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General theory of renormalization of gauge theories in nonlinear gauges

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Abstract. We discuss the general theory of renormalization of unbroken gauge theories in the nonlinear gauges in which the gauge-fixing term is of the form

$$-\tfrac{1}{2}\sum_{\alpha} f_\alpha^2[A] = -\tfrac{1}{2}\sum_{\alpha}\frac{1}{\eta_\alpha}(\partial^\mu A_\mu^\alpha + \zeta_{\beta\gamma}^\alpha A_\mu^\beta A^{\gamma\mu})^2$$

We show that higher loop renormalization modifies $f_\alpha[A]$ to contain ghost terms of the form

$$f_\alpha[A,c,\bar{c}] = \eta_\alpha^{-\frac{1}{2}}(\partial^\mu A_\mu^\alpha + \zeta_{\beta\gamma}^\alpha A_\mu^\beta A^{\gamma\mu} + \tau_\gamma^{\alpha\beta}\bar{c}_\beta c_\gamma)$$

and show how the corresponding ghost terms are deduced from $f_\alpha[A,c,\bar{c}]$ uniquely. We show that the theory can be renormalized while preserving a modified form of BRS invariance by multiplicative and independent renormalizations on A, c, g, η, ζ, τ. We briefly discuss the independence of the renormalized S-matrix from η, ζ, τ.

Keywords. Gauge theory; renormalization; nonlinear gauge.

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1. Introduction

Renormalization of gauge theories in the linear gauges has been thoroughly discussed in the literature since the original work of 'tHooft and Veltman (1972b) and of Lee and Zinn–Justin (1972, 1973). The discussion of renormalizability has been greatly simplified since, due first to the introduction of the treatment based on the generating function for the proper vertices by Lee (1974) and then especially by the use of BRS invariance (Becchi *et al* 1975) of gauge theories by Zinn–Justin (1974) and by Lee (1975). Since then the Becchi-Rouet-Stora (BRS) invariance has been exploited in other renormalization problems in linear gauges such as renormalization of gauge-invariant operators in gauge theories (Dixon and Taylor 1974; Kluberg–Stern and Zuber 1975; Joglekar and Lee 1976).

Renormalization of gauge theories in quadratic gauges has, however, not been thoroughly discussed presumably because such gauges must seem unnecessarily complicated on the whole in practical calculations. The present author worked out (Joglekar 1974) renormalization of scalar and spinor electrodynamics in quadratic gauges; since then it has received some attention (Das 1981, 1982). However, recently, many calculations in spontaneously broken gauge theories have been performed in non-linear R gauges (Deshpande and Nazerimonford 1983). These gauges have several calculational advantages, such as the absence of certain vertices in the Lagrangian and simplified electromagnetic Ward–Takahashi (WT) identities etc. Use of this gauge in higher order calculations will, of course, call for an understanding of renormalization in quadratic gauges.

What is perhaps equally important from a theoretical point of view is the simultaneous presence of two kinds of BRS invariances in the nonlinear R_ξ gauges and their analogues in unbroken gauge theories. This poses an interesting problem as to whether gauge theories in such gauges can be renormalized so as to preserve both kinds of BRS invariances simultaneously. We shall deal with this particular problem in a separate publication (Joglekar 1988) while devoting the present work to the general discussion of renormalization of gauge theories with quadratic gauge conditions in which the gauge-fixing term in the Lagrange density has the form

$$-\tfrac{1}{2}\sum_{\alpha} f_\alpha[A]^2 = -\tfrac{1}{2}\sum_{\alpha} (\partial^\mu A^\alpha_\mu + \zeta^\alpha_{\beta\gamma} A^{\beta\mu} A^\gamma_\mu)^2.$$

We note that quadratic gauges have also been used by Fujikawa (1973) where he has established a relationship between the ξ-limiting process of Lee and Yang (1962) and quadratic gauges.

We now briefly outline the content of the paper. In §2 we define our notations and conventions. We also introduce BRS transformations and BRS invariance. In §3 we introduce generating functionals for Green's functions and proper vertices and use BRS invariance to derive WT identities for proper vertices. In §4 we discuss the structure of one-loop divergence using the WT identity and show that a subtractive renormalization of the gauge-fixing and ghost terms is needed. We discuss how the gauge-fixing and ghost terms are modified by renormalization. In §5 we discuss the BRS invariance of the modified action under modified BRS transformations, which preserve the form of the WT identity. In §6, we prove renormalizability. We show that the theory can be renormalized by multiplicative renormalizations on A, c, g, η, ζ, τ and the sources. Section 7 contains a brief discussion of gauge independence of the renormalized S-matrix.

2. Preliminary

In this work, we shall restrict our discussion to the case of the unbroken Yang–Mills theories with a simple group G. (Generalizations to semisimple groups are obvious.) We shall simplify the problem also by considering a theory not containing fermions and scalars.

The gauge-invariant action in such a theory is given by

$$S_0[A] = \tfrac{1}{4}\int \mathrm{d}^4x F^\alpha_{\mu\nu} F^{\alpha\mu\nu}.$$

where

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + g f^{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu \tag{1}$$

are the field strengths. Here g is the unrenormalized coupling constant of the group G. A is the unrenormalized gauge field.

One can fix the gauge by adding to $S_0\,[A]$ a gauge-fixing action of the form

$$S_{g.f.}[A] = -\int \tfrac{1}{2}\sum_{\alpha} f_{\alpha}^{2}[A]\,\mathrm{d}^{n}x. \tag{2}$$

In this work we shall discuss the case when $f_\alpha[A]$ is of the form:

$$\eta_{\alpha}^{-1/2}(\partial^{\mu}A_{\mu}^{\alpha} + \zeta_{\beta\gamma}^{\alpha}A_{\mu}^{\beta}A^{\gamma\mu}), \tag{3}$$

where $\zeta_{\beta\gamma}^{\alpha}$ are certain arbitrary constants. We shall assume implicitly that $\zeta_{\beta\gamma}^{\alpha}A_{\mu}^{\beta}A^{\gamma\mu}$ is the most general Lorentz-invariant quadratic polynomial in A. Special cases may be obtained by imposing restrictions on $\zeta_{\beta\gamma}^{\alpha}$ later; though such restrictions may or may not be preserved by renormalizations. We shall mainly deal with the case when $\zeta_{\beta\gamma}^{\alpha}A_{\mu}^{\beta}A^{\gamma\mu}$ does not break global invariance.

Henceforth, throughout the rest of this paper, we shall find it convenient to adopt the compact notation and the summation-integration convention adopted by Lee (1974). We shall introduce indices α, β...which will stand collectively for a group index α, β,...etc. *and* a space time point x_α, x_β,... etc. We shall use letters $i, j,$... to stand for a group index α, Lorentz index μ and a spacetime point x_i. Thus, $i \leftrightarrow (\alpha, \mu, x_i)$. We further introduce coefficients

$$\zeta_{ij}^{\alpha} \equiv \zeta_{\beta\gamma}^{\alpha}\delta^{4}(x_{\alpha} - x_{i})\delta^{4}(x_{\alpha} - x_{j})g_{\mu\nu}$$

and introduce $\partial_{i}^{\alpha} = \partial_{\mu}^{x^{\alpha}}\delta^{4}(x_{i} - x_{\alpha})$. We can then write

$$f_{\alpha}[A] = \eta_{\alpha}^{-\frac{1}{2}}(\partial_{i}^{\alpha}A_{i} + \zeta_{ij}^{\alpha}A_{i}A_{j}), \tag{4}$$

where η_α are the gauge parameters which may depend upon α. Henceforth, we shall generally suppress suffix α on η_α.

One must add the Faddeev-Popov ghost action (Faddeev and Popov 1967)

$$S_{g}[A, c, \bar{c}] = \eta_{\alpha}^{1/2}\bar{c}_{\beta}\frac{\delta f_{\alpha}}{\delta A_{i}}D_{i}^{\alpha}[A]c_{\alpha}$$

$$\equiv \bar{c}_{\beta}M_{\beta\alpha}c_{\alpha}, \tag{5}$$

where

$$D_{i}^{\alpha}[A] = \partial_{i}^{\alpha} + g t_{ij}^{\alpha}A_{j} \tag{6}$$

and t_{ij}^{α} are real antisymmetric representation of G of the form

$$t_{ij}^{\alpha} = f^{\alpha\beta\gamma}\delta^{4}(x_{i} - x_{\alpha})\delta^{4}(x_{j} - x_{\alpha}).$$

The total effective action

$$S_{\text{eff}}[A, c, \bar{c}] = S_{0}[A] + S_{g.f.}[A] + S_{g}[A, c, \bar{c}] \tag{7}$$

has the invariance under BRS transformations (Becchi *et al* 1975):

$$\delta A_{i} = D_{i}^{\alpha}c_{\alpha\lambda}, \tag{8a}$$

$$\delta c_{\alpha} = -\tfrac{1}{2}g f^{\alpha\beta\gamma}c_{\beta}c_{\gamma}\lambda, \tag{8b}$$

$$\delta\bar{c}_{\alpha} = -\eta_{\alpha}^{-\frac{1}{2}}f_{\alpha}[A]\lambda, \tag{8c}$$

where λ is an x-independent anticommuting c-number. We note that under BRS transformations,

$$\delta(S_0)=0;\quad \delta(c^\alpha D_i^\alpha)=0;\quad \delta(\tfrac{1}{2}f^{\alpha\beta\gamma}c_\beta c_\gamma)=0;$$

$$\delta(-\tfrac{1}{2}f_\alpha^2[A]+\bar{c}_\alpha M_{\alpha\beta}c_\beta)=0;\quad \delta S_{\rm eff}[A,c,\bar{c}]=0. \tag{9}$$

In order to discuss renormalizations of $S_{\rm eff}[A,c,\bar{c}]$ it is convenient (Zinn–Justin 1974) to introduce sources K_i, L_α, R_α for the operators $c^\alpha D_i^\alpha$, $\frac{1}{2}gf^{\alpha\beta\gamma}c_\beta c_\gamma$ and $f_\alpha[A]$ that appear in (8). Note that in the bilinear gauges $f_\alpha[A]$ becomes a composite operator and it becomes necessary to introduce a source for it. We let

$$S[A,c,\bar{c},K,L,R]\equiv S_0[A]-\tfrac{1}{2}\{f_\alpha[A]-\eta_\alpha^{-\frac{1}{2}}R_\alpha\}^2+S_g[A,c,\bar{c}]+K_iD_i^\alpha c_\alpha$$
$$+\tfrac{1}{2}gf^{\alpha\beta\gamma}c_\beta c_\gamma L_\alpha. \tag{10}$$

We also need to modify the BRS transformation on $\bar{c}$ given by (8c):

$$\delta\bar{c}_\alpha=-\eta_\alpha^{-1/2}(f_\alpha[A]-\eta_\alpha^{-\frac{1}{2}}R_\alpha). \tag{8d}$$

Then S of (10) is invariant under the BRS transformations of (8a), (8b) and (8d).

The aim of this work is to discuss the renormalization of $S[A,c,\bar{c},K,L,R]$ of (10).

3. Generating functionals and WT identities

We introduce the dimensionally regularized ('tHooft and Veltman 1972a) generating functional for the unrenormalized Green's functions as in Joglekar and Lee (1976)

$$W[J,\xi,\bar{\xi},K,L,R]=[{\rm d}A{\rm d}c{\rm d}\bar{c}]\exp i\{S[A,c,\bar{c},K,L,R]$$
$$+J_iA_i+\bar{\xi}_\alpha c_\alpha+\bar{c}_\alpha\xi_\alpha\} \tag{11}$$

where ξ, $\bar{\xi}$ are anticommuting sources for the ghost fields.

The generating functional of connected Green's functions is given by

$$Z[J,\xi,\bar{\xi},K,L,R]=-i\ln W[J,\xi,\bar{\xi},K,L,R]. \tag{12}$$

We define the expectation values in the presence of sources by†

$$\langle A_i\rangle=\delta Z/\delta J_i,\quad \langle c_\alpha\rangle=\delta Z/\delta\bar{\xi}_\alpha,\quad \langle\bar{c}_\alpha\rangle=-\partial Z/\partial\xi_\alpha \tag{13}$$

and introduce the generating functional of proper vertices by

$$\Gamma[\langle A\rangle\langle c\rangle\langle\bar{c}\rangle,K,L,R]=Z-J_i\langle A_i\rangle-\bar{\xi}_\alpha\langle c_\alpha\rangle-\langle\bar{c}_\alpha\rangle\xi_\alpha. \tag{14}$$

As we shall deal with Γ exclusively, no confusion will arise if we drop the brackets around the fields in the expectation values $\langle A\rangle$ etc.

Further, we have the relations

$$J_i=-\delta\Gamma/\delta A_i,\quad \xi_\alpha=-\delta\Gamma/\delta\bar{c}_\alpha,\quad \bar{\xi}_\alpha=\delta\Gamma/\delta c_\alpha \tag{15}$$

†We use left functional derivatives through out.

and[†]

$$\delta\Gamma[A, c, \bar{c}, K, L, R]/\delta M = \delta Z[J, \xi, \bar{\xi}, K, L, R]/\delta M, \tag{16}$$

where M stands for K, L or R.

By considering the BRS transformations of (8a, b, d) on the integration variables in (11) and equating the change to zero one obtains, on account of BRS invariance,

$$0 = [\mathrm{d}A\mathrm{d}c\mathrm{d}\bar{c}]\{J_i D_i^{\alpha} c_\alpha - \bar{\xi}_\alpha \tfrac{1}{2} g f^{\alpha\beta\gamma} c_\beta c_\gamma + \eta \tfrac{1}{2}(f_\alpha[A] - \eta^{1/2} R_\alpha)\xi_\alpha\} \times \exp i\{S + \cdots\}, \tag{17}$$

which can be expressed as an identity for Γ viz:

$$\frac{\delta\Gamma}{\delta A_i}\frac{\delta\Gamma}{\delta K_i} + \frac{\delta\Gamma}{\delta c_\alpha}\frac{\delta\Gamma}{\delta L_\alpha} + \frac{\delta\Gamma}{\delta \bar{c}_\alpha}\frac{\delta\Gamma}{\delta R_\alpha} = 0. \tag{18}$$

Or, writing collectively[‡] $\Phi_i = \{A_i, c_\alpha, \bar{c}_\alpha\}$ and $X_i = \{K_i, L_\alpha, R_\alpha\}$ we can write the WT identity in short:

$$(\delta\Gamma/\delta\Phi_i)(\delta\Gamma/\delta X_i) = 0. \tag{19}$$

In particular, in the zero loop approximation, $\Gamma = S$ and thus

$$(\delta S/\delta\Phi_i)(\delta S/\delta X_i) = 0. \tag{20}$$

4. Structure of the counter terms

In this section we shall analyse the structure of the divergence in the one-loop approximation, very much in the same manner as was done by Lee (1975). Equation (18) is the WT identity satisfied the unrenormalized proper vertices of the theory. Consider (18) in the one-loop approximation. We consider the divergent part of Γ viz $\{\Gamma\}_1^{\mathrm{div}}$. It satisfies

$$\mathscr{G}\{\Gamma\}_1^{\mathrm{div}} = 0, \tag{21}$$

where

$$\mathscr{G} = (\delta S/\delta\Phi_i)(\delta/\delta X_i) + (\delta S/\delta X_i)(\delta/\delta\Phi_i). \tag{22}$$

We note that

$$\mathscr{G}^2 = \frac{\delta S}{\delta X_j}\frac{\delta^2 S}{\delta\Phi_j \delta\Phi_i}\frac{\delta}{\delta X_i} + \frac{\delta S}{\delta\Phi_i}\frac{\delta^2 S}{\delta X_j \delta\Phi_i}\frac{\delta}{\delta X_i} + \frac{\delta S}{\delta X_j}\frac{\delta^2 S}{\delta\Phi_j \delta X_i}\frac{\delta}{\delta\Phi_i} + \frac{\delta S}{\delta\Phi_j}\frac{\delta^2 S}{\delta X_j \delta X_i}\frac{\delta}{\delta\Phi_i} \tag{23}$$

$$= p_{(i)}\frac{\delta}{\delta\Phi_i}\left(\frac{\delta S}{\delta X_j}\frac{\delta S}{\delta\Phi_j}\right)\frac{\delta}{\delta X_i} - p_{(i)}\frac{\delta}{\delta X_i}\left(\frac{\delta S}{\delta X_j}\frac{\delta S}{\delta\Phi_j}\right)\frac{\delta}{\delta\Phi_i}, \tag{24}$$

$$= 0. \tag{25}$$

[†]It is understood here that all other (different) arguments of the respective generating functionals are held fixed

[‡]Note here that the suffix i of Φ_i does not contain Lorentz index when referring to c_α and $\bar{c}_\alpha$ This slight deviation from the notation is to be able to present compact proofs

In obtaining (23) we note that the terms proportional to two derivative operators such as $\delta^2/\delta X_i \delta X_j$ etc drop out because of the anticommutivity of the variables. In (24) $p_{(i)}$ is ± 1 depending on whether Φ_i refers to a commuting field (A_i) or an anticommuting field $(c_\alpha, \bar{c}_\alpha)$. Equation (25) then follows because of (18) which expresses the BRS invariance of S.

Thus as with linear gauges (Lee 1975) one encounters (21) with $\mathscr{G}$, another nilpotent operator. Solving the equation as was done by Joglekar and Lee (1976) is extremely difficult. However one can also verify by direct construction that the solution to this equation is again of the same form as in Joglekar and Lee (1976) viz.,

$$\{\Gamma\}_1^{\mathrm{div}} = \alpha_1(\varepsilon) S_0[A] + \mathscr{G} X[A, c, \bar{c}, K, L, R; \varepsilon], \tag{26}$$

where X is a Lorentz-invariant local functional of dimension 3 and ghost number (Joglekar and Lee 1976) minus one. (Here we note that K, L, R have dimensions two each. The ghost number is an additive number and has values 0, 1, -1, -2 for (A, R); c; $(\bar{c}, K)$; L respectively). Therefore, X must have the general structure† ($\varepsilon = 4 - n$),

$$\begin{aligned} X = {} & \alpha_2(\varepsilon) K_i A_i + \alpha_3(\varepsilon) L_\alpha c_\alpha + \alpha_4(\varepsilon) R_\alpha \bar{c}_\alpha + \alpha_5(\varepsilon) \partial_i^\alpha \bar{c}_\alpha A_i \\ & + \sum \alpha_6^{(\alpha i j)}(\varepsilon) \tfrac{1}{2} \bar{c}_\alpha \lambda_{ij}^\alpha A_i A_j + \sum \alpha_7^{(\alpha\beta\gamma)}(\varepsilon) \tfrac{1}{4} \rho_\gamma^{\alpha\beta} \bar{c}_\alpha \bar{c}_\beta c_\gamma, \end{aligned} \tag{27}$$

where in the last two terms the summation is over all polynomials of the given form; and $\rho_\gamma^{\alpha\beta}$ are antisymmetric in (α, β) and depend on $\zeta_{\beta\gamma}^\alpha$.

Exhibiting the loop expansion parameter a explicitly the divergence may be expressed as (we suppress summations in the last two terms of (27) henceforth)

$$\{\Gamma\}_1^{\mathrm{div}} = a \sum_i \alpha_i(\varepsilon) D_i, \tag{28}$$

where

$$\begin{aligned} D_1 &= S_0[A], \\ D_2 &= (\delta S/\delta A_i) A_i - K_i (\delta S/\delta K_i), \\ D_3 &= (\delta S/\delta c_\alpha) c_\alpha + L_\alpha (\delta S/\delta L_\alpha), \\ D_4 &= (\delta S/\delta \bar{c}_\alpha) \bar{c}_\alpha + (\delta S/\delta R_\alpha) R_\alpha, \\ D_5 &= (\delta S/\partial K_i) \partial_i \bar{c}_\alpha + \eta^{-1/2} (f_\alpha[A] - \eta^{-1/2} R_\alpha) \partial_i^\alpha A_i, \\ D_6 &= \tfrac{1}{2} \eta^{-1/2} (f_\alpha[A] - \eta^{-1/2} R_\alpha) \lambda_{ij}^\alpha A_i A_j + \frac{\delta S}{\delta K_i} \bar{c}_\alpha \lambda_{ij}^\alpha A_j, \\ D_7 &= \eta^{-\frac{1}{2}} (f_\alpha[A] - \eta^{-\frac{1}{2}} R_\alpha) \rho_\gamma^{\alpha\beta} \bar{c}_\beta c_\gamma + \tfrac{1}{2} \frac{\delta S}{\delta L_\alpha} \rho_\gamma^{\alpha\beta} \bar{c}_\alpha \bar{c}_\beta. \end{aligned} \tag{29}$$

We note the presence of a counter-term D_7 is quartic in ghosts and certainly cannot be obtained by multiplicative renormalizations. In fact, it can be interpreted as a subtractive renormalization of the gauge-fixing and the ghost term. To see this, we

†If the term $\zeta_{ij}^\alpha A_i A_j$ in $f_\alpha[A]$ breaks global invariance, one does not have that as a guidance in writing the structure of X However, it is clear that all such deviations must be expressible in terms of ζ_{ij}^α. This case is discussed later.

note that we could write to the first order in the loop expansion parameter a:

$$-\tfrac{1}{2}(f_\alpha - \eta^{-1/2}R_\alpha)^2 + S_g - a\alpha_7(\varepsilon)D_7$$
$$= -\tfrac{1}{2}[f_\alpha - \eta^{-\frac{1}{2}}R_\alpha + a\alpha_7(\varepsilon)\eta^{-\frac{1}{2}}\rho_\gamma^{\alpha\beta}\bar{c}_\beta c_\gamma]^2 + \{S_g - a\alpha_7(\varepsilon)\tfrac{1}{2}\frac{\delta S}{\delta L_\alpha}\rho_\gamma^{\alpha\beta}\bar{c}_\alpha\bar{c}_\beta\}. \quad (30)$$

Thus, the new counter terms, in fact, modify the f_α to a functional of A, c, $\bar{c}$ and modify the ghost term in a certain fashion to be explained below.

This modification of gauge-fixing and ghost terms is precisely as encountered in the renormalization of gauge-invariant operators (Joglekar and Lee 1976). The general rule is that the gauge-fixing term is modified to be of the form

$$f_\alpha[A, c, \bar{c}] = f_\alpha[A] + \eta^{-\frac{1}{2}}(\delta F/\delta\bar{c}_\alpha) \quad (31)$$

and the corresponding ghost terms are

$$S_g' = S_g - \mathscr{G}_0 F, \quad (32)$$

where $\mathscr{G}_0$ is the BRS variational operator on c and A introduced in Joglekar and Lee (1976):

$$\mathscr{G}_0 = c^\alpha D_i^\alpha(\delta/\delta A_i) + \tfrac{1}{2}g f^{\alpha\beta\gamma}c_\beta c_\gamma(\delta/\delta c_\alpha) \quad (33)$$

with,

$$\mathscr{G}_0^2 = 0. \quad (34)$$

In (30), F happens to be

$$F = \tfrac{1}{2}a\alpha_7(\varepsilon)\rho_\gamma^{\alpha\beta}\bar{c}_\alpha\bar{c}_\beta c_\gamma. \quad (35)$$

Thus, before we fully discuss one-loop renormalization, we note this subtractive renormalization of the gauge-fixing and the ghost terms. Thus even if we started with the action of (10), renormalization will alter the form of the action. Hence the BRS invariance of the action and the consequent WT identities will have to be discussed anew when discussing two and higher loop renormalizability.

All these problems can be avoided if we instead modify the gauge-fixing term f_α to contain an arbitrary polynomial, linear in c and $\bar{c}$ each, from the beginning with the understanding that the coefficients of such terms are functions of the loop expansion parameter and vanish in the zero-loop approximation.

We shall therefore modify $f_\alpha[A] \to f_\alpha\,[A, \bar{c}, c]$ with

$$f_\alpha[A, c, \bar{c}] = \eta^{-\frac{1}{2}}(\partial_i^\alpha A_i + \zeta_{ij}^\alpha A_i A_j + \tau_\gamma^{\alpha\beta}\bar{c}_\beta c_\gamma) \quad (36)$$
$$\equiv \eta^{-\frac{1}{2}}\left(\partial_i A_i + \zeta_{ij}^\alpha A_i A_j + \frac{\delta F'}{\delta\bar{c}_\alpha}\right)$$
$$= \eta^{-\frac{1}{2}}\frac{\delta}{\delta\bar{c}_\alpha}(\bar{c}_\alpha\partial_i^\alpha A_i + \bar{c}_\alpha\zeta_{ij}^\alpha A_i A_j + F')$$
$$\equiv \eta^{-\frac{1}{2}}\frac{\delta}{\delta\bar{c}_\alpha}H[A, c, \bar{c}] \quad (37)$$

and modify the ghost term to

$$
\begin{aligned}
S'_g &= S_g - \mathscr{G}_0 F' \\
&= -\mathscr{G}_0[\bar{c}_\alpha \partial^\alpha_i A_i + \bar{c}_\alpha \zeta^\alpha_{ij} A_i A_j + F'] \\
&= -\mathscr{G}_0 H[A, c, \bar{c}]
\end{aligned}
\tag{38}
$$

and discuss the new action directly

$$
S' = S_0 - \tfrac{1}{2}(f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}} R_\alpha)^2 + S'_g + K_i D^\alpha_i c_\alpha + \tfrac{1}{2} g L_\alpha f^{\alpha\beta\gamma} c_\beta c_\gamma \tag{39}
$$

with f_α and S'_g given in the form of (37) and (38). Note that $\tau^{\alpha\beta}_\gamma$ is antisymmetric in α and β as only such counterterms are induced by renormalization.

For future reference, we note

$$
(\delta/\delta\bar{c}_\alpha) S'_g = \mathscr{G}_0(\delta H/\delta\bar{c}_\alpha) = \eta^{\frac{1}{2}} \mathscr{G}_0 f_\alpha. \tag{40}
$$

5. BRS invariance of the new action

We show that the modified action of (39) is invariant under the new modified BRS transformations

$$
\begin{aligned}
\delta A_i &= D^\alpha_i c_\alpha \lambda, \\
\delta c_\alpha &= -\tfrac{1}{2} g f^{\alpha\beta\gamma} c_\beta c_\gamma, \\
\delta \bar{c}_\alpha &= -\eta^{-\frac{1}{2}} \{ f_\alpha[A, c, \bar{c}] - \eta^{-1/2} R_\alpha \} \lambda.
\end{aligned}
\tag{41}
$$

Note the modification in the last equation for $\delta\bar{c}_\alpha$.

To show the BRS invariance of S' we only have to show anew that

$$
\delta(S'_{g\,f.} + S'_g) = 0. \tag{42}
$$

To show this we note:

$$
\begin{aligned}
&\delta\{-\tfrac{1}{2}(f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}} R_\alpha)^2\} \\
&= -(f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}} R_\alpha) \Big\{ \mathscr{G}_0 f_\alpha[A,c,\bar{c}] \lambda \\
&\quad + \eta^{-\frac{1}{2}} (f_\beta[A,c,\bar{c}] - \eta^{-\frac{1}{2}} R_\beta) \frac{\delta f_\alpha[A,c,\bar{c}]}{\delta \bar{c}_\beta} \lambda \Big\} \\
&= -(f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}} R_\alpha) \mathscr{G}_0 f_\alpha[A,c,\bar{c}] \lambda,
\end{aligned}
\tag{43}
$$

the second term vanishes on account of antisymmetry in α, β when the form of $f_\alpha[A,c,\bar{c}]$ in (37) is taken into consideration.

Whereas,

$$
\begin{aligned}
\delta S'_g &= \eta^{-\frac{1}{2}} (f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}} R_\alpha) \frac{\delta}{\delta \bar{c}_\alpha} S'_g \lambda + \mathscr{G}_0 S'_g \lambda \\
&= (f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}} R_\alpha) \mathscr{G}_0 f_\alpha \lambda + \mathscr{G}_0^2 H \lambda,
\end{aligned}
\tag{44}
$$

where we have used (38) and (40). The second term on the right hand side vanishes as $\mathscr{G}_0^2 = 0$. Equations (43) and (44) together prove the result of (41) and thus the BRS invariance of S'.

Using S', we could consider a set of new generating functionals W', Z', as done earlier in §3. One can then use the modified BRS invariance of S' to obtain the WT identities and they will be as before of the same form as (20) viz.,

$$(\delta\Gamma'/\delta\Phi_i)(\delta\Gamma'/\delta X_i) = 0. \tag{45}$$

We shall now discuss the renormalization of S' using (45) in the next section and prove the renormalizability of the theory.

6. Proof of renormalizability

We define the following renormalization transformation:[†]

$$\begin{aligned} &A_i = Z^{\frac{1}{2}} A_i^{(r)}; \quad c_\alpha = \tilde{Z}^{\frac{1}{2}} c_\alpha^{(r)}; \quad \bar{c}_\alpha = \tilde{Z}^{\frac{1}{2}} \bar{c}_\alpha^{(r)}, \\ &K_i = (V\tilde{Z}Z^{-1})^{\frac{1}{2}} K_i^{(r)}; \quad L_\alpha = V^{\frac{1}{2}} L_\alpha^{(r)}; \quad R_\alpha = V^{\frac{1}{2}} R_\alpha^{(r)}, \\ &g = X(\tilde{Z}^2 Z)^{-\frac{1}{2}} g^{(r)}; \quad \eta = W Z^{-1} \eta^{(r)}, \\ &\zeta_{ij}^\alpha = Y_{(\alpha ij)} \zeta_{ij}^{\alpha(r)}; \quad \tau = T_{(\alpha\beta\gamma)} \tau_\gamma^{\alpha\beta(r)}. \end{aligned} \tag{46}$$

In the last two equations Y and T could depend on the indices of ζ and τ. All the renormalization constants are functions of ε and have a loop expansion of the form

$$Z = \sum_{\gamma=0}^{\infty} a^r z_r \tag{47}$$

etc. For future reference we introduce a symbol

$$Z^{(n)} = \sum_{r=0}^{n} a^r z_r \tag{48}$$

and represents a renormalization constant when renormalization is done upto n loops.

The renormalized generating functional of proper vertices is defined by

$$\begin{aligned} &\Gamma'(r)[A^{(r)}, c^{(r)}, \bar{c}^{(r)}, K^{(r)}, L^{(r)}, R^{(r)}, g^{(r)}, \eta^{(r)}, \zeta^{(r)}, \tau^{(r)}] \\ &\quad = \Gamma'[A, c, \bar{c}, K, L, R; g, \eta, \zeta, \tau]. \end{aligned} \tag{49}$$

Our aim is to prove that the renormalization constants in the renormalization transformations of (46) can be chosen in each loop order, so that $\Gamma'^{(r)}$ is a finite functional of its arguments. We prove this by induction.

Suppose we have chosen renormalization constants upto $(n-1)$ loops, i.e. constructed $Z^{(n-1)}$ etc. so that $\Gamma'^{(r)}$ is a finite functional of its arguments upto $O(a^{n-1})$ when expanded in powers of a. We note that we have essentially performed renormalization

[†]If η_α are different for different α's, Z will be different for different A_μ^α The same remark applies to $\tilde{Z}_\alpha$ and V_α. However $V_\alpha \tilde{Z}_\alpha$ will be independent of α. Also if $A_i = Z_{(i)}^{\frac{1}{2}} A_i^{(r)}$, then $K_i = (V\tilde{Z}Z_{(i)}^{-1})^{\frac{1}{2}} K_i^{(r)}$

transformations of (46) on the arguments of Γ with Z etc replaced by $Z^{(n-1)}$ etc. We note that these transformations are such that the WT identity for $\Gamma'^{(r)}$ becomes

$$\frac{\delta\Gamma'^{(r)}}{\delta\Phi_i^{(r)}}\frac{\delta\Gamma'^{(r)}}{\delta X_i^{(r)}}=0. \tag{50}$$

Now consider the divergent part of $\Gamma^{(r)}$. It is entirely of $O(a^n)$. Thus considering the divergent part of the terms of $O(a^n)$ in (50) we get

$$\mathscr{G}'\{\Gamma'^{(r)}\}_n^{\rm div}=0, \tag{51}$$

where

$$\mathscr{G}'=\frac{\delta S'}{\delta\Phi_i^{(r)}}\frac{\delta}{\delta X_i^{(r)}}+\frac{\delta S'}{\delta X_i^{(r)}}\frac{\delta}{\delta\Phi_i^{(r)}} \tag{52}$$

and has the solution of the same form as (28) viz:

$$\{\Gamma'\}_n^{\rm div}=a^n\sum_i\alpha_i(\varepsilon)D_i', \tag{53}$$

where D_i' are obtained from D_i by replacing S by S' and $f_\alpha[A]$ by $f_\alpha[A,c,\bar{c}]$ in (29).

We have to show that we can choose renormalization constants in the n-loop approximation so that $\{\Gamma^{(r)}\}_n^{\rm div}$ is cancelled. Note that this amounts to doing the following new renormalization transformation:

$$[A_i^{(r)}]^{(n)}=\{Z^{(n)}/Z^{(n-1)}\}^{\frac{1}{2}}\,[A_i^{(r)}]^{(n-1)} \tag{54}$$

etc in the obvious notations.

We write down the n-loop counterterms. They are of the form (omitting a^n)

$$\tfrac{1}{2}z_nC_1+\tfrac{1}{2}\tilde{z}_nC_2+\tfrac{1}{2}v_nC_3+(x_n-\tilde{z}_n-\tfrac{1}{2}z_n)C_4$$
$$+(\omega_n-z_n)C_5+y_nC_6+t_nC_7, \tag{55}$$

where

$$C_1=\frac{\delta S'}{\delta A_i}A_i+\frac{\delta S'}{\delta K_i}K_i, \tag{55}$$

$$C_2=c_\alpha\frac{\delta S'}{\delta c_\alpha}+\bar{c}_\alpha\frac{\delta S'}{\delta\bar{c}_\alpha}-\frac{\delta S'}{\delta K_i}K_i,$$

$$C_3=\frac{\delta S'}{\delta R_\alpha}R_\alpha-\frac{\delta S'}{\delta K_i}K_i+\frac{\delta S'}{\delta L_\alpha}L_\alpha,$$

$$C_4=g(\partial S'/\partial g),$$

$$C_5=\eta(\partial S'/\partial\eta),$$

$$C_6=\partial S'/\partial\zeta^\alpha_{\beta\gamma},$$

$$C_7=\partial S'/\partial\tau^{\alpha\beta}_\gamma, \tag{56}$$

To prove renormalizability, we must show that D_i' can be expressed in terms of C_i's. To prove that all the renormalization constants in (46) are in fact independent, we must show that the above relations are unique. To this end, we need these relations and they are obtained with the help of following identities†:

$$(\delta S_0/\delta A_i)A_i = g(\partial S_0/\partial g) + 2S_0,$$

$$\frac{\delta S'_{g.f}}{\delta A_i}A_i = \left\{-2\eta\frac{\partial}{\partial\eta} + \zeta\frac{\partial}{\partial\zeta} - \tau\frac{\partial}{\partial\tau} - R\frac{\partial}{\partial R} + g\frac{\partial}{\partial g}\right\}S'_{g\,f},$$

$$\frac{\delta S'_g}{\delta A_i}A_i = \left(g\frac{\partial}{\partial g} + \zeta\frac{\partial}{\partial\zeta} - \tau\frac{\partial}{\partial\tau}\right)S'_g,$$

$$\frac{\delta}{\delta A_i}\{K_i(D_i^\alpha c_\alpha)\}A_i = g\frac{\partial}{\partial g}\{K_i D_i^\alpha c_\alpha\},$$

$$\frac{\delta}{\delta A_i}\{\tfrac{1}{2}L_\alpha g f^{\alpha\beta\gamma}c_\beta c_\gamma\}A_i = g\frac{\partial}{\partial g}\{\tfrac{1}{2}L_\alpha g f^{\alpha\beta\gamma}c_\beta c_\gamma\} - L_\alpha\frac{\delta S'}{\delta L_\alpha}. \tag{57}$$

Adding these equations and rearranging the terms one obtains

$$2D'_1 = 2S_0 = \left\{-g\frac{\partial}{\partial g} + A_i\frac{\delta}{\delta A_i} + L_\alpha\frac{\delta}{\delta L_\alpha} + 2\eta\frac{\partial}{\partial\eta} - \zeta\frac{\partial}{\partial\zeta} + \tau\frac{\partial}{\partial\tau} + R_\alpha\frac{\delta}{\delta R_\alpha}\right\}S'. \tag{58}$$

Also we have,

$$D'_2 = \left(A_i\frac{\delta}{\delta A_i} - K_i\frac{\delta}{\delta K_i}\right)S',$$

$$D'_3 = \left(-c_\alpha\frac{\delta}{\delta c_\alpha} + L_\alpha\frac{\delta}{\delta L_\alpha}\right)S',$$

$$D'_4 = \left(-\bar{c}_\alpha\frac{\delta}{\delta\bar{c}_\alpha} + R_\alpha\frac{\delta}{\delta R_\alpha}\right)S'. \tag{59}$$

One can establish the following identities in a similar fashion:

$$\begin{aligned}D'_5 &= \frac{\delta S'}{\delta K_i}(\partial_i^\alpha\bar{c}_\alpha) + \eta^{-\frac{1}{2}}(f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}}R_\alpha)\partial_i^\alpha A_i\\ &= \left\{-2\eta\frac{\partial}{\partial\eta} + \zeta\frac{\partial}{\partial\zeta} + 2\tau\frac{\partial}{\partial\tau} + R_\alpha\frac{\delta}{\delta R_\alpha} - \bar{c}_\alpha\frac{\delta}{\delta\bar{c}_\alpha}\right\}S',\end{aligned} \tag{60}$$

$$\begin{aligned}D'_6 &= \tfrac{1}{2}(f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}}R_\alpha)\lambda_{ij}^\alpha A_i A_j\\ &\quad + \frac{\delta S}{\delta K_i}\bar{c}_\alpha\lambda_{ij}^\alpha A_j = \sum_{(\alpha ij)} -\frac{1}{2\sqrt{\eta}}\lambda_{ij}^\alpha\frac{\partial}{\partial\zeta_{ij}^\alpha}S'^{\ddagger},\end{aligned} \tag{61}$$

†Indices of ζ and τ are being suppressed.

‡Summation-integration convention is being suppressed here.

$$D'_7 = (f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}}R_\alpha)\rho_\gamma^{\alpha\beta}\bar{c}_\beta c_\gamma + \tfrac{1}{2}\frac{\delta S}{\delta L_\alpha}\rho_\gamma^{\alpha\beta}\bar{c}_\alpha\bar{c}_\beta$$

$$= \sum_{(\alpha\beta\gamma)} -\frac{1}{\sqrt{\eta}}\rho_\gamma^{\alpha\beta}\frac{\partial}{\partial\tau_\gamma^{\alpha\beta}}S'. \tag{62}$$

The above relations together with the following identity among the derivatives of S' viz.

$$\left(K_i\frac{\delta}{\delta K_i} + 2L_\alpha\frac{\delta}{\delta L_\alpha} + \bar{c}_\alpha\frac{\delta}{\delta\bar{c}_\alpha} - c_\alpha\frac{\delta}{\delta c_\alpha}\right)S' = 0 \tag{63}$$

allow us to express $D'_1 \ldots D'_7$ uniquely in terms of $C_1, C_2 \ldots C_7$ provided we insist (by our free choice) that they lead to identical renormalization constants $\tilde{Z}$ for c and $\bar{c}$. These relations allow us to determine uniquely $z_n, \tilde{z}_n \ldots$ etc. in terms of $\alpha_1, \alpha_2, \ldots, \alpha_7$ proving the renormalizability up to nth order and thus by induction to all orders. These relations are:

$$D'_1 = C_1 + C_3 - C_4 + 2C_5 - \zeta C_6 + \tau C_7; \quad D'_2 = C'_1,$$

$$D'_3 + D'_4 = C_3 - C_2; \quad D'_3 - D'_4 = -C_3,$$

$$D'_5 - D'_4 = -2C_5 + \zeta C_6 + 2\tau C_7; \quad D'_6 = -\frac{\lambda}{2\sqrt{\eta}}C_6; \quad D'_7 = -\frac{\rho}{\sqrt{\eta}}C_7. \tag{64}$$

7. Independence of the S-matrix from variations in ζ and τ

Finally, we shall show that the renormalized S-matrix is independent of the parameters ζ and τ. To do this we compute the change in W' when $\zeta \to \zeta + \delta\zeta$

$$\delta\zeta\frac{\partial W'}{\partial\zeta} = \langle \eta^{-\frac{1}{2}}(f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}}R_\alpha)\delta\zeta_{ij}^\alpha A_i A_j + 2\bar{c}_\beta\delta\zeta_{ij}^\beta A_j D_i^\alpha c_\alpha\rangle, \tag{65}$$

where $\langle 0 \rangle$ stands for

$$\langle 0[A,c,\bar{c}]\rangle = \int [\mathrm{d}A\,\mathrm{d}c\,\mathrm{d}\bar{c}]0[A,c,\bar{c}]\exp i\{S' + \cdots\} \tag{66}$$

and the change in W' when $\tau \to \tau + \delta\tau$

$$\delta\tau\frac{\partial W'}{\partial\tau} = \langle -\eta^{-1/2}(f_\alpha[A,c,\bar{c}] - \eta^{-\frac{1}{2}}R_\alpha)\delta\tau_\gamma^{\alpha\beta}\bar{c}_\beta c_\gamma$$

$$+ \tfrac{1}{4}\bar{c}_\alpha\delta\tau_\gamma^{\alpha\beta}\bar{c}_\beta g f^{\gamma\delta\sigma}c_\delta c_\sigma\rangle. \tag{67}$$

We simplify expressions of (65) and (67) with the help of the WT identity of (17) evaluated at $K = L = R = 0$:

$$0 = \langle J_i D_i^\alpha c_\alpha - \tfrac{1}{2}\bar{\xi}_\alpha g f^{\alpha\beta\gamma}c_\beta c_\gamma + \eta^{-\frac{1}{2}}f_\alpha[A,c,\bar{c}]\xi_\alpha\rangle. \tag{68}$$

Operating on (68) by

$$\delta\zeta^{\beta}_{jk}\frac{\delta^2}{\delta J_j\delta J_k}\frac{\delta}{\delta\xi_\beta}$$

and then setting $\xi=\bar{\xi}=0$, we obtain

$$\begin{aligned}&\langle -\eta^{-\frac{1}{2}}f_\beta[A,c,\bar{c}]\delta\zeta^{\beta}_{jk}A_jA_k+2\bar{c}_\beta\delta\zeta^{\beta}_{ij}A_jD^{\alpha}_{i}c_\alpha\rangle\\&\quad=-i\langle J_iD^{\alpha}_{i}c_\alpha\bar{c}_\beta\delta\zeta^{\beta}_{jk}A_jA_k\rangle.\end{aligned}\qquad(69)$$

Similarly operating on (68) by

$$\tfrac{1}{2}\tau^{\alpha\beta}_{\gamma}\frac{\delta^3}{\delta\xi_\beta\delta\bar{\xi}_\gamma\delta\xi_\alpha}$$

and then setting $\xi=\bar{\xi}=0$, we obtain

$$\begin{aligned}&\langle -\eta^{-\frac{1}{2}}f_\alpha[A,c,\bar{c}]\delta\tau^{\alpha\beta}_{\gamma}\bar{c}_\beta c_\gamma+\tfrac{1}{4}\bar{c}_\alpha\delta\tau^{\alpha\beta}_{\gamma}\bar{c}_\beta g f^{\gamma\delta\sigma}c_\delta c_\sigma\rangle\\&\quad=-\tfrac{1}{2}\langle J_iD^{\delta}_{i}c_\delta\bar{c}_\alpha\delta\tau^{\alpha\beta}_{\gamma}\bar{c}_\beta c_\gamma\rangle.\end{aligned}\qquad(70)$$

Given (69) and (70), the discussion of the gauge independence of the renormalized S-matrix proceeds exactly as in linear gauges (Lee and Zinn–Justin 1972, 1973; Lee 1975). A similar discussion for the η- independence of the renormalized S-matrix can also be given. Since the whole discussion is now essentially similar to that given for linear gauges it is not being repeated here.

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