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An analysis of the mass formulae for S- and P-wave mesons

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Abstract. Mass regularities for S- and P-wave mesons and relations between their masses are discussed. A detailed analysis is given for S-wave mesons which extends the investigations on P-wave mesons reported earlier. Masses for the S- and P-states of all interesting $q\bar{q}$-systems (including toponium states) are predicted. Partial understanding of the mass formulae is obtained within a general potential model approach. Scaling arguments are presented which support the empirical scaling behaviour found for the expectation values determining the spin-splittings in the potential picture.

Keywords. Quarkonium; mesons; mass formulae.

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1. Introduction

An analysis of the masses of the low-lying quarkonia states reveals some striking regularities (Gupta and Kögerler 1988). The mass $M(q_1\bar{q}_2)$ of a bound state of a quark q_1 and an antiquark $\bar{q}_2$ depends very simply on $x \equiv (m_1 + m_2)$, where m_1 and m_2 are the masses of the constituents. The generic mass formula

$$M(q_1\bar{q}_2) = A + Bx, \tag{1}$$

seems to give the dominant behaviour of the masses of the S- and P-wave mesons for a given J^P, where A and B are constants independent of the flavour quantum numbers. For the four $J^P = 2^{++}, 1^{++}, 1^{+-}, 0^{++}$, P-wave mesons A increases with J and lies between 0·52 and 0·7 GeV. Remarkably enough, the slope $B = 0{\cdot}92$–$0{\cdot}93$ is found to be practically independent of J^P. For the two $J^P = 1^{--}$ and 0^{-+}, S-wave states (Gupta and Kögerler 1988) hereafter referred to as [GK] the linear formula (1) works well with the same value B as for the P-wave mesons, if one excludes the two lowest pseudoscalar mesons, the π and the K.

The other regularity found was for the expectation values which determine the masses if a potential is used to describe the $q\bar{q}$-interactions.

In a potential model approach (Harrington *et al* 1975; Eichten *et al* 1975; Barbieri *et al* 1975), which has been fairly successful in describing quarkonia spectra, the $q\bar{q}$-system is taken to interact non-relativistically through a potential of the form

$$U = U_0 + U_{\text{spin}}. \tag{2}$$

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The spin-independent part U_0 (which may be velocity-dependent) will essentially yield $\bar{m}_L$, the centre of gravity (CoG) of the states with the same orbital angular momentum L, while U_{spin} contains the spin-dependent terms which cause spin-splittings of these levels. In this framework the meson mass for given L and J will have the form,

$$M(q_1\bar{q}_2) = \bar{m}_L + \langle U_{\text{spin}} \rangle_{L,J}. \tag{3}$$

Earlier, in [GK], the mass splittings of mesons with $L \geqslant 1$ were discussed on the basis of a general U_{spin} generated by a static potential U_{NR} (contained in U_0) which consisted of a vector part $V(r)$ and a scalar part $S(r)$. There are strong reasons, both theoretical (Eichten and Feinberg 1981; Gromes 1984) and phenomenological (Martin 1986; Buchmüller 1985), that U_{NR} has such a mixed Lorentz character.

In this case, assuming that U results from a non-relativistic expansion of the Bethe-Salpeter equation, to order $(v/c)^2$, the spin-dependent potential is given as

$$U_{\text{spin}} = \tfrac{1}{2}(\tilde{c}_1 + \tilde{c}_2)\mathbf{L}\cdot(\mathbf{S}_1 + \mathbf{S}_2) + \tfrac{1}{2}(\tilde{c}_1 - \tilde{c}_2)\mathbf{L}\cdot(\mathbf{S}_1 - \mathbf{S}_2) + \tilde{c}_3(3\mathbf{S}_1\cdot\hat{r}\mathbf{S}_2\cdot\hat{r} - \mathbf{S}_1\cdot\mathbf{S}_2) + \tilde{c}_4\mathbf{S}_1\cdot\mathbf{S}_2, \tag{4}$$

where

$$\tilde{c}_i = \frac{1}{m_1 m_2}\frac{V'}{r} + \frac{1}{2m_i^2}\left(\frac{V'}{r} - \frac{S'}{r}\right), \quad i = 1,2 \tag{4a}$$

$$\tilde{c}_3 = \frac{1}{3m_1 m_2}\left(-V'' + \frac{V'}{r}\right), \tag{4b}$$

$$\tilde{c}_4 = \frac{2}{3m_1 m_2}\nabla^2 V. \tag{4c}$$

Prime denotes derivative with respect to r. The four masses for a given $L \geqslant 1$, in (3), on using (4) then depend only on three unknown expectation values

$$e_1 \equiv \left\langle \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}r} \right\rangle, \quad e_2 \equiv \left\langle \frac{\mathrm{d}^2 V}{\mathrm{d}r^2} \right\rangle \quad \text{and} \quad e_3 \equiv \left\langle \frac{1}{r}\frac{\mathrm{d}S}{\mathrm{d}r} \right\rangle \tag{5}$$

contained in the second term in (3). This is because $\langle \nabla^2 V \rangle = \langle V'' + 2V'/r \rangle$, if V is no more singular than $1/r$ near $r = 0$ and a $\delta^3(\mathbf{r})$ term—if present in $\nabla^2 V$—will not contribute to the expectation value for $L \geqslant 1$. Note in [GK], we had used the notation $v_i = e_i/m_1 m_2$ $(i = 1,2)$ and $s = e_3/m_1 m_2$. These expectation values, when extracted from the experimentally known P-wave meson masses, for the $u\bar{s}$, $c\bar{c}$, $b\bar{b}$ and the $I = 1$ $u\bar{u}$ systems,* were all found to scale as

$$e_i = a_i + b_i \mu, \quad i = 1,2,3 \tag{6}$$

where μ is the reduced mass of the system and a_i and b_i are constants.

Predictions for the P-wave meson masses of other $q\bar{q}$-systems ($u\bar{c}$, $u\bar{b}$ etc.) based on mass formulae (1) and those based on the potential approach (3) and (6) have already been given [GK].

* Throughout the paper we do not consider the isosinglet $u\bar{u}$- nor the $s\bar{s}$-states since they are affected by complications like octet-singlet mixing and/or strong mixing with glueball-groundstates.

In this note our object is firstly to extend the above type of analysis to the S-wave mesons. This is done in §2. Secondly, an attempt is made, to understand the scaling behaviour of the expectation values using general scaling arguments in terms of simple forms for $V(r)$ and $S(r)$, which are currently popular. These suggest the possibility that e_3 may scale as $a + b\mu^{1/3}$, while e_1 and e_2 still obey (6). New predictions for P-wave masses are given for this scenario in §3. The scaling arguments are discussed in §4. In §5 general consequences of the mass formulae, like (1), are discussed and relations between the meson masses for the S- and P-wave mesons are given.

2. Analysis of S-wave mesons

Experimentally (Particle data group 1986), the masses of the seven $1S$ vector mesons for the $u\bar{u}$, $u\bar{s}$, $u\bar{c}$, $s\bar{c}$, $c\bar{c}$, $u\bar{b}$ and $b\bar{b}$ systems are known. The corresponding pseudoscalar masses are available only for the first six systems. We use M_V and M_P to denote the mass of the 1^{--} and 0^{-+} mesons. Their centre of gravity is defined by

$$\bar{m}_0 = \frac{3M_V + M_P}{4}. \tag{7}$$

The possibility that the S-wave meson masses satisfy a linear mass formula like (1) was shown in [GK] but was not analysed in detail nor predictions for other $q\bar{q}$ systems were given. Here we investigate S-wave masses both from the point of view of possible simple mass formulae and of a potential model approach.

Mass formulae As mentioned above, the linear formula does not work well for the pseudoscalars if one wants to fit the π and K mesons, too. Phenomenologically,

$$M_P(q_1\bar{q}_2) = A_P + B_P x + C_P/x, \tag{8}$$

gives a satisfactory fit. The motivation for choosing the third term is two-fold. Firstly, for large x, (8) would become linear in x and, secondly, such a term is expected from the quark mass dependence of the expectation value which would determine $M_V - M_P$ in a potential model approach. Using the $u\bar{u}$, $u\bar{s}$ and $c\bar{c}$ systems as inputs we get

$$A_P = 0{\cdot}2969\,\text{GeV}, \quad B_P = 0{\cdot}9099 \quad \text{and} \quad C_P = -0{\cdot}4233\,\text{GeV}^2. \tag{8a}$$

The quark masses (in GeV) used for the fit were $m_u = m_d = 0{\cdot}3\,\text{GeV}$, $m_s = 0{\cdot}5\,\text{GeV}$, $m_c = 1{\cdot}55\,\text{GeV}$ and $m_b = 5\,\text{GeV}$. These values have been used earlier [GK] to obtain linear mass formulae for P-wave mesons and we stick to them throughout this paper. This choice for the quark masses is what is normally used. It is conceivable that by varying them one may be able to further improve the fits given here.

For the vector mesons, the linear formula given before [GK]

$$M_V(q_1\bar{q}_2) = A_V + B_V x = 0{\cdot}2381 + 0{\cdot}9222x, \tag{9}$$

works very well. The constants A_V (in GeV) and B_V were determined using $M_V(c\bar{c})$ and $M_V(b\bar{b})$ masses as inputs. Inclusion of a x^{-1} term does not really improve the fit

Table 1. Predictions for the masses of the 1*S* vector (M_V) and pseudoscalar (M_P) mesons. (a) From the mass formulae (8) and (9). The input masses are marked by an asterisk. (b) From the fit (12b) to the expectation value and the parametrization (10) of the centre of gravity $\bar{m}_0$ (given in column 1). In brackets are the experimental values. In case of charge multiplets we quote the average mass. $s\bar{s}$-states have been omitted because they are affected by octet-singlet-mixing.

	$\bar{m}_0$ (MeV)	M_V (MeV)			M_P (MeV)		
		(a)		(b)	(a)		(b)
$u\bar{u}$	628	791	(ρ(770))	786	137·3*	(π(137·3))	154
$u\bar{s}$	856	976	(K^*(894))	951	495·7*	(K(495·7))	569
$u\bar{c}$	1896	1944	(D^*(2008))	1931	1752	(D(1867))	1790
$s\bar{c}$	2085	2129	(D_s^*(2110))	2118	1956	(D_s(1971))	1988
$c\bar{c}$	3068	3096·9*	(J/ψ(3096·9))	3097	2981·1*	(η_c(2981·1))	2981
$u\bar{b}$	5104	5126	(B^*(5325)?)	5116	5040	(B(5273))	5068
$s\bar{b}$	5289	5310		5302	5224		5250
$c\bar{b}$	6257	6279		6273	6192		6208
$b\bar{b}$	9433	9460*	(Υ(9460))	9447	9354		9393

given by (9). So, taking (8) and (9), one obtains from (7)

$$\bar{m}_0(q_1\bar{q}_2) = \bar{A}_0 + \bar{B}_0 x + \bar{C}_0/x$$
$$= 0{\cdot}2528 + 0{\cdot}9191x - 0{\cdot}1058/x. \tag{10}$$

The predictions for the masses of the other quarkonium systems on the basis of (8) and (9) are presented in table 1 (entry a). It is seen, that the predictions from these purely phenomenological fits deviate not more than 5% from the experimental data, except for the $u\bar{s}$, where the prediction is too high by 9%.

Although we are unable to give an explanation why these simple mass formulae work so well, we should at least try to understand the different x dependences of the fits in (8) and (9). Such an understanding can be obtained within the potential model approach discussed below.

Potential model approach For *S*-wave mesons, using (3), (4) and (7), the masses for the $q_1\bar{q}_2$-system are given by

$$M_P = \bar{m}_0 - \frac{1}{2m_1 m_2} e_0 \tag{11a}$$

$$M_V = \bar{m}_0 + \frac{1}{6m_1 m_2} e_0, \tag{11b}$$

with

$$e_0 \equiv \langle \nabla^2 V \rangle_s. \tag{11c}$$

The mass difference $M_V - M_P$ is determined solely by the spin-spin interaction through the S-state expectation value e_0. Earlier, such a mass formula has been investigated (Zeldovich and Sakharov 1967; Lipkin 1987) using $\bar{m}_0 = m_1 + m_2$ for S-wave mesons and baryons. Recently, in an attempt to understand the empirical regularity (Martin 1982) $M_V^2 - M_P^2 \simeq$ constant it was suggested (Frank and O'Donnell 1985; Lichtenberg 1987) that e_0 is proportional to μ. However, on extracting e_0 from the data, one finds that a linear behaviour of e_0 with respect to μ is too simplistic as can be seen from figure 1. An approximate linear behaviour would be suggested only if one was to exclude the points corresponding to the $u\bar{u}$ and $u\bar{s}$-systems. To take the curvature of e_0 with μ into account we have used a parametrization of the form

$$e_0 = a_0 + b_0\mu + c_0\mu^\beta. \tag{12}$$

Of course, one has many choices for β, the power of μ in the "correction"-term, which could do the job. We consider two simple choices.

(a) $\beta = -1$ Using the values of e_0 for the $u\bar{u}$, $u\bar{c}$ and the $c\bar{c}$ systems, one obtains for e_0 (in MeV GeV^2) and μ (in GeV)

$$e_0 = -172 + 724\mu + 22{\cdot}3/\mu. \tag{12a}$$

This gives a good fit for $s\bar{c}$ within 1%, but gives a too low value, by about 8% (5%) for the $u\bar{s}$ ($u\bar{b}$) system. Note that the term linear in μ dominates.

Since $m_1 m_2 = \mu x$, it is clear from (11) that the linear term in μ in (12) will contribute a $1/x$ term to M_V and M_P with opposite signs. This term almost cancels the $1/x$ term coming from $\bar{m}_0$ (10) in M_V but not in M_P. Since the other terms in (12a) are small, this helps us to understand why M_V has a simpler behaviour in x than M_P.

Though the choice $\beta = -1$ is fine phenomenologically, it is not easy to motivate using scaling arguments (see §4).

(b) $\beta = 1/3$ This particular choice is motivated by the scaling arguments given in §4 below and is expected for $V(r) \sim \alpha(r)/r$ arising from one gluon exchange. Again, using $u\bar{u}$, $u\bar{c}$ and $c\bar{c}$ systems as inputs for e_0 (in MeV GeV^2) and μ (in GeV), one obtains

$$e_0 = 487 + 1212\mu - 1098{\cdot}1\,\mu^{1/3}. \tag{12b}$$

This also gives a good overall fit. The values obtained for $u\bar{s}$, $s\bar{c}$ and $u\bar{b}$ systems are smaller by about 4%, 6% and 7% respectively. The term in μ is largest and, for sufficiently large μ, it will dominate the $\mu^{1/3}$ term.

When (12b) is used in (11) together with (10), the combined effect of the terms with different μ and x dependence, is to nullify the effect of the $1/x$ term in $\bar{m}_0$ for M_V but not for M_P. This again shows the quantitative consistency of the mass formulae ((8) and (9)) and the potential model approach.

For comparison, the two parametrizations (12a) and (12b) are plotted in figure 1 together with the experimentally deduced values of e_0. It is clear that (12b) predicts larger values for $M_V - M_P$ (with increasing μ) as compared to (12a), even though both are about the same, quantitatively, till about $\mu = \mu(c\bar{c})$. It is interesting that (12a)

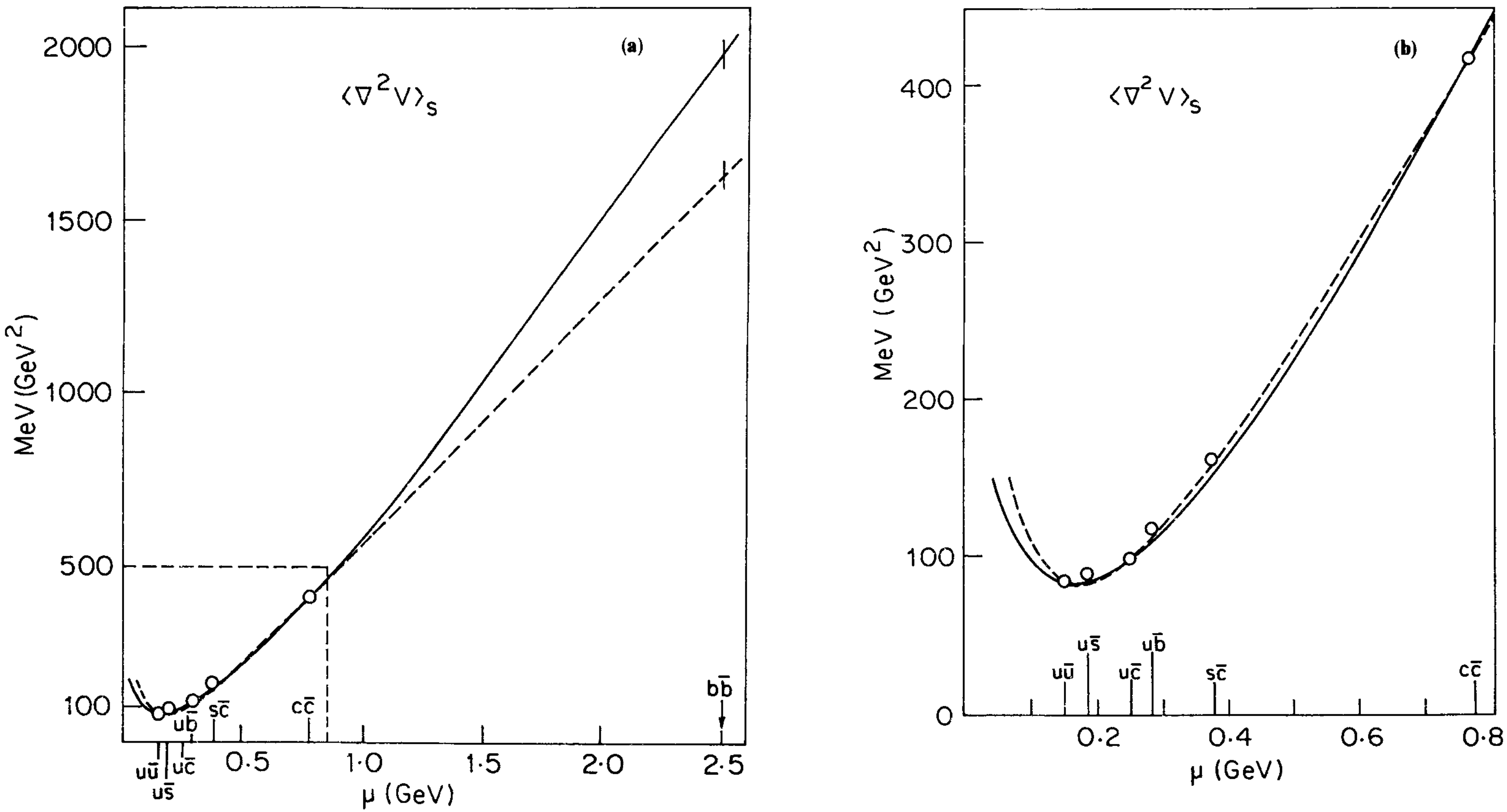

Figure 1. Expectation value $e_0 = \langle \nabla^2 V \rangle_S$ in MeV GeV^2 for S-wave mesons as a function of the reduced mass μ in GeV. The dotted and solid curves represent the parametrizations (12a, b). The circles denote the known experimental values.

(**a**) μ scale up to $\mu(b\bar{b}) = 2{\cdot}5$ GeV, to show the difference in predictions for large μ
(**b**) μ scale up to $\mu(c\bar{c}) = 0{\cdot}775$, to bring out the difference for small μ.

Table 2. Predictions for the $1S$ and $1P$ toponium ($t\bar{t}$) states for top-quark-mass m_t between 30 and 60 GeV. The entries for the $1S$-states are based on (10) and (12b), while for the $1P$-states they are based on (15), (16) and (17). All masses are given in GeV.

m_t (GeV)	$1S$		$1P$			
	$M_V(1^{--})$	$M_P(0^{-+})$	$M_2(2^{++})$	$M_0(0^{++})$	$M_1(1^{++})$	$M_1'(1^{+-})$
30	55·4000	55·3882	56·0861	56·0750	56·0814	56·0834
40	73·7817	73·7727	74·5593	74·5509	74·5557	74·5572
50	92·1636	92·1562	93·0327	93·0259	93·0298	93·0310
60	110·5455	110·5393	111·5062	111·5005	111·5037	111·5048

predicts $M_P(b\bar{b}) = \eta_b = 9{\cdot}400\,\text{GeV}$ which is about 7 MeV higher than the value obtained from (12b). An experimental discovery of this state in the near future would help to distinguish the two fits.

The potential model only specifies the mass difference $M_V - M_P$ in terms of e_0. To calculate masses, without specifying $V(r)$ and $S(r)$, one needs a parametrization for $\bar{m}_0$. Predictions for the S-wave masses, using (10) and (12) are also given in table 1. The parametrization of (12b) was chosen since it can be motivated by scaling arguments. The new predictions are slighly nearer to the experimental values in most cases than those given by the mass formulae (8) and (9). For higher mass $q\bar{q}$-states both types of predictions are nearly the same.

For future reference, predictions for the toponium based on (10) and (12b) are given in table 2 for the top quark mass $m_t = 30$–$60\,\text{GeV}$.

3. P-wave mesons

We denote the masses of the four $J^{PC} = 2^{++}$, 1^{++}, 0^{++} and 1^{+-} $1P$-states by M_2, M_1, M_0 and M_1', respectively. Their centre of gravity $\bar{m}_1$ is defined by

$$\bar{m}_1 = \frac{5M_2 + 3M_1 + 3M_1' + M_0}{12}. \tag{13}$$

Mass formulae The isospin $I = 1$ $u\bar{u}$, $I = 1/2\, u\bar{s}$, $c\bar{c}$ and $b\bar{b}$ systems conidered earlier [GK] satisfy linear mass formulae in x as mentioned in the introduction. With data for only these four cases there is no cause at present to consider any modification of (1). Our old results stand. The main urgent theoretical problem is to understand, why the constant $B = 0{\cdot}92$–$0{\cdot}93$ is independent of the J^P of the P-state and is also same for the two S-states. We have no clear answer to this question at present.

New results in potential model approach For P-wave mesons, from (3), (4) and (5), the four masses for the $q_1\bar{q}_2$-system are given by

$$M_2 = \bar{m}_1 + \frac{1}{m_1 m_2}\left[\tfrac{13}{10}e_1 + \tfrac{1}{5}e_2 + \frac{1}{4}\left(\frac{m_1}{m_2} + \frac{m_2}{m_1}\right)(e_1 - e_3)\right], \tag{14a}$$

$$M_0 = \bar{m}_1 + \frac{1}{m_1 m_2}\left[-2e_1 + \tfrac{1}{2}e_2 - \frac{1}{2}\left(\frac{m_1}{m_2} + \frac{m_2}{m_1}\right)(e_1 - e_3)\right], \tag{14b}$$

$$M_1 = \bar{m}_1 + \frac{1}{m_1 m_2}\left[\tfrac{1}{2}e_1 + \frac{1}{4}\left(\frac{m_1}{m_2} + \frac{m_2}{m_1}\right)(e_1 - e_3)\right], \tag{14c}$$

$$M'_1 = \bar{m}_1 - \frac{1}{m_1 m_2}[e_1 + \tfrac{1}{2}e_2]. \tag{14d}$$

Furthermore,

$$\delta = \frac{1}{2\sqrt{2}}\frac{1}{m_1 m_2}\left(\frac{m_2}{m_1} - \frac{m_1}{m_2}\right)(e_1 - e_3). \tag{14e}$$

The parameter δ determines the mixing between the 1^{++} and 1^{+-} states arising from the second term in (4). It is clearly non-zero only for the unequal mass case, that is, when $m_1 \neq m_2$. In these cases the masses of the physical states are denoted by

$$M_{\substack{\alpha \\ \beta}} = \frac{(M_1 + M'_1) \mp \sqrt{(M_1 - M'_1)^2 + 4\delta^2}}{2}. \tag{14f}$$

For details see [GK].

Knowing the four masses, the three expectation values e_i were extracted in [GK] from (14) and found to scale linearly with μ as in (6). For this parametrization the tentative experimental value $9894{\cdot}8 \pm 1{\cdot}5$ MeV was used for $M'_1(b\bar{b})$. However, it has been recently argued (Gupta and Kögerler 1988) that overall data for the $c\bar{c}$ and $b\bar{b}$ systems suggest that the ${}^1P_1(b\bar{b})$ state (like the ${}^1P_1(c\bar{c})$ level) should lie at centre of gravity of the three ${}^3P_J$-levels, that is at $9900{\cdot}2 \pm 0{\cdot}7$ MeV. This few MeV difference crucially influences the value of $e_3(b\bar{b})$. In fact, the new value is half the old one, and as a result e_3 does not admit a straight line anymore. Furthermore, the scaling arguments given in §4 also indicate that e_3 does not scale linearly like e_1 and e_2. In fact, they suggest that

$$e_i = a_i + b_i\mu, \quad i = 1, 2, \tag{15a}$$

but,

$$e_3 = a_3 + b_3\mu^{1/3}. \tag{15b}$$

These μ-dependences are expected from a Coulomb-like $V(r)$ and a linear $S(r)$, responsible for quark confinement.

Using the $u\bar{u}$* and $c\bar{c}$ systems as inputs, we find (with e_i in MeV GeV^2 and μ in GeV)

$$a_1 = -4{\cdot}77; \quad b_1 = 132{\cdot}02 \tag{16a}$$

$$a_2 = 15{\cdot}09, \quad b_2 = -270{\cdot}88 \tag{16b}$$

$$a_3 = -91{\cdot}80, \quad b_3 = 235{\cdot}18. \tag{16c}$$

The results of this parametrization for e_i are plotted in figure 2. On comparing them

*We interpret the $a_0(980)$ (the former $\delta(980)$) as the 0^{++} $(u\bar{u})$ state, although this interpretation is not beyond doubts, mainly because of the small width of a_0.

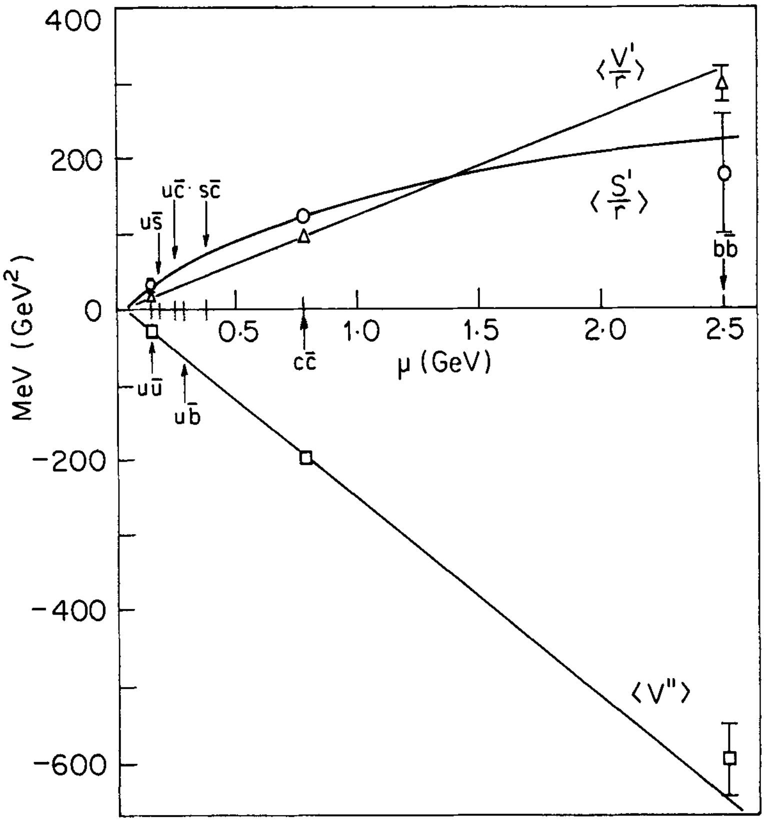

Figure 2. Expectation values e_i ($i = 1, 2, 3$) in MeV GeV^2 for P-wave mesons defined in (5) as a function of the reduced mass μ in GeV. The two straight lines and the curve represent the parametrization given by (15) and (16). Triangles denote the values of e_1, squares those of e_2 and circles those of e_3, respectively, as deduced from experimental masses. The central values shown for $b\bar{b}$ were extracted from the data assuming that the $^1P_1(b\bar{b})$ state coincides with the centre of gravity of the $^3P_J(b\bar{b})$ states.

with the earlier parametrizations*, the main points to be noted are that the two parametrizations for e_1 and e_2 are nearly the same whereas that for e_3 is very different. The new fits ((15) and (16)) predict values for $e_i(b\bar{b})$ which agree well with the assumption that $^1P_1(b\bar{b})$ coincides with the centre of gravity of the $^3P_J(b\bar{b})$-levels as can be seen from figure 2.

The numerical predictions for the masses for quarkonia systems based on (14), (15) and (16) are given in table 3. For these predictions we have used

$$\bar{m}_1 = 0{\cdot}662 + 0{\cdot}924x, \tag{17}$$

which was obtained from the direct fit of (1) to the masses in [GK]. We do not give predictions based on mass formulae (1) as they can already be found in [GK].

These new results for the $1P$-wave masses are in better agreement with data than

* For reference, the values for the linear fits given in [GK], were $a_1 = -5{\cdot}7$, $b_1 = 133{\cdot}3$, $a_2 = 12{\cdot}2$, $b_2 = -263{\cdot}9$, $a_3 = 11{\cdot}1$ and $b_3 = 145{\cdot}8$.

Table 3. Predictions for the masses of various P-wave mesons (in MeV). These are based on (17) and the new parametrizations (15) and (16) of the expectation values. Experimental values are given in brackets. Note that the prediction for the $^1P_1(b\bar{b})$ state is very near to the CoG of the $^3P_J(b\bar{b})$-levels $M_{CoG} = 9900{\cdot}2$. For clarity, J^{PC} of the state is indicated in brackets in the top row. However, C is relevant only for the $u\bar{u}$, $c\bar{c}$ and $b\bar{b}$ mesons.

	$M_2(2^{++})$	$M_0(0^{++})$	$M_1(1^{++})$	$M_1'(1^{+-})$	M_α	M_β
$u\bar{u}$	1276 (1318 ± 5)	942 (983 ± 2)	1233 (1275 ± 28)	1191 (1233 ± 10)		
$u\bar{s}$	1440 (1426 ± 2)	1188 (1350 ?)	1421	1387	1344 (1270 ± 10)	1463 (1406 ± 10)
$u\bar{c}$	2346	2354	2422 (2420 ?)	2367	2284	2504
$s\bar{c}$	2572	2456	2563	2554	2516	2601
$c\bar{c}$	3556 (3556·3 ± 0·4)	3415 (3414·9 ± 1·1)	3511 (3510·7 ± 0·5)	3525 (3525·4 ± 0·8)		
$u\bar{b}$	5494	5661	5630	5556	5470	5716
$s\bar{b}$	5728	5745	5765	5742	5705	5802
$c\bar{b}$	6729	6655	6703	6712	6703	6712
$b\bar{b}$	9912 (9913·3 ± 0·6)	9856 (9859·8 ± 1·3)	9890 (9891·9 ± 0·7)	9899 (9894·8 ± 1·5)?		

those given earlier [GK] using purely linear fits. It is very gratifying that the input of $u\bar{u}$ and $c\bar{c}$ (to determine the expectation values), predicts the $b\bar{b}$ masses so well. The predicted values for the $u\bar{s}$ system are also in good agreement and suggest that the poorly measured mass (~ 1350 MeV) of the $0^{++}(u\bar{s})$-state should be near 1200 MeV. This coincides with the results of the analysis presented in [GK].

A particular point to be noticed, in table 3, is the multiplet inversion, that is $M_2 < M_0$, predicted when $m_2 \gg m_1$ (e.g. $u\bar{b}$, $s\bar{b}$). Such an effect is well-known and expected when the scalar part of the potential dominates the vector part. This can be seen from (14a) and (14b), where the last term becomes large, since $m_2 \gg m_1$, and negative because $e_3 > e_1$ (see figure 2) in this region; so it decreases M_2 compared to M_0.

In table 2, with the same inputs as in table 3, the predictions for the $1P$ $(t\bar{t})$ states of toponium are also presented as they may be of interest in the near future.

This completes the presentation of the phenomenological fits to the expectation values which determine the spin-splitting of the S- and P-wave states. We now turn to general scaling arguments in order to see if we can understand the μ-dependences of these fits.

4. Scaling arguments for expectation values

The simple dependence on the reduced mass μ of the expectation values obtained above from experiment, using a potential model, requires a deeper understanding or a theoretical basis. In the potential approach, we have throughout assumed that the static part of U_{NR} (which generates U_{spin}) is flavour independent and, consequently, the expectation values we deal with will depend on the quark masses only via the reduced mass μ. Moreover, U_{NR} is assumed to be of mixed Lorentz character with a vector part $V(r)$ and a scalar part $S(r)$. As a result, only the four expectation values (defined in (5) and (11c)) determine the mass-splittings of the S and P wave mesons. For general $V(r)$ and $S(r)$ it is not possible to give arguments for the expected scaling behaviour of e_i. Nor is it possible to deduce or infer the precise form of $V(r)$ and $S(r)$ from the phenomenological fits. So we consider simple power-like forms for $V(r)$ and $S(r)$ which seem to have a theoretical and phenomenological basis, and which can lead to the observed phenomenological scaling behaviour.

We first consider the case when both $V(r)$ and $S(r)$ are pure power potentials,

$$V(r) = \lambda r^{\varepsilon_V}, \quad S(r) = \lambda r^{\varepsilon_S}, \tag{17}$$

where λ_V and λ_S are constants. For a *single* power law potential ($\varepsilon_V = \varepsilon_S$), the exponent in the potential would govern the scaling behaviour. If there are two power law potential parts we assume that there is an effective scaling power ν such that a typical length scales (Eichten *et al* 1978; Quigg and Rosner 1979) (with μ) like $\mu^{-1/2+\nu}$, consequently, for an arbitrary power ρ,

$$\langle r^{\rho} \rangle_L \simeq c_L \mu^{-\rho/(2+\nu)}$$

The value of ν would be nearer ε_V or ε_S depending on the relative strength of the two potentials and will, in general, also depend on L.

Given that effective scaling power, we expect for the P-waves:

$$e_1 = \left\langle \frac{V'}{r} \right\rangle_1 = c_1 \lambda_V \varepsilon_V (\mu)^{P_V} \tag{18a}$$

$$e_2 = \langle V'' \rangle_1 = c_1 \lambda_V \varepsilon_V (\varepsilon_V - 1)(\mu)^{P_V} \tag{18b}$$

$$e_3 = \left\langle \frac{S'}{r} \right\rangle_1 = c_1 \lambda_S \varepsilon_S (\mu)^{P_S} \tag{18c}$$

where

$$P_i = \frac{2 - \varepsilon_i}{2 + \nu}, \quad i = V \text{ or } S. \tag{18d}$$

The experimental finding that both e_1 and e_2 dominantly scale linearly with μ agrees with (18a) and (18b) and suggests $P_V \simeq 1$. If e_3 also scaled linearly (as in [GK]), then $P_S \simeq 1$ and we would expect

$$\nu = -\varepsilon_V = -\varepsilon_S. \tag{19}$$

But if the two power laws are the same ($\varepsilon_V = \varepsilon_S$), then ν should be equal to $\varepsilon_V = \varepsilon_S$. Consequently, (19) implies that all the three are very near to zero. This would be consistent with a Martin type potential (Martin 1980) which has small $\varepsilon_S = \varepsilon_V = \varepsilon > 0$ and which also chooses $\lambda_V = f\lambda$ and $\lambda_S = (1-f)\lambda$, with $0 < f < 1$. Explicit model calculations (Barik and Jena 1982) find the empirical fits $\varepsilon = 0{\cdot}11$ and $f = 0{\cdot}58$. This would require $\nu = 0{\cdot}11$ and e_i ($i = 1, 2, 3$) to scale with the power $P_V = P_S = 0{\cdot}9$. The data on e_i can certainly be fitted by $a_i + b_i \mu^{0{\cdot}9}$, with magnitudes of a_i and b_i differing by only a few per cent from those obtained in the purely linear fit. Nevertheless, it leads to a manifest problem: Because from the empirical fits ($a + b\mu^{\alpha}$, $\alpha = 1$ or 0·9), one finds

$$r_1 \equiv e_2/e_1 \simeq b_2/b_1 \simeq -(2{\cdot}05 \text{ or } 2{\cdot}06) \tag{20a}$$

$$r_2 \equiv e_1/e_3 \simeq b_1/b_3 \simeq (0{\cdot}93 \text{ or } 0{\cdot}91). \tag{20b}$$

(The first and second number correspond to $\alpha = 1$ or 0·9). While from (18), for a Martin-type potential with $\varepsilon = 0{\cdot}11$ and $f = 0{\cdot}58$ we expect

$$r_1 = \varepsilon - 1 \simeq -0{\cdot}9, \quad r_2 = \frac{f}{1-f} \simeq 1{\cdot}4. \tag{21}$$

The manifest difference between the values of r_1 and r_2 in (20) and (21) suggests such a potential cannot adequately describe the P-wave meson data. This is also true for the S-wave case because, as $\varepsilon_V > 0$, one would expect e_0 to scale as $a_0 + b_0\mu^{0{\cdot}9}$. However, as can be seen from figure 1, e_0 requires a function with positive curvature (e.g. (12a) or (12b)).*

The above discussion suggests that $\varepsilon_V \neq \varepsilon_S$. Furthermore, if $V(r)$ is a simple power-like potential then from the linearity of e_1 and e_2 we need $\varepsilon_V \simeq -1$ and $v \simeq 1$. On the other hand, quark confinement requires that $\varepsilon_S > 0$ and consequent from (18d), $P_S < 2/3$ if $v = 1$. This means that $e_3(\mu)$ is not linear but has negative curvature. To further fix ε_S we appeal to the currently popular value of 1, giving $P_S = 1/3$ which leads to the fit (15b). As we have seen in §3, this provides a very good description of the data.

We thus assume in the following, that $S(r)$ is dominantly linear in r. Specifically, we consider the case

$$V(r) = \frac{\alpha(r)}{r} \quad \text{and} \quad S(r) = \lambda_S r \tag{22}$$

where $\alpha(r)$ is a slowly varying function of r. A form of $\alpha(r)$ suggested by quantum chromodynamics and used and tested in model calculations (Richardson 1979; Igi and Ono 1986) is

$$\alpha(r) = \frac{1}{\ln(\Lambda r + c)}. \tag{23}$$

We have introduced a constant c in order to fix $\alpha(0) \neq 0$.

The leading dominant behaviour of e_1 and e_2 for small r will be essentially given by (18a) and (18b) with $\varepsilon_V = -1$, $v = 1$, i.e. $P_V = 1$. Though e_1 and e_2 behave linearly with μ, in contrast, e_3 is expected to scale as $\mu^{1/3}$. This μ-dependence of e_3 which we have already anticipated in §3 is therefore justified to the extent the choice for $S(r)$ in (22) has a theoretical basis in lattice calculations and/or string models. Note that the fit to the data for e_3 (see (15c) and (16c)) requires also a sizable constant term a_3 which may be an indication that the scalar potential contains, in addition to the dominant linear term, a small quadratic term.

* It is interesting to note that the Martin-type potential is also excluded by the observation (Gromes 1988) that the confining part of the potential (unless it grows only logarithmically) cannot be of vector origin.

The scaling behaviour of e_0 is more complicated because $V(r)$ is singular at $r=0$. One finds now

$$\nabla^2 V(r) = -4\pi\alpha(r)\delta^3(\mathbf{r}) + \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\alpha(r) \tag{24}$$

if $r(\mathrm{d}\alpha/\mathrm{d}r)\to 0$ for $r\to 0$. Using (23), the first term behaves as $1/r$ for $r\to 0$. This leads to the dominant scaling behaviour for e_0 to be given by

$$e_0 = \langle \nabla^2 V \rangle_S \simeq b_0\mu^{3/(2+\nu)} + c_0\mu^{1/(2+\nu)} \tag{25a}$$

where the first and second terms correspond to the first and second terms of (24). With an effective scaling $\nu=1$, (25a) yields the form (12b) used before for e_0. Note for a constant α in (22) we would get a pure linear μ-dependence for e_0, so the curvature manifest in the e_0-data indicate that a pure Coulomb-vector potential is not sufficient.

For P-waves, the δ-function-term in (24) is not effective and the contribution of the spin-spin interaction to the mass splittings is small. According to (24) and (23) one expects

$$\langle \nabla^2 V(r) \rangle_1 = \left\langle V'' + 2\frac{V'}{r} \right\rangle \sim \mu^{1/3}. \tag{25b}$$

From the fits given in (15) and (16) one finds $\langle \nabla^2 V \rangle_1$ to be small with a seemingly linear scaling behaviour but having a very small slope ($b_2 + 2b_1 = -6{\cdot}84$). There is no real discrepancy since for e_1 and e_2 we have only kept the leading behaviour (steming from $1/r$) to keep the fits simple, and ignored the small nonleading terms ($\mu^{2/3}$ for e_1; $\mu^{2/3}$ and $\mu^{1/3}$ for e_2). Fits which include the latter corrections would have satisfied (25b). However, at present, there are not enough P-wave systems known to warrant such detailed fits.

5. General mass relations for *S* and *P*-wave mesons

On a direct analysis of the S and P-wave meson masses, we found that their dominant behaviour is given by a mass formula of the generic form

$$M(q_1\bar{q}_2) = A + Bx + C/x \quad (x = m_1 + m_2) \tag{26}$$

where a non-zero C seems to be required only for the 0^{-+} states (and for the centre of gravity $\bar{m}_0$ of the two S-states). Because of its simple algebraic structure (specially if $C=0$), these mass formulae yield relations between the masses of states with different J^P. Also, one may hope to determine quark masses in terms of the observed masses. This would remove the uncertainty inherent in the choice of the input masses. Given the amount and quality of the data available and given the accuracy of the mass formulae (within a few per cent only) such an attempt would be premature as it would not determine the quark masses too accurately. However, we present some typical relations expected from the mass formulae. The ones which connect the S and P state masses, follow from the implicit assumption that a given constituent quark has the same mass in different hadrons.

For the cases when the mass formula is simply $A+Bx$ ($C=0$ in (26)) one obtains general equalities, which are of the form,

$$M(q_1\bar{q}_2)+M(q_3\bar{q}_4)=M(q_1\bar{q}_4)+M(q_2\bar{q}_3)=M(q_1\bar{q}_3)+M(q_2\bar{q}_4) \quad (27)$$

For n flavours these give $n(n-1)/2$ independent relations. Equations (27) apply to S-wave vector mesons and to P-wave mesons for each J^P as well as to $\bar{m}_1$. Particular cases, for example give

$$M(D^*)+M(D_s^*)=M(J/\psi)+M(K^*)$$
$$2M(B^*)=M(\Upsilon)+M(\rho) \quad (28)$$

These relations for the 1^{--} vector mesons are satisfied within a few per cent and just reflect the accuracy of the mass formula itself. Relations for P-wave mesons deducible from (27) cannot be checked with the available data. However, since the slope B is independent of J^P for the P-wave mesons, one has relations connecting states with different J^P. For example,

$$M_2(b\bar{b})+M_0(c\bar{c})=M_2(c\bar{c})+M_0(b\bar{b}),\quad \text{etc.,} \quad (29)$$

which experimentally hold to within 1%. Another type of relations which connect S and P states follow from the observation that $B_V\simeq B_2\simeq\bar{B}_1\simeq\bar{B}_0$. For example, from (27),

$$M_2(b\bar{b})+M(J/\psi)=M_2(c\bar{c})+M(\Upsilon) \quad (30)$$

which is satisfied to within 0·05%.

In the case $C\neq 0$ in (26) it is hard to obtain relations in form of equalities which are independent of the quark masses, though inequalities can be derived. These would be near equalities in systems with large x (e.g. $c\bar{b}$, $u\bar{t}\ldots$) since C_P and C_0 in (8a) and (10) are small. In general, one obtains, since C_P and C_0 are <0,

$$M(q\bar{Q})+m(q'\bar{Q}')\gtreqless M(q'\bar{Q}')+m(q\bar{Q}) \quad (31)$$

etc., if $m_q+m_Q\gtreqless m_{q'}+m_{Q'}$ and where we used the fact the slopes are the same in all cases. In (31), M stands for a mass which satisfies a $A+Bx$ formula, while m stands for either M_P or $\bar{m}_0$. Clearly the equalities (27) are a special case of (31). Two examples of these inequalities are

$$2\bar{m}_0(u\bar{c})>\bar{m}_0(u\bar{u})+\bar{m}_0(c\bar{c}) \quad (32)$$

and

$$\bar{m}_1(u\bar{u})+\bar{m}_0(c\bar{c})>\bar{m}_1(c\bar{c})+\bar{m}_0(u\bar{u}) \quad (33)$$

which are satisfied experimentally.

Relations involving the centres of gravity are interesting because they should be satisfied in a potential model. Inequalities like (31) for individual J^P can be obtained from more general considerations (Nussinov 1984). However, many of the equalities and inequalities given above are new. Even though they do not yield as much information as the precise mass formulae, they may be of interest in providing a clue to some underlying symmetry principle which might lead to a deeper understanding of the nature of the mass formulae.

6. Concluding remarks

We have analyzed meson masses by using two different approaches. One is based on the strikingly simple phenomenological mass formulae and the other on the standard potential picture for $q\bar{q}$-systems. The latter gives a partial understanding of the S-wave meson mass formulae. Predictions based on the two approaches are consistent with each other. It is possible that the mass formulae may have a more general validity than the potential model used. It would be very interesting to see if simple mass formulae like (1) found for the $1S$ and $1P$ states also hold for other J^P states and higher radial excitations.

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