Pramāṇa – J. Phys., Vol. 32, No. 4, April 1989, pp. 563–572. © Printed in India.

Gribov ambiguity in gauge theories

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Abstract. Gribov ambiguity in gauge field theories is discussed and it is shown that such an ambiguity exists even for Abelian theories in covariant gauge at finite temperature. Both geometric and algebraic proofs are presented. In view of the importance of non-perturbative methods, some special gauges are given in which such ambiguities do not exist or are not relevant. The significance of these in the study of confinement in QCD is pointed out.

Keywords. Gribov ambiguity; gauge theory; QCD confinement; Abelian field theory.

PACS No. 11·15

1. Introduction

In quantizing a gauge theory, one needs to fix the gauge. This is partly to eliminate the unphysical degrees of freedom of the gauge field. In the language of path integrals, wherein one starts from the vacuum-to-vacuum transition amplitude

$$Z = \int [\mathrm{d}A] \exp\left[i \int \mathrm{d}^4 x \mathscr{L}(A) \right],$$

it can be seen that the term in the action quadratic in the vector potential is singular i.e. the corresponding wave operator has no inverse. Thus there is an insufficient gaussian (on going to the Euclidean version) damping of the integral. Further under a gauge transformation the action $\int \mathrm{d}^4 x \mathscr{L}(A)$ remains invariant. As the above functional integral is over the space of gauge potentials, one is forced to consider gauge-inequivalent potentials to avoid any possible double counting. Usually a Coulomb or Lorentz gauge is used. Once a gauge is fixed, the question naturally arises, how far this gauge-fixing is unique. Gribov (1978) showed that in non-Abelian gauge theories, the above gauge-fixing schemes are not unique. In Abelian gauge theories it is usually assumed that such problems do not exist. We illustrate this now. For an Abelian gauge theory (gauge group is U(1)) the gauge transformation is

$$A_\mu \to A_\mu + \partial_\mu \Lambda,$$

with $\Lambda(x) \to 0$ as $|x| \to \infty$. Consider a Lorentz gauge $\partial^\mu A_\mu = 0$. Then the uniqueness is determined by the non-existence of a non-trivial and regular solution vanishing as $|\mathrm{x}| \to \infty$ to the equation $\partial^\mu \partial_\mu \Lambda(x) = 0$. This has no such solution and hence the gauge-fixing is unique. A similar conclusion is reached for Coulomb gauge as well. The situation for non-Abelian theories is quite different. We will demonstrate that even for *Abelian theories* at *finite temperature* the situation is different.

2. Gribov ambiguity

In the case of non-Abelian theories the gauge transformation is

$$A^a_\mu \to A^a_\mu + D^{ab}_\mu \omega^b,$$

$$D^{ab}_\mu = \partial_\mu \delta^{ab} + g f^{acb} A^c_\mu. \tag{1}$$

Let A^a_μ and A'^a_μ be the two gauge-equivalent potentials (having same $F^a_{\mu\nu}$) both satisfying the Lorentz gauge condition $\partial^\mu A^a_\mu = \partial^\mu A'^a_\mu = 0$. Then we get the following equation for the gauge function ω^a

$$\partial_\mu \partial^\mu \omega^a + g f^{acb} A^c_\mu \partial^\mu \omega^b = 0, \tag{2}$$

known as the Gribov equation. Gribov (1978) showed that the above equation admits non-trivial regular solution for large enough A^a_μ. This means the gauge-fixing condition fails to avoid possible double counting. The existence of solutions for (2) for large $A^a_\mu(x)$ can be seen, for example following Itzykson and Zuber (1985). So one cannot fix the gauge globally. However, this is no impediment to calculating small fluctuations about some classical background field which includes the perturbation theory. For understanding the non-perturbative effects, the gauge ambiguity problem raises serious difficulties.

Singer (1978) generalized the above result for any smooth gauge-fixing function when the vector potential continued to 4-dimensional Euclidean space can be compactified on S^4. This is briefly outlined below. Let M be an oriented Riemannian base space and G be a compact Lie group (the gauge group). We fix a principal G-bundle over M (Daniel and Viallet 1980) with the canonical projection π. Let $\mathscr{G}$ be the Lie algebra of G.

$$\begin{array}{l} G \to P \\ \quad \downarrow \pi \\ \quad M \end{array} \tag{3}$$

Let ω^1 be the Lie algebra $\tilde{\mathscr{G}}$ valued connection 1-form on P, with horizontal kernel which transforms equivariantly under the action of G on P and let Ω^2 be the curvature (Lie algebra $\tilde{\mathscr{G}}$ valued 2-form on P) of ω^1; $\Omega^2 = \mathrm{d}\omega^1 + \frac{1}{2}[\omega^1 + \omega^1]$. The total space P of the bundle (3) has a free G-action defined on it ($p \to pg$ for $p \in P$, $g \in G$). Transformations of P, preserving this G-action are automorphisms of P, denoted by Aut P. The sub-group of Aut P which induces the identity transformation on M is called the *group of gauge transformations* $\mathscr{G}$. Let us take P to be a trivial product bundle, $P = M \times G$. Then we have the following (Kobayashi and Nomizu 1963, 1969; Narasimhan and Ramadas 1979 and Mitter and Viallet 1981): (i) the group of gauge transformations $\mathscr{G} \simeq \mathrm{Map}(M, G)$ and (ii) there exists a global section $\sigma: M \to P$ such that $\pi \cdot \sigma =$ identity on M, which can be used to pull down the connection 1-form ω^1 and the curvature 2-form Ω^2 from P to M, identified respectively as gauge potential 1-form A and the field strength 2 form F on M. It follows that A on M is in one-to-one correspondence with ω^1 on $P = M \times G$.

Now consider the space of gauge potentials A and denote it by $\mathscr{A}$. The vacuum-to-vacuum amplitude mentioned in §1 is a functional integral over $\mathscr{A}$. The gauge group $\mathscr{G}$ has a natural action on $\mathscr{A}$. For a point $A \in \mathscr{A}$, the action of $g \in \mathscr{G}$ is $g \cdot A = g^{-1}Ag + g^{-1}dg$ (which is the non-Abelian gauge transformation). Such an action is not in general free as there might be gauge-equivalent points in $\mathscr{A}$ for which $g \cdot A = A$ (gauge-equivalent potentials). We now fix a base point, say $x_0 \in M$ and call it the point at infinity. Consider the subgroup $\mathscr{G}_*$ of $\mathscr{G}$ for which $g(x_0) = g(\infty) = e$, the identity of G. The $\mathscr{G}_*$ has a free action on $\mathscr{A}$. Thus we consider base-point preserving *smooth* maps from M to G.

$$\mathscr{G}_* \simeq \mathrm{Map}_*(M, G) \tag{4}$$

(we consider the class of gauge transformations whose gauge function (the exponent in the usual representation) $\to 0$ as $|x| \to \infty$). All other gauge functions can be generated from this by infinitesimal group transformations as done in any Lie group). The free $\mathscr{G}_*$ action on $\mathscr{A}$ results in the sequence

$$\begin{array}{l} \mathscr{G}_* \to \mathscr{A} \\ \qquad \downarrow p \\ \qquad \mathscr{A}/\mathscr{G}_* \end{array} \tag{5}$$

with the canonical projection $p: \mathscr{A} \to \mathscr{A}/\mathscr{G}_*$, a principal $\mathscr{G}_*$ fibre bundle over $\mathscr{A}/\mathscr{G}_*$. So far we have given the setting up of a gauge theory on a mathematical basis.

In the quantization of gauge theories, for reasons spelt out earlier we need to fix the gauge. The gauge-fixing is expected to choose in a continuous manner one unique gauge potential on each fibre (the absence of non-trivial regular solution to the Gribov equation is equivalent to this statement). It is a hyper-surface in the space of gauge fields intersecting each fibre *exactly* at one point. It is a *smooth* map $q: \mathscr{A}/\mathscr{G}_* \to \mathscr{A}$ such that $q \cdot p = $ identity on $\mathscr{A}/\mathscr{G}_*$. Thus q is a *global* section of the bundle (5). *It is the existence of such a global section to* (5) *that is the main theme here.*

For a principal bundle (5) the existence of a global section q is equivalent to the bundle being trivial i.e. a product bundle. (Localy one can always have trivial or product structure to any fibre bundle and so locally there always exists a section q. This is equivalent to the statement that the Gribov ambiguity is no impediment for perturbation calculations which mainly concentrate on small fluctuations. We can in principle split up the space $\mathscr{A}$ into different regions, each consisting of a set of orbits and fix the gauge locally in each region. Locally as said we have always a direct product structure (local triviality) for and so the F.P. determinant is *univalent*.) So the existence of a global section implies

$$\mathscr{A} = \mathscr{G}_* \times \mathscr{A}/\mathscr{G}_*. \tag{6}$$

For a trivial bundle, the homotopy groups of $\mathscr{A}$, $\mathscr{G}_*$ and $\mathscr{A}/\mathscr{G}_*$ are related by

$$\pi_n(\mathscr{A}) = \pi_n(\mathscr{G}_*) \oplus \pi_n(\mathscr{A}/\mathscr{G}_*); \quad n \geqslant 0. \tag{7}$$

$\mathscr{A}$ the space of gauge potentials is an affine space and contractable. Detailed considerations of this space led Singer (1978) to conclude $\pi_n(\mathscr{A}) = 0, \forall n \geqslant 0$ (theorem 2 of Singer 1978). Hence we have

$$0 = \pi_n(\mathscr{G}_*) \oplus \pi_n(\mathscr{A}/\mathscr{G}_*). \tag{8}$$

This is the condition for the existence of a *global* section q. If this condition is *not* satisfied, even for one n, then there *does not exist a global section* or the *gauge-fixing condition fails* to choose in a continuous manner one *unique* gauge potential on each fibre (gauge orbit). This is the Gribov ambiguity.

Now Singer's (1978) proof becomes apparent. When $M = S^r$ (S^3 for Coulomb gauge-fixing and S^4 for covariant gauge-fixing) with one base point on M fixed at ∞, one has $\mathbb{R}^3 \cup \infty = S^3$ for Coulomb gauge and $\mathbb{R}^4 \cup \infty = S^4$ for covariant gauge: the condition at ∞ imposed by Gribov (fixing the base point on M) is that the gauge transformation from $\mathbb{R}^3$ (or $\mathbb{R}^4$) to G extends to S^3 (or S^4) with $g(\infty) = e$), $\pi_n(\mathscr{G}_*) = \pi_{n+r}(SU(N)) \neq 0$ for $G = SU(N)$. As an example for $n = 0$, $r = 3$, $\pi_0(\mathscr{G}_*) = \pi_3(SU(N)) = Z$; $n = 0$, $r = 4$, $\pi_0(\mathscr{G}_*) = \pi_4(SU(2)) = Z_2$. Hence (8) is not satisfied. This completes Singer's proof of the existence of Gribov ambiguity for non-Abelian theories.

3. Abelian field theory at finite temperature

In the study of field theory at finite temperature one goes to Euclidean space by writing $it = \tau$ and imposes periodic boundary conditions for Bose fields (anti-periodic for Fermi fields) with respect to τ i.e. $\phi(\mathbf{x}, \tau) = \phi(\mathbf{x}, \tau + \beta)$ with $\beta = 1/kT$ where k is the Boltzmann constant and T is the absolute temperature (Bernard 1974; Dolan and Jackiw 1974). This implies that τ-co-ordinate is being mapped on to S^1 and so the base space $\mathbb{R}^4$ becomes $\mathbb{R}^3 \times S^1$. For Abelian gauge theory $G = U(1)$ the gauge group is a map, $\mathscr{G} \simeq$ map $(\mathbb{R}^3 \times S^1, U(1))$. To have a free action of $\mathscr{G}$ on the space of gauge potentials $\mathscr{A}$, we said that we have to fix a point in the base space. This is taken to be the point at infinity. Using $\mathbb{R}^3 \cup \infty = S^3$, we have

$$\mathscr{G}_* \simeq \mathrm{Map}_*(S^3 \times S^1, U(1)). \tag{9}$$

The gauge theory is now defined on the product manifold $M = S^3 \times S^1$. To examine the existence of Gribov ambiguity we consider (8) with $\mathscr{G}_*$ defined as (9). We prove below (Parthasarathy 1988) $\pi_0(\mathscr{G}_*) \neq 0$ and hence (8) is not satisfied. Thus we prove that there is no global section to the bundle or we show that there is the Gribov ambiguity.

Theorem. *For* $\mathscr{G}_* \simeq \mathrm{Map}_*(S^3 \times S^1, U(1))$, $\pi_0(\mathscr{G}_*) \neq 0$.

Proof. For a group of smooth maps from $S^3 \times S^1$ to the gauge group G, we have (Killingback 1984),

$$\pi_0(\mathscr{G}_*) = [S^3 \times S^1; G]_*, \tag{10}$$

where $[S^3 \times S^1; G]_*$ is the group of homotopy classes of base point preserving maps from $S^3 \times S^1$ to G. We need only to prove the existence of non-trivial subgroups of $[S^3 \times S^1; G]_*$. We quote a theorem following Whitehead (1978) without proof.

Theorem. *If* $\Gamma = [S^{n_1} \times S^{n_2} \times \cdots S^{n_k}: G]_*$ *with* S^{n_i} *as the* n_i *sphere, then* Γ *has a central chain of length* k, $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \cdots \supset \Gamma_k$ *and* $\Gamma_k = \{0\}$, *where* $\Gamma_{i-1}/\Gamma_i = \Pi_{|\alpha|=i}\pi_{n(\alpha)}(G)$, *with* $\Pi_{|\alpha|=i}$ *as the direct products of the homotopy groups* $\pi_{n(\alpha)}(G)$ *over those subsets* α *of* $\{1, 2, 3, \ldots, k\}$ *having exactly* i *elements and* $n_{(\alpha)} = \sum_{i\in\alpha} n_i$.

For $\mathscr{G}_* \simeq \mathrm{Map}_*(S^3 \times S^1; U(1))$, we have (10) to which the above theorem is applied. We have then $n_1 = 3$, $n_2 = 1$, $k = 2$ and so the central chain of

$$\pi_0(\mathscr{G}_*) = [S^3 \times S^1; U(1)]_* = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \quad \text{with} \quad \Gamma_2 = \{0\}.$$

$$\Gamma_1/\Gamma_2 = \Pi_{|\alpha|=2}\pi_{n(\alpha)}(U(1)) \quad \text{with} \quad \Pi_{|\alpha|=2}$$

as the direct products of the homotopy groups of $\pi_{n(\alpha)}(U(1))$ over those subsets of α of $\{1,2\}$ having exactly '2' elements. There is only one such subset i.e. $\{1,2\}$ itself. Then $n_{(\alpha)} = n_1 + n_2 = 4$. Hence

$$\Gamma_1/\Gamma_2 = \pi_4(U(1)).$$

$$\Gamma_0/\Gamma_1 = \Pi_{|\alpha|=1}\pi_{n(\alpha)}(U(1)) \quad \text{with} \quad \Pi_{|\alpha|=1}$$

as the direct product of the homotopy groups of $\pi_{n(\alpha)}(U(1))$ over those subsets of α of $\{1,2\}$ having exactly one element. There are two subsets $\{1\}$ and $\{2\}$ for which $n_{(\alpha)} = 3$ and 1 respectively. Hence $\Gamma_0/\Gamma_1 = \Pi_3(U(1)) \oplus \Pi_1(U(1))$.

Thus there is a non-trivial subgroup of $[S^3 \times S^1; U(1)]_*$ and so $\Pi_0(\mathscr{G}_*) \neq 0$. This completes the proof.

It follows from the above that (8) is not satisfied for $G = U(1)$ when the base manifold is $S^3 \times S^1$ or when $\mathscr{G}_* \simeq \mathrm{Map}_*(S^3 \times S^1; U(1))$ and so there is Gribov ambiguity in Abelian theories at finite temperature. In general one can adopt the above reasoning to prove that there will be Gribov ambiguity in gauge theories in general and Abelian gauge theory in particular, when the gauge theory is defined on $S^3 \times S^1$. A similar result is obtained for $S^1 \times S^1 \times S^1 \times S^1$ in Killingback (1984).

We now give an explicit construction of non-trivial solutions to the Gribov equation in our case, finite temperature Abelian gauge theory. For covariant gauge $\partial^\mu A_\mu(\mathbf{x}, \tau) = 0$ to examine whether or not the points on the gauge orbits intersecting the hypersurface defined by the above gauge-fixing condition, are unique, make the gauge transformation $A_\mu(\mathbf{x}, \tau) \to A_\mu(\mathbf{x}, \tau) + \partial_\mu \Lambda(\mathbf{x}, \tau)$. For the transformed potential satisfy the same covariant gauge (so that the new A_μ are on the gauge orbits intersecting the hypersurface) we have the requirement $\partial_\mu \partial^\mu \Lambda(\mathbf{x}, \tau) = 0$. In our case we can expand $\Lambda(\mathbf{x}, \tau)$ as

$$\Lambda(\mathbf{x}, \tau) = \sum_{n=0}^{\infty} \Lambda_n(\mathbf{x}) \sin(n\tau/\beta); \quad 0 \leqslant \tau \leqslant 2\pi\beta, \tag{11}$$

which gives

$$\left(\nabla^2 - \frac{n^2}{\beta^2}\right)\Lambda_n(\mathbf{x}) = 0. \tag{11a}$$

A non-trivial solution may be of the form

$$\Lambda_n(\mathbf{x}) = \exp\left(-\frac{n}{\beta}\hat{\alpha}\cdot\mathbf{x}\right), \tag{11b}$$

where $\hat{\alpha}$ is a constant unit vector. Then

$$\Lambda(\mathbf{x}, \tau) = \sin(\tau/\beta) \Big/ \left[\exp\left(\frac{\hat{\alpha}\cdot\mathbf{x}}{\beta}\right) + \exp\left(-\frac{\hat{\alpha}\cdot\mathbf{x}}{\beta}\right) - 2\cos(\tau/\beta)\right]. \tag{12}$$

This solution is regular and goes to zero as $|\mathbf{x}| \to \infty$ irrespective of the direction $\mathbf{x}$ takes to approach ∞. It is to be noted that the absolute value of the terms in (11) with $\Lambda_n(\mathbf{x})$ as given in (11b) is divergent as $|\mathbf{x}| \to \infty$ in the second and third quadrants for the orientation of $\mathbf{x}$. However, the summed expression (12) is convergent. So we have conditional convergence and not absolute convergence. Thus there exists a non-trivial solution for $\Lambda(\mathbf{x}, \tau)$ thereby supplementing the geometrical proof for the existence of Gribov ambiguity in finite temperature Abelian gauge theory*. We close this section by recalling other related difficulties in quantizing the Abelian theory at finite temperature in the covariant gauge. Bernard (1974) noted that the usual partition function $\mathrm{Tr}\exp(-\beta H)$ in the covariant gauge yields a description of a Bose gas with three positive and one negative metric states. The unphysical modes do not come to equilibrium with a physical heat bath. He rightly observed that once the Faddeev-Popov determinant is included to write the generating functional as

$$\int [\mathrm{d}A]\Delta_{\mathrm{FP}}\Pi(\delta F)\exp(iS(A)) \int \Pi_x \mathrm{d}\Lambda(x), \tag{13}$$

with F as the gauge-fixing condition and $\Delta_{\mathrm{FP}} = \det(\delta F/\delta\Lambda)$, then the crucial term $\Delta_{\mathrm{FP}} = \det(-\Box)$ for $F = \partial_\mu A^\mu(x)$ at finite temperature yields a β-dependent factor which neutralises the contribution from the unphysical modes. At zero temperature, Δ_{FP} just comes out. Hata and Kugo (1980) rigorously proved Bernard's suggestion. In gauge theories the standard expression $\exp(-\beta H)$ is a gauge-dependent quantity and in general no longer the correct expression for the partition function (Kugo and Ojima 1978a, b, 1979a, b). This is due to the Faddeev-Popov ghosts and the unphysical modes of the gauge fields. To obtain the correct partition function one should work in a special gauge or restrict the trace operation to a subspace of states which consists of physical particles only. Hata and Kugo (1980) start from

$$Z(\beta) = \mathrm{Tr}(P\exp(-\beta H)), \tag{14}$$

with P as the projection on to a subspace of physical particles. Then a statistical operator $\exp(-\beta H - \Pi Q_c)$ is proposed instead of the usual form $\exp(-\beta H)$ with Q_c as the Faddeev-Popov ghost charge. The ghosts have periodic temperature Green functions despite their (Fermi) statistics. Ojima (1981) confirms the above proposal which correctly expresses the fact that the physical degrees of freedom of photons are indeed the two transverse modes only. This could not be achieved without the Δ_{FP} factor which is indispensable in the covariant formulation of gauge theories at a finite temperature.

The demonstration that there is Gribov ambiguity at finite temperature when the gauge theory (Abelian) is defined on the full gauge orbit space implies that there are β-dependent zero modes in Δ_{FP}.

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* Spherically symmetric solutions to $\Box\Lambda(\mathbf{x}, \tau) = 0$ which are periodic in τ do not have this problem of convergence. Such solutions are of the form

$$\Lambda(\mathbf{x}, \tau) = \Lambda(|\mathbf{x}| = r, \tau) = \frac{1}{2r}\sinh r/(\cosh r - \cos\tau).$$

(see for details Harrington and Shepard 1978).

4. Some special gauges

A question naturally arises: Is it possible to work in a gauge in which the Gribov ambiguity is absent so that one can study the gauge theory non-perturbatively? In this section the answer to this is discussed.

4.1. *Abelian theory at finite temperature*

Abelian theory at zero temperature does not at all have this difficulty. In §3 it has been shown that at finite temperature, covariant gauges have this Gribov problem, i.e. the Lorentz gauge-fixing condition fails to select a unique representative gauge potential in each class of gauge equivalent potentials. Let us consider a non-covariant gauge such as Coulomb gauge,

$$\nabla\cdot\mathbf{A}=0 \quad \text{and} \quad A_0=0. \tag{15}$$

PROPOSITION (Singer 1978)

Let $\mathscr{A}_0=[A_\mu\in\mathscr{A}: A_0=0]$ be the space of vector potentials for the trivial $U(1)$ bundle over $\mathbb{R}^3$, depending upon a parameter τ. Then the subgroup of $\mathscr{G}_*$ leaving $\mathscr{A}_0$ invariant as a set is the set of smooth functions of $\mathbb{R}^3$ into $U(1)$. The condition at infinity imposed on $g(x)$ is that the gauge transformations from $\mathbb{R}^3$ to $U(1)$ extends to S^3 with $g(\infty)=1$.

Then the group of transformations appropriate here is

$$\mathscr{G}_*^C \simeq \mathrm{Map}_*(S^3, U(1)), \tag{16}$$

which is to be compared with (9). The superfix C in (16) is for Coulomb gauge. The gauge potentials A_1, A_2 and A_3 will depend upon $\mathbf{x}$ and τ with τ as a parameter. Thus the base space in (5) is the quotient $\mathscr{A}_0/\mathscr{G}_*^C$ in $A_0=0$ gauge modulo the three-dimensional gauge transformations. Following §2, it is seen that there is no Gribov ambiguity in Abelian field theory. In the language of physics; Consider $A_\mu(\mathbf{x},\tau)$ and choose Coulomb gauge. To examine whether or not the points on the gauge orbits intersecting the hypersurface defined by $\nabla\cdot\mathbf{A}=0$ and $A_0=0$, are unique, make a gauge transformation $A_\mu(\mathbf{x},\tau)\to A_\mu(\mathbf{x},\tau)+\partial_\mu\Lambda(\mathbf{x},\tau)$. For the new potential to satisfy the Coulomb gauge (to be on the hypersurface) one finds that $\Lambda(\mathbf{x},\tau)$ is independent of τ and $\nabla^2\Lambda(\mathbf{x})=0$. This has no *regular* solution vanishing at $|\mathbf{x}|\to\infty$. So there is no Gribov ambiguity in the Coulomb gauge. In the spirit of the discussion that followed (13), $\Delta_{\mathrm{FP}}(=\det(-\Box))$ becomes a constant, independent of β. So the usual form $\exp(-\beta H)$ correctly gives the statistics of photon gas in the Coulomb gauge. In fact the Coulomb gauge is mostly preferred albeit its non-covariant form.

4.2. *Non-Abelian theories in axial gauge*

Singer (1978) proved that no continuous choice of exactly one connection on each orbit can be made and so the Gribov ambiguity will occur in all other gauges. Consequently no gauge-fixing is possible. In the above proof, the gauge transformation is a *smooth* map of *compactified* 4-sphere onto the gauge group. Potentials in axial gauge are of special nature which cannot be compactified. For example, consider $A_\mu^a(x)$ and its gauge transform $A_\mu^a(x)+D_\mu^{ab}\omega^b(x)$. In axial gauge $A_3^a=0$, we have

$D_3^{ab}\omega^b = \partial_3\omega^a + gf^{acb}A_3^c\omega^b = \partial_3\omega^a = 0$. So the gauge function ω^a is independent of x^3-co-ordinate and hence the above statement is illustrated. In this gauge Δ_{FP} is gauge field-independent and may be dropped. There is no Gribov ambiguity. This gauge has been used by Arnowitt and Fickler (1962), Fradkin and Tyutin (1970), Mohapatra (1972), Bernstein (1977) and Weisberger (1988).

4.3. *Background gauge*

In the study of 1-loop potential in gauge theory, one expands the fields around the classical solutions $\bar{A}_\mu$ as

$$A_\mu \to \bar{A}_\mu + a_\mu \tag{17}$$

with a_μ as quantum fluctuations. This is substituted in the generating function $\int[dA_\mu]\exp(iS)$ and the action is expanded keeping terms up to quadratic in a_μ. This is then integrated over a_μ to obtain the effective potential. In this background field method one usually chooses to work in the background gauge $\bar{D}_\mu(\bar{A})a_\mu = 0$ where $\bar{D}_\mu$ is the covariant derivative taken with $\bar{A}_\mu$ field only. One then has a quantization of gauge theory around a classical solution. Amati and Rouet (1978) showed that the Gribov ambiguity is an irrelevant problem for quantization in the background gauge. This well-defined prescription is given below. The various fields transform in the following manner.

$$\begin{aligned} A_\mu &\to UA_\mu U^{-1} + iU\partial_\mu U^{-1}, \\ \bar{A}_\mu &\to U\bar{A}_\mu U^{-1} + iU\partial_\mu U^{-1}, \\ a_\mu &\to Ua_\mu U^{-1}, \end{aligned} \tag{18}$$

with U as the gauge transformation $\subset \mathscr{G}_*$. It is to be noted that a_μ transforms homogeneously. This is required by (17). Amati and Rouet (1978) prove that the zero modes of the 1-loop operator (the terms of quadratic in fluctuations a_μ) are indeed $\delta\bar{A}_\mu/\delta\omega$. Expanding the field a_μ in terms of these eign modes, one obtains the generating functional precisely in the background gauge. The result is independent of the gauge chosen to obtain the classical solution $\bar{A}_\mu$. The zero modes parametrize the orbit and so do not give rise to ambiguity. The crucial step is (18), which tells the well-known property that connections (gauge fields) generate an *affine space* (the sum of two connections is not a connection). In this sense the functional integral $\int[d\bar{A}_\mu][da_\mu]$ with $A_\mu = \bar{A}_\mu + a_\mu$ is not well-defined. However, the fluctuations a_μ which are the deviations with respect to a connection on a given fibre bundle generate a *vector space* and the functional integral $\int[da_\mu]$ is well-defined. Here the infinity due to the gauge volume that exists in the usual quantization scheme is eliminated. The Gribov ambiguity arising in gauge-fixing for the solutions of classical solutions is irrelevant in the quantization using the background field method. The background gauge $\bar{D}_\mu a_\mu = 0$ is free from gauge-fixing ambiguities as $\partial_\mu \to Ua_\mu U^{-1}$.

5. Summary and conclusion

The problem of gauge-fixing in the quantization of gauge field theories is discussed. After reviewing the Gribov ambiguity and giving a geometrical setting, it has been proved that such ambiguity exists even for Abelian gauge theories in covariant gauge at a finite temperature. Even at zero temperature the use of nonlinear gauges has this problem in both Abelian and non-Abelian theories (Parthasarathy 1988a). However, it is pointed out that for perturbation theory the Gribov ambiguity is not an impediment. This is understood by means of the local triviality of the principle fibre bundle. For non-perturbative studies this ambiguity poses a serious problem, as gauge-fixing is not possible. Nevertheless there are some special gauges such as axial gauge or background gauge in which one can quantize the theory and study its properties non-perturbatively.

The above scenario is of direct relevance to QCD, a non-Abelian theory based on $SU(3)$ colour group. In QCD theory of strong interaction, it is strongly suspected that one has confinement of quarks and gluons. This important realization thus far has not been proved. It is also believed that confinement is a non-perturbative phenomenon. Therefore it appears that the best method to discuss confinement is to use background field method wherein one can make the background gauge in which Gribov problem does not pose a threat. It is encouraging to point out that studies of QCD in the background field method (Saviddy 1977; Nielson and Olesen 1978; Nielson and Ninomiya 1979; Anishetty 1982; Parthasarathy *et al* 1983; Kay *et al* 1983; Parthasarathy and Pasupathy 1988) demonstrate the non-triviality of the QCD-vacuum such as a non-zero value for $\langle \text{QCD-vacuum}|F^a_{\mu\nu}F^{\mu\nu a}|\text{QCD-vacuum}\rangle$ which is just the gluon condensate. It is hoped that we will soon have a better understanding of the confinement properly.

Acknowledgements

This paper was presented at the Nuclear and Particle Physics Symposium in June 1988 in Ahmedabad. I take this opportunity to express my deep appreciation and gratitude to Professor S P Pandya who has devoted himself to growth of physics in general and nuclear physics in particular. I am extremely thankful to Professor J C Parikh and the organisers for the opportunity provided.

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