frac{8\Lambda_f}{3}\right)^{1/2} (M + \tfrac{3}{2}).$$

Thus we see that the energy levels are equispaced like the spectrum of an harmonic oscillator. As there is no continuum, it would not be possible to liberate a quark by supplying energy from outside. As λ is proportional to M, the exponential factor in the solution of the wave function indicates that the higher energy states are closer to the centre than the lower ones. Thus the gravitational field provides a trap for the confinement of quarks i.e., increasing the energy of a quark brings it closer to the centre rather than taking it away. We note that the energy levels are degenerate, except for ground state $M = 0$. So in our model the mass of the proton will be

$$m_p = 3 \times \tfrac{3}{2}(8\Lambda_f/3)^{1/2}$$

so that $\Lambda_f = 0{\cdot}016\,\mathrm{GeV} = m_\pi^2$ revealing a characteristic feature of strong interaction. As our solution is valid for regions $(\Lambda_f)^{1/2} r \ll 1$, it suggests that the quark orbits will well lie inside the hadron.

In retrospect, we have considered the motion of quarks in a strong gravitational background and find that quarks are confined in this tensor field background. The field equations of the massless quarks in the strong gravitational background is equivalent to the field equations in a colour dielectric medium. We find also that the colour dielectric constant is a tensor field and simulate the effect of colour dielectric medium.

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