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The Vaidya solution in higher dimensions

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Abstract. The Vaidya metric representing the gravitational field of a radiating star is generalized to spacetimes of dimensions greater than four.

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The possible existence of dimensions greater than four has been seriously considered in recent times. This has come about from approaches in particle physics to the unification of all forces including gravitation like Kaluza-Klein theories and more recently superstrings. This was followed by a number of investigations into possible observational consequences of the extra dimensions both in the cosmological and in the black hole contexts. Issues addressed in the Kaluza-Klein cosmologies include, for instance, dimensional reduction via generalized Kasner solutions (Chodos and Detweiler 1980), the entropy problem (Alvarez and Gavela 1983), variation of fundamental constants on cosmological time scales (Chodos and Detweiler 1980), effect on thermal history of the early universe (Sahdev 1984), classification of homogeneous cosmologies (Demianski *et al* 1987) and so on. In connection with localized sources, higher dimensional versions of the spherically symmetric Schwarzschild and Reissner-Nordström black holes (Chodos and Detweiler 1982; Gibbons and Wiltshire 1986) have been obtained, as also generalization of the rotating Kerr black hole (Myers and Perry 1986; Mazur 1987; Frolov *et al* 1987; Xu Dianyan 1988) and black holes in compactified spacetime (Myers 1987). Questions of linearized stability (Gregory and Laflamme 1988), no hair theorems (Sokolowski and Carr 1986), thermodynamics and Hawking radiation (Myers and Perry 1986) have also been investigated. In four dimensions, the only exact solution representing a radiating star is the Vaidya solution (Vaidya 1951). Although it cannot describe the radiation of angular momentum, it has nevertheless been used in model computations of the evolution of a black hole under Hawking radiation (Hiscock 1981). It would thus be of interest to obtain the generalization of the Vaidya metric in higher dimensions. And this is what we derive in this paper. As we shall demonstrate, Vaidya's treatment may be adapted to higher dimensions in a straightforward manner. Our conventions are as follows: The spacetime dimensions are $D = n + 2$; signature $(- + + + + \cdots)$; coordinates $1, 2, 3 \ldots n+2$ respectively being $t, r, \theta_n \ldots \theta_1$; $R^a_{bcd} = \Gamma^a_{bd,c} - \cdots$, $R_{ab} = R^c_{acb}$; units $8\pi G_D = c = 1$.

We are looking for spherically symmetric solutions of the Einstein field equations

$$G_{ab} = T_{ab} \tag{1}$$

in D dimensions, where

$$T_{ab} = \rho V_a V_b, \quad V_a V^a = 0. \tag{2}$$

Thus,

$$ds^2 = -\exp(2\phi(r,t))\,dt^2 + \exp(2\lambda(r,t))\,dr^2 + r^2\,d\Omega_n^2,$$

where

$$d\Omega_n^2 = d\theta_n^2 + \sin^2\theta_n(d\theta_{n-1}^2 + \sin^2\theta_{n-1}(d\theta_{n-2}^2 + \cdots + \sin^2\theta_2 d\theta_1^2))\cdots)), \tag{3b}$$

is the metric on the n-sphere in polar coordinates.

Introducing the basis one-forms

$$\begin{aligned}
\omega^1 &= \exp(\phi)\,dt\\
\omega^2 &= \exp(\lambda)\,dr\\
\omega^3 &= r\,d\theta_n\\
&\vdots\\
\omega^{n+2} &= r\sin\theta_n \ldots \sin\theta_2\,d\theta_1,
\end{aligned} \tag{4}$$

one can conveniently compute the Einstein tensor and write down the field equations (1) explicitly. For radial outflow of radiation $V^a = (1,1,0,0\ldots 0)$ and the non-vanishing vielbein components turn out to be

$$G_1^1 = -\frac{n\exp(-2\lambda)\lambda'}{r} - \frac{n(n-1)}{2}\psi = T_1^1 = -\rho \tag{5}$$

$$G_2^2 = -\frac{n\exp(-2\lambda)\phi'}{r} - \frac{n(n-1)}{2}\psi = T_2^2 = +\rho \tag{6}$$

$$\begin{aligned}
G_3^3 = G_4^4 = \cdots = G_{n+2}^{n+2} &= \exp(-2\lambda)\left(\phi'' + \phi'^2 - \phi'\lambda' - \frac{(n-1)(\lambda' - \phi')}{r}\right)\\
&\quad - \exp(-2\phi)(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\phi}) - \frac{(n-1)(n-2)}{2}\psi = 0\\
&= T_3^3 = T_4^4 = \cdots = T_{n+2}^{n+2}
\end{aligned} \tag{7}$$

$$G_2^1 = -\frac{n\exp(-(\lambda+\phi))\dot{\lambda}}{r} = T_2^1 = \rho \tag{8}$$

where $'$ and $.$ refer to differentiation with respect to r and t respectively, and

$$\psi = \frac{1}{r^2} - \frac{\exp(-2\lambda)}{r^2}. \tag{9}$$

From eqs (5) and (6) we have

$$T_1^1 + T_2^2 = 0, \tag{10a}$$

i.e.

$$\frac{n\exp(-2\lambda)}{r}(\lambda' - \phi') + n(n-1)\psi = 0, \tag{10b}$$

while (5) and (8) give

$$T_1^1 + T_2^1 = 0, \tag{11a}$$

$$\frac{n\exp(2\lambda)\lambda'}{r} + \frac{n(n-1)}{2}\psi + \frac{n\exp(-(\lambda+\phi))}{r}\dot{\lambda} = 0.$$

Thus eqs (10b), (11b) and (8) are three equations for three unknowns ϕ, λ and ρ. To solve them we start with the ansatz

$$\exp(-2\lambda) = 1 - \frac{2m(r,t)}{r^{n-1}}. \tag{12}$$

Equation (11b) then implies

$$\exp(\phi) = -\frac{\dot{m}}{m'}\exp(\lambda) \tag{13}$$

which may be rewritten as

$$\frac{\mathrm{d}m}{\mathrm{d}\tau} = 0; \quad \frac{\mathrm{d}}{\mathrm{d}\tau} \equiv \exp(-\phi)\frac{\partial}{\partial t} + \exp(-\lambda)\frac{\partial}{\partial r}. \tag{14}$$

Substituting (13) in (10b) yields

$$\frac{m'}{\dot{m}}\left(\frac{\dot{m}}{m'}\right)' - \frac{2m(n-1)}{r^n}\left(1 - \frac{2m}{r^{n-1}}\right)^{-1} = 0. \tag{15}$$

Recalling that ′ is partial derivative with respect to r keeping t constant, and is partial derivative with respect to t keeping r constant whereas it is more convenient to evaluate the partial derivatives holding m constant one obtains

$$\frac{1}{m'}\left(\frac{\partial m'}{\partial r}\right)_m = -\frac{2m(n-1)}{r^n}\bigg/\left(1 - \frac{2m}{r^{n-1}}\right),$$

which can be integrated to give

$$m'\left(1 - \frac{2m}{r^{n-1}}\right) = f(m). \tag{16}$$

It now remains to verify that on using the eqs (12), (13) and (16), eq. (7) is also satisfied. As in four dimensions, this is tractable if instead of direct substitution one uses the covariant conservation of T^a_b. Writing down the r-th component of

$$T^a_{b;a} = 0 \tag{17}$$

and converting to the vielbien components, we have

$$T^2_{2,r} + \exp(\lambda - \phi)T^1_{2,t} + T^1_2\exp(\lambda-\phi)(\dot{\lambda} - \phi) + \phi'(T^2_2 - T^1_1)$$
$$+ (\dot{\phi} + \dot{\lambda})T^1_2\exp(\lambda - \phi) + \frac{n}{r}(T^2_2 - T^3_3) = 0. \tag{18}$$

Using eqs (10), (11) and (5) we obtain after some manipulation,

$$nT^3_3 = -r^{n+1}\exp(3\lambda)\frac{\mathrm{d}}{\mathrm{d}\tau}\left(m'\left(1-\frac{2m}{r^{n-1}}\right)\right), \tag{19}$$

which as a consequence of eqs (16) and (14) yields

$$T^3_3 = 0. \tag{20}$$

Thus eq. (7) is identically satisfied.

In conclusion, we have shown that the 'shining star' Vaidya metric in n-dimensions takes the form

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r^{n-1}}\right)\frac{\dot{m}^2}{f^2}\mathrm{d}t^2 + \left(1-\frac{2m}{r^{n-1}}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2_n, \tag{21a}$$

where

$$m = m(r,t), \quad m'\left(1-\frac{2m}{r^{n-1}}\right) = f(m). \tag{21b}$$

Introducing a coordinate $u \equiv u(m)$ defined by

$$\mathrm{d}u \equiv -\frac{\mathrm{d}m}{f(m)} = -\left(\mathrm{d}r + \frac{\dot{m}}{m'}\mathrm{d}t\right)\left(1-\frac{2m}{r^{n-1}}\right)^{-1}, \tag{22}$$

the $(n+2)$-dimensional metric may alternatively be written as

$$\mathrm{d}s^2 = r^2\,\mathrm{d}\Omega^2_n - 2\,\mathrm{d}u\,\mathrm{d}r - \left(1-\frac{2m(u)}{r^{n-1}}\right)\mathrm{d}u^2\,\cdot \tag{24}$$

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