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Variational calculation of the sine-Gordon effective potential

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Abstract. The Gaussian effective potential of the sine-Gordon model is calculated in $1+1$ and $2+1$ dimensions. Issues like renormalization, vacuum energy and stability of the vacuum are discussed in detail.

Keywords. Sine-Gordon model; Gaussian effective potential; stability of the vacuum.

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1. Introduction

The sine-Gordon (SG) model finds applications in different branches of physics. In $D=1$ dimension the model is exactly solvable (Rajaraman 1975; Fadeev and Korepin 1978) and is equivalent to a number of other models such as massive Thirring model, two dimensional Coulomb gas model, etc. Using a variational argument, Coleman (1975) showed that the vacuum of the model is stable only if $\beta^2 < 8\pi$. Perturbative calculations show that the model is renormalizable only for $D=1$. The impact of quantum effects on the classical potential is represented by the effective potential which is usually evaluated by the well-known loop expansion method (Coleman and Weinberg 1973). This method is a perturbative one as the calculations are not exact and usually the calculations are done only at the one loop level. The effective potential of the SG model was also calculated earlier using this approach (Babu Joseph and Kuriakose 1982).

It has been realized recently that the effective potential of quantum field theories can be calculated nonperturbatively by making use of the variational calculation of the ground state of quantum mechanical systems (Barnes and Ghandour 1980; Bardeen and Moshe 1983; Stevenson 1985). In this approach a trial wave functional of the Gaussian form is used and hence the effective potential so calculated is known as the Gaussian effective potential (GEP). This method is superior to the loop expansion method in the sense that the calculation can be made exactly and hence the method is nonperturbative. Though in the GEP approach the harmonic oscillator wave function is chosen as trial wave function, it was shown earlier that the method works well for φ^4 and φ^6 models (Stevenson 1985; Stevenson and Roditi 1986).

In this paper, following Stevenson (1985), we calculate the GEP of the SG model in $D=1$ and $D=2$ dimensions and the study of 2-dimensional SG model is reported as it is useful in the study of 2-dimensional Josephson effect (Elibeck *et al* 1985). The Coleman restriction on the β parameter, viz., $\beta^2 < 8\pi$ for $D=1$ case follows automatically in this formalism and for $D=2$ case the restriction gets modified as $\beta^2 < 16\pi$. Earlier Ingermanson (1986) using a different approach calculated the GEP of the SG model and our results agree with his results.

2. The GEP for SG model

2.1 *Formulation*

The Lagrangian for the SG model in $D+1$ dimension can be written as

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \varphi)^2 + \frac{\alpha_0}{\beta^2}\cos(\beta\varphi) + \gamma \tag{1}$$

where α_0 and β are bare parameters. The Hamiltonian density of the model is

$$\mathscr{H} = \tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}(\nabla\varphi)^2 - \frac{\alpha_0}{\beta^2}\cos(\beta\varphi) - \gamma. \tag{2}$$

The effective potential $V(\varphi_0)$ is defined as the expectation value of the Hamiltonian density in a normalized state $|\varphi_0\rangle$ in which the field φ has a constant expectation value $\langle\varphi\rangle = \varphi_0$ and for which $V(\varphi_0)$ is a minimum. To evaluate $V(\varphi_0)$ variationally we need consider a set of states $|\varphi_0\rangle$ with some parametrization μ and minimization of $\langle\varphi_0|\mathscr{H}|\varphi_0\rangle$ with respect to the parameter then yields the effective potential. We will consider the set $|0\rangle_{\mu,\varphi_0}$ which is a normalized Gaussian wave functional centred on $\varphi = \varphi_0$ as trial wave function such that

$${}_{\varphi_0,\mu}\langle 0|\varphi|0\rangle_{\mu,\varphi_0} = \varphi_0 \quad \text{and} \quad {}_{\varphi_0,\mu}\langle 0|0\rangle_{\mu,\varphi_0} = 1.$$

The variable parameter μ, the mass parameter, must be positive ($0 < \mu < \infty$) for the wave functional to be normalizable.

Let us write now $\varphi = \varphi_0 + \tilde{\varphi}$ where φ_0 is a constant classical field and $\tilde{\varphi}$ is a quantum field of mass μ and $|0\rangle_{\mu,\varphi_0}$ is the vacuum state of the free field. Then

$$\varphi = \varphi_0 + \int (\mathrm{d}k)_\mu [a_\mu \exp(-ikx) + a_\mu^+ \exp(ikx)] \tag{3}$$

$$\partial_\mu \varphi = \int (\mathrm{d}k)_\mu (-ik_\mu)[a_\mu \exp(-ikx) - a_\mu^+ \exp(ikx)] \tag{4}$$

$$k^0 = \omega_k(\mu) = (k^2 + \mu^2)^{1/2} \tag{5}$$

$$(\mathrm{d}k)_\mu = \frac{\mathrm{d}^D k}{(2\pi)^D 2\omega_k(\mu)} \tag{6}$$

$$[a_\mu(k), a_\mu^+(k')] = \delta_{kk'}; \quad a_\mu^{(k)}|0\rangle_\mu = 0 \tag{7}$$

The GEP is given by

$$\begin{aligned} V(\varphi_0) &= \min_\mu V_G(\varphi_0, \mu) \\ &= \min_\mu {}_{\varphi_0,\mu}\langle 0|\mathscr{H}|0\rangle_{\mu,\varphi_0}. \end{aligned} \tag{8}$$

$V_G(\varphi_0, \mu)$ is obtained from $\langle 0|\mathscr{H}|0\rangle$ which is calculated straightforwardly by substituting (3) and (4) in (2):

$$_{\varphi_0,\mu}\langle 0|\mathscr{H}|0\rangle_{\mu,\varphi_0} = \int (\mathrm{d}k)_\mu [\omega_k^2(\mu) - \tfrac{1}{2}\mu^2] - \gamma - \frac{\alpha_0}{\beta^2}\cos(\beta\varphi_0)\exp\left[-\int \frac{\mathrm{d}^D k}{(2\pi)^D}\left(\frac{1}{2(k^2+\mu^2)^{1/2}}\right)\right]. \tag{9}$$

Introducing the notation

$$I_N(\mu) = \int (\mathrm{d}k)_\mu [\omega_k^2(\mu)]^N \tag{10}$$

where N is an integer, we rewrite (9) as

$$V_G(\varphi_0, \mu) = I_1 - \tfrac{1}{2}\mu^2 I_0 - \frac{\alpha_0}{\beta^2}\cos(\beta\varphi_0)\exp\left[-\frac{\beta^2 I_0}{2}\right] - \gamma. \tag{11}$$

The GEP is obtained by minimising $V_G(\varphi_0, \mu)$ with respect to μ $(0 < \mu < \infty)$. $\mathrm{d}V_G/\mathrm{d}\mu^2 = 0$ gives the value of μ which minimises V_G and let μ_0 be the optimum value of μ. Thus we find

$$\left.\frac{\mathrm{d}V_G}{\mathrm{d}\mu^2}\right|_{\mu^2=\mu_0^2} = 0$$

gives

$$\mu_0^2 = \alpha_0 \cos(\beta\varphi_0)\exp\left(-\frac{\beta^2 I_0}{2}\right) \tag{12}$$

where we have made use of the result

$$\frac{\mathrm{d}I_N}{\mathrm{d}\mu^2} = \frac{(2N-1)}{2} I_{N-1}. \tag{13}$$

Substituting (12) in (11) we obtain the GEP:

$$V(\varphi_0) = I_1 + \alpha_0\left(\frac{1}{\beta^2} + \frac{I_0}{2}\right)\cos(\beta\varphi_0)\exp\left(-\frac{\beta^2 I_0}{2}\right) - \gamma. \tag{14}$$

This equation contains divergent integrals and bare parameters. A finite expression for $V(\varphi_0)$ can be obtained by removing the divergent integrals and renormalising the bare parameters. From the effective potential the renormalised parameters α and β_R can be calculated as

$$\alpha = \left.\frac{\mathrm{d}^2 V(\varphi_0)}{\mathrm{d}\varphi_0^2}\right|_{\varphi_0=0}; \quad \alpha\beta_R^2 = \left.\frac{\mathrm{d}^4 V(\varphi_0)}{\mathrm{d}\varphi_0^4}\right|_{\varphi_0=0} \tag{15}$$

As μ_0 depends on φ_0, from (12) we can find

$$\frac{\mathrm{d}\mu_0^2}{\mathrm{d}\varphi_0} = \frac{\alpha_0\beta\sin(\beta\varphi_0)\exp\left(-\dfrac{\beta^2 I_0}{2}\right)}{\dfrac{\alpha_0\beta^2}{2}\cos(\beta\varphi_0)\exp\left(-\dfrac{\beta^2 I_0}{2}\right)\cdot I_{-1}} \tag{16}$$

Differentiating (14) with respect to φ_0 twice and making use of the last equation, we have

$$\alpha = \alpha_0 \exp\left(-\frac{\beta^2 I_0(\bar{\mu}_0)}{2} \right) \tag{17}$$

when $\bar{\mu}_0$ is the value of μ_0 at $\varphi_0 = 0$. But the μ_0 equation (12) shows that the right hand side of the above equation is just $\bar{\mu}_0^2$ and therefore we find that $\alpha = \bar{\mu}_0^2$. This identification implies that α is positive definite. Thus we may write

$$\alpha = \alpha_0 \exp\left(-\frac{\beta^2}{2} I_0(\alpha) \right) \tag{18}$$

The expression for $V_G(\varphi_0, \mu)$ in terms of α now reads

$$V_G(\varphi_0, \mu) = I_1 - \tfrac{1}{2}\mu^2 I_0 - \frac{\alpha}{\beta^2}\cos(\beta\varphi_0)\exp\left(-\frac{\beta^2}{2}(I_0 - I_0(\alpha)) \right) - \gamma. \tag{19}$$

2.2 *1 + 1 dimension*

The I_N integrals are divergent and therefore $I_0(\alpha)$ in (18) can be evaluated by introducing an U.V. cut off Λ. Then we find

$$\begin{aligned} \alpha &= \alpha_0 \left(\frac{\alpha_0}{\Lambda} \right)^{\beta^2/(8\pi - \beta^2)} \\ &= \alpha_0 \left(\frac{\alpha_0}{\Lambda} \right)^{\beta'^2/8\pi} \end{aligned} \tag{20}$$

where

$$\beta'^2 = \frac{\beta^2}{(1 - \beta^2/8\pi)} \,. \tag{21}$$

If $\beta^2/8\pi$ is small compared to one, we obtain the Coleman's (1975) result for mass renormalization, and (20) can be treated as an improvement over the perturbative calculation.

Following the programme of Stevenson (1985) to handle the divergent I_N integrals, we can rewrite (19) as

$$\begin{aligned} V_G(\varphi_0, \mu) = {} & I_1(\alpha) - \tfrac{1}{2}\alpha^2 I_0(\alpha) + \frac{\mu^2}{8\pi} L_1(x) - \frac{\alpha^2}{8\pi} L_2(x) \\ & - \frac{\alpha}{\beta^2}\cos(\beta\varphi_0)\exp\left(\frac{\beta^2}{8\pi} L_1(x) \right) - \gamma \end{aligned}$$

where

$$x = \frac{\mu^2}{\alpha^2}; \quad L_1(x) = \ln x$$

$$L_2(x) = x \ln x - (x - 1) \,.$$

The expression for V_G still contains divergent integrals. The divergent expression $D = I_1(\alpha) - \frac{1}{2}\alpha^2 I_0(\alpha)$ represents the vacuum energy density of the field. The presence of this divergent expression has no physical significance as the energy differences are only measurable and therefore D may be subtracted from $V_G(\varphi_0, \mu)$ to yield a finite result, namely,

$$\bar{V}_G(\varphi_0) = -\frac{\alpha}{\beta^2}\left(\frac{\mu^2}{\alpha^2}\right)^{\beta^2/8\pi} \cos(\beta\varphi_0) + \frac{1}{8\pi}(\mu^2 - \alpha^2) - \gamma. \tag{22}$$

This expression gives the GEP for SG model in $1+1$ dimension. Now let us check it for the end point values of μ viz., $\mu = 0$ and $\mu = \infty$. The $\mu = 0$ gives a finite result while in the case of $\mu \to \infty$, the GEP is bounded below only if $\beta^2 < 8\pi$. This result coincides with the Coleman result.

A root of the μ_0 equation (12) will correspond to a minimum of V_G if

$$\left.\frac{\mathrm{d}^2 V_G}{\mathrm{d}(\mu^2)^2}\right|_{\mu^2 = \mu_0^2} > 0.$$

The I_{-1} integral gives a finite value in $1+1$ dimension:

$$I_{-1} = \frac{1}{2\pi\mu_0^2}.$$

Thus we find

$$\frac{\mathrm{d}^2 V_G}{\mathrm{d}(\mu^2)^2} = \frac{I_{-1}}{4}\left(1 - \frac{\beta^2}{8\pi}\right). \tag{23}$$

This means that the vacuum is stable only if

$$1 - \frac{\beta^2}{8\pi} > 0$$

i.e.,

$$\beta^2 < 8\pi. \tag{24}$$

From this condition we can see that the nature of the vacuum depends on the value of $\beta^2 \cdot \beta^2 > 8\pi$ implies an unstable vacuum and $\beta^2 = 8\pi$ can be treated as the transition point.

2.3 *2 + 1 dimension*

In this case the renormalized α-parameter is given by

$$\alpha = \alpha_0 \exp\left(-\frac{\beta^2 \Lambda}{16\pi}\right) \tag{25}$$

where Λ is the U.V. cut off parameter. This form is entirely different from (20). This means that the form of the α-renormalization is dimension dependent. The expression for the GEP is given by (Stevenson 1985)

$$\bar{V}_G(\varphi_0) = -\frac{\alpha}{\beta^2}\cos(\beta\varphi_0)\exp\left(\frac{\alpha\beta^2}{8\pi}(x^{1/2}-1)\right) - \frac{\alpha}{8\pi}(\mu^2(x^{1/2}-1)) + \frac{\alpha^2}{3}(x^{1/2}-1)^2(2x^{1/2}+1) - \gamma \tag{26}$$

where

$$x = \mu^2/\alpha^2 \, .$$

From this expression it follows that the $\mu \to 0$ end point yields a finite result; however, the GEP is not bounded below when $\mu \to \infty$.

Now we will examine the stability of the vacuum. In 2 + 1 dimension also the I_{-1} integral is finite and $I_{-1} = 1/4\pi\mu_0^2$. Proceeding exactly as before we find that in the present, the vacuum is stable only if

$$\beta^2 < 16\pi. \tag{27}$$

This means that the criterion for the stability of the vacuum is dimension-dependent.

3. Conclusion

We have calculated the effective potential for the sine-Gordon theory in 1 + 1 and 2 + 1 dimensions. Just as in the Ingermanson approach, the present analysis also cannot be extended to higher spatial dimensions as the I_{-1} integral develops divergence for $D \geqslant 3$. For $D = 1$, the present calculation reproduces exactly the Coleman restriction on the β parameter, viz., $\beta^2 < 8\pi$ for the energy density to be bounded below. Moreover, vacuum of the model is stable in 1 + 1 dimension only if $\beta^2 < 8\pi$, while in 2 + 1 dimension this condition gets modified as $\beta^2 < 16\pi$. The α-renormalization result also changes with the spatial dimension.

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