

Temperature-modified Coulomb potential for the electron–proton system

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Abstract. The finite-temperature Schrödinger equation, derived recently from the Bethe-Salpeter equation for the bound states of an electron and a proton interacting via the instantaneous Coulomb interaction, is studied in the coordinate space. An expression for the temperature-modified Coulomb potential is obtained and briefly discussed.

Keywords. Finite temperature theory; Schrödinger equation; spectra in keV region; X-ray astronomy.

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The Bethe-Salpeter equation for the Wick-Cutkosky model (Wick 1954; Cutkosky 1954) in the instantaneous approximation was recently generalized to arbitrary temperatures, and was shown to possess temperature-dependent discrete spectra in the large-temperature limit (Malik and Pande 1988). This motivated us to carry out a similar investigation for the electron–proton bound state system interacting via the instantaneous Coulomb interaction. The finite temperature Bethe-Salpeter equation (FTBSE) so obtained is, however, very difficult to handle, but can be reduced, in a non-relativistic approximation, to a finite temperature Schrödinger equation (FTSE). Using momentum space methods, we have shown that the latter equation possesses solutions which seem to be of great relevance to X-ray astronomy (Malik *et al* 1988—MPVI hereafter). The purpose of this note is to study the same FTSE in coordinate space and obtain an expression for the temperature-modified Coulomb potential which governs its spectra.

The FTBSE for the electron–proton bound state system is (MPVI):

$$S_{ij}(\mathbf{p}) = \frac{e^2}{4\pi^2} \Lambda_i^a(p) \Lambda_j^b(p) \int \frac{d^3\mathbf{q}}{(\mathbf{q} - \mathbf{p})^2} \sum_{k,l} S_{kl}(\mathbf{q}) \times \frac{(\tanh A_k + \tanh B_l)}{[E - E_k^a(\mathbf{q}) - E_l^b(\mathbf{q})]}, \quad (1)$$

where i, j, k and l take on values $+$ or $-$ and Λ_{\pm} are the usual positive and negative energy projection operators given by

$$\Lambda_{\pm}^{a,b}(p) = [E^{a,b}(\mathbf{p}) \pm H^{a,b}(\mathbf{p})]/2E^{a,b}(p), \quad (2)$$

with

$$\pm E^{a,b}(\mathbf{p}) \equiv E_{\pm}^{a,b}(\mathbf{p}) = \pm (p^2 + m_{a,b}^2)^{1/2}, \quad (3)$$

$$H^{a,b}(\mathbf{p}) = (m_{a,b}\beta^{a,b} \pm \mathbf{p} \cdot \boldsymbol{\alpha}^{a,b}), \quad (4)$$

where α and β are the standard Dirac matrices, associated with the electron if they carry the superscript a and with the proton if they carry the superscript b . The positive sign in (4) goes with a and the negative sign with b . m_a and m_b are, respectively, the masses of the electron and the proton and \mathbf{p} is their relative three-momentum. E represents the energy eigenvalue of (1). The functions A_{\pm} and B_{\pm} are given by

$$A_{\pm} = (1/2)\beta[\mu_a E - E_{\pm}^a(\mathbf{p})], \quad (5)$$

$$B_{\pm} = (1/2)\beta[\mu_b E - E_{\pm}^b(\mathbf{p})], \quad (6)$$

with

$$\mu_{a,b} = m_{a,b}/(m_a + m_b) \quad (7)$$

and

$$\beta = 1/kT, \quad (8)$$

where T is temperature and k the Boltzmann constant; $\hbar = c = 1$.

If we now neglect spin and negative energy components, only an approximated version, say S_{++}^0 , of the function S_{++} will survive in (1). In terms of a function $S(\mathbf{p})$ defined by

$$S(\mathbf{p}) = \frac{\tanh A_{++} + \tanh B_{++}}{E - E^a - E^b} S_{++}^0(\mathbf{p}), \quad (9)$$

the equation can then be written as

$$\left(W - \frac{p^2}{2\mu}\right) S(\mathbf{p}) = \frac{e^2}{4\pi^2} \cdot Q_{\beta}(W, p^2) \int \frac{d^3\mathbf{q}}{(\mathbf{q} - \mathbf{p})^2} S(\mathbf{q}). \quad (10)$$

Note that simultaneous with the neglect of spin etc., we have introduced the appropriate non-relativistic kinematics:

$$E - E^a(p) - E^b(p) \simeq W - (p^2/2\mu), \quad (11)$$

$$W \equiv -|W| = E - m_a - m_b, \quad (12)$$

where

$$\mu = m_a m_b / (m_a + m_b) \quad (13)$$

and

$$\begin{aligned} & \tanh A_{++} + \tanh B_{++} \\ & \simeq \tanh\left(\frac{1}{2}\beta\mu_a W - \frac{\beta p^2}{4m_a}\right) + \tanh\left(\frac{1}{2}\beta\mu_b W - \frac{\beta p^2}{4m_b}\right) \\ & \equiv Q_{\beta}(W, p^2). \end{aligned} \quad (14)$$

For zero temperature, $Q_{\beta}(W, p^2) = -2$. Equation (10) then reduces to the normal

Schrödinger equation. We now write

$$\begin{aligned}
 Q_\beta(W, p^2) = & \frac{\tanh\left(\frac{\beta\mu_a W}{2}\right) - \tanh\left(\frac{p^2\beta}{4m_a}\right)}{1 - \tanh\left(\frac{\beta\mu_a W}{2}\right)\tanh\left(\frac{\beta p^2}{4m_a}\right)} \\
 & + \frac{\tanh\left(\frac{\beta\mu_b W}{2}\right) - \tanh\left(\frac{p^2\beta}{4m_b}\right)}{1 - \tanh\left(\frac{\beta\mu_b W}{2}\right)\tanh\left(\frac{\beta p^2}{4m_b}\right)}
 \end{aligned} \tag{15}$$

and make the approximation

$$\tanh(\beta p^2/4m_{a,b}) \simeq p^2\beta/4m_{a,b}; \tag{16}$$

such an approximation is in keeping with the spirit of (10). We thus get

$$\begin{aligned}
 Q_\beta(W, p^2) & \simeq (d_1 - d_2 p^2)/(1 - d_3 p^2), \\
 & \equiv \tilde{Q}_\beta(W, p^2),
 \end{aligned} \tag{17}$$

where

$$d_1 = a + b; \quad d_2 = \frac{\beta}{4\mu}(1 + ab); \quad d_3 = \frac{\beta}{4}\left(\frac{a}{m_a} + \frac{b}{m_b}\right)$$

and

$$a = \tanh \frac{1}{2}(\beta\mu_a W), \quad b = \tanh \frac{1}{2}(\beta\mu_b W). \tag{18}$$

We now write

$$\tilde{Q}_\beta(W, p^2) = D_1 + \frac{D_2 D_3}{D_2 + p^2}, \tag{19}$$

where

$$D_1 = d_2/d_3, \quad D_2 = -1/d_3, \quad D_3 = [d_1 - (d_2/d_3)] \tag{20}$$

and consider (10) with $Q_\beta(W, p^2)$ replaced by $\tilde{Q}_\beta(W, p^2)$ as given in (19). We obtain

$$\begin{aligned}
 \left(W - \frac{p^2}{2\mu}\right)\psi(\mathbf{p}) & = \frac{e^2}{4\pi^2} D_1 \int \frac{d^3\mathbf{q}\psi(\mathbf{q})}{(\mathbf{q} - \mathbf{p})^2} \\
 & + \frac{e^2 D_2 D_3}{4\pi^2} \int \frac{d^3\mathbf{q}\psi(\mathbf{q})}{(\mathbf{q} - \mathbf{p})^2(\mathbf{q}^2 + D_2)} \\
 & \equiv T_1 + T_2,
 \end{aligned} \tag{21}$$

where $\psi(\mathbf{p})$ is related to $S(\mathbf{p})$ through

$$\psi(\mathbf{p}) = [\tilde{Q}_\beta(W, p^2)]^{-1} S(\mathbf{p}). \tag{22}$$

To investigate (21) in coordinate space, let $u(\mathbf{r})$ be the fourier transform (FT) of $\psi(\mathbf{p})$,

i.e.,

$$u(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{p} \exp(-i\mathbf{p}\cdot\mathbf{r})\psi(\mathbf{p}) \quad (23)$$

and

$$\psi(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{r} \exp(i\mathbf{p}\cdot\mathbf{r})u(\mathbf{r}). \quad (24)$$

We now take the FT of (21). The FT of the term on the left hand side is trivial to calculate. To calculate the FT on the right hand side, we note that the functions $\exp(-mr/r)$ and $(4\pi)(2\pi)^{-3/2}(\mathbf{p}^2 + \mathbf{m}^2)^{-1}$ form a FT pair, where $r = |\mathbf{r}|$. For the case of the Coulomb potential, we simply put $m = 0$. Thus,

$$\begin{aligned} T_1 &\rightarrow \frac{e^2 D_1}{4\pi^2} \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{q} \psi(\mathbf{q}) \cdot \int \frac{\exp(-i\mathbf{p}\cdot\mathbf{r}) d^3\mathbf{p}}{(\mathbf{q} - \mathbf{p})^2} \\ &= \frac{e^2 D_1}{4\pi^2} \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{q} \exp(i\mathbf{q}\cdot\mathbf{r})\psi(\mathbf{q}) \cdot \int \frac{\exp(-i\mathbf{p}'\cdot\mathbf{r}) d^3\mathbf{p}'}{p'^2} \\ &= \frac{e^2 D_1}{4\pi^2} \frac{2\pi^2}{r} \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{q} \exp(-i\mathbf{q}\cdot\mathbf{r})\psi(\mathbf{q}) \\ &= \frac{e^2 D_1}{2r} u(\mathbf{r}); \\ T_2 &\rightarrow \frac{e^2 D_2 D_3}{4\pi^2} \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{q} \psi(\mathbf{q})}{(\mathbf{q}^2 + D_2)} \int \frac{\exp(-i\mathbf{p}\cdot\mathbf{r})}{(\mathbf{q} - \mathbf{p})^2} d^3\mathbf{p} \\ &= \frac{e^2 D_2 D_3}{4\pi^2} \frac{2\pi^2}{r} \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{q} \exp(-i\mathbf{q}\cdot\mathbf{r})\psi(\mathbf{q})}{(\mathbf{q}^2 + D_2)} \\ &= \frac{e^2 D_2 D_3}{2r} \frac{1}{4\pi} \int d^3\mathbf{r}' u(\mathbf{r} - \mathbf{r}') \exp(-\sqrt{D_2} r')/r', \end{aligned}$$

where in the last step we have used the convolution theorem for FTs. In coordinate space, (21) may therefore be written as

$$\begin{aligned} \left(W + \frac{1}{2\mu} \nabla^2 \right) u(\mathbf{r}) &= \frac{e^2 D_1}{2r} u(\mathbf{r}) + \frac{e^2 D_2 D_3}{8\pi r} \\ &\quad \times \int d^3\mathbf{r}' [\exp(-\sqrt{D_2} r')/r'] u(\mathbf{r} - \mathbf{r}'). \end{aligned} \quad (25)$$

Equation (25) shows how, for the bound state problem, the Coulomb potential gets modified by temperature. In addition to an attractive Coulomb term whose coupling is $(e^2 D_1/2)$ —note that D_1 is always negative, we now have another $(1/r)$ -type term. The latter term involves an integration of a Yukawa-like term over the space of the wavefunction. The characteristic energy of the associated Yukawa particle, a phonon, is given by $(D_2)^{1/2}$. It may be noted that the $T \rightarrow 0$ limit of (25) yields the equation for the normal hydrogen; this is either fortuitous (because the equation was obtained *after* the approximation in (16) had been made), or it tells us that $p^2 \rightarrow 0$ faster than $T \rightarrow 0$.

Further insight into the structure of (25) can be obtained by directly solving it. This will not be attempted here. An alternative method consists of using the information obtained from the solutions of the corresponding equation in momentum space. The details of such solutions have been reported in MPVI. We note here that:

- (i) The sign of $D_2 D_3$ as well as the overall sign of the second term on the right side of (25) is always positive. This term therefore causes repulsion.
- (ii) The system has an infinite number of energy levels for each value of the angular momentum, l . Let these levels be labelled as E_{nl} . It is seen that—refer to table 1, which corresponds to $T = 1.6 \times 10^6$ K and is reproduced from MPVI—the energy levels become degenerate for large n and/or large l . We then have the constraint $n \geq l$ and essentially a l/n^2 -pattern for the eigenvalues. For such a situation, (25) may be written as

$$\left(W - \frac{p^2}{2\mu}\right)u(\mathbf{r}) = (-e_{\text{eff}}^2/r)u(\mathbf{r}), \tag{26}$$

where

$$-e_{\text{eff}}^2 = e^2\left(\frac{1}{2}D_1 + \frac{1}{2}D_2 D_3 X\right), \tag{27}$$

which follows by writing the integral on the right side of (25) as

$$\int d^3r' [\exp(-D_2 r')/r']u(\mathbf{r} - \mathbf{r}') \simeq 4\pi X u(\mathbf{r}). \tag{28}$$

The energy levels $E_{nl} \equiv E_n$ of (26) are given by

$$E_n = -\frac{1}{2}\mu(e_{\text{eff}}^4/n^2).$$

For $E_{15,0} = 13.893$,

$$e_{\text{eff}}^2 \simeq 0.1106396. \tag{29}$$

Table 1. Energy eigenvalues, E_{nl} , of equation (10) at 1.6×10^6 K (for details, MPVI).

n	E_{n0} (eV)	E_{n1}	E_{n2}	E_{n3}
1	4333.555			
2	1178.483	1244.684		
3	432.737	438.005	439.645	
4	211.778	212.143	212.240	212.282
5	129.295	129.341	129.352	129.357
6	88.233	88.242	88.244	88.245
7	64.336	64.338	64.339	64.339
8	49.076	49.077	49.077	49.077
9	38.700	38.700	38.701	38.701
10	31.312	31.312	31.312	31.312
11	25.860	25.860	25.860	25.860
12	21.720	21.720	21.720	21.720
13	18.502	18.502	18.502	18.502
14	15.950	15.950	15.950	15.950
15	13.893	13.893	13.893	13.893

Table 2. D_1, D_2, D_3 [see equation (25)] and X [see equation (28)] etc. for the degenerate eigenvalues of table 1.

n	E_n (MeV) ($\times 10^{-6}$)	D_1	$D_2(\text{MeV}^2)$	D_3	X (cm^2)	$D_2 D_3 X$	$M = \sqrt{D_2}$ (MeV)
8	49.076	-5191.355	1.462	5191.179	0.680	5160.863	1.209
9	38.700	-6570.283	1.851	6570.143	0.538	6542.798	1.360
10	31.312	-8111.400	2.285	8111.287	0.436	8080.951	1.512
11	25.860	-9814.830	2.765	9814.736	0.361	9796.726	1.663
12	21.720	-11680.581	3.290	11680.503	0.303	11643.943	1.814
13	18.502	-13708.267	3.861	13708.200	0.258	13655.259	1.965
14	15.950	-15898.526	4.478	15898.468	0.223	15876.115	2.116
15	13.893	-18250.008	5.140	18249.958	0.194	18198.128	2.267

For any degenerate eigenvalue of table 1, we may now calculate the quantity X occurring in (27). Our results are given in table 2. We also give, in this table, the values of the terms D_1 and $D_2 D_3 X$. These values show that both the terms on the right side of (25) are equally important.

Unfortunately, at this stage, we are not able to give an estimate of the integral in (25) for energy levels which do not follow an effective Coulomb pattern. This aspect and the other ramifications of temperature-dependent approach to explain the discrete spectra of hot plasmas are currently under investigation.

References

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