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On the perfect penrose tiling and its basic building blocks

V C SAHNI and M K SANYAL
Nuclear Physics Division, Bhabha Atomic Research Centre, Bombay 400085, India

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Abstract. We show that the basic building blocks of a perfect Penrose pattern (PPT) in two dimensions can be established by adding another condition to the Penrose's original edge rules. The implications of this result are discussed in the context of recent papers by Onada, Jaric, Ronchetti and others concerning growth algorithm for PPT's.

Keywords. Quasicrystals; Penrose-tiling.

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1. Introduction

Since the discovery of quasicrystals (Shechtman *et al* 1984; Levine and Steinhardt 1986), there has been a discussion on whether or not purely "local" growth rules can lead to perfect quasi-periodic structures or certain "global" rules are essential (Henley 1987). This issue is of importance since questions about the physical realization of such structures could be raised if global rules were found essential. In a recent paper Onada *et al* (1988) have examined this issue in the context of (arrowed rhombus) Penrose pattern (Penrose 1974) in two dimensions and claimed to have found a growth algorithm "for aggregation of Penrose tiles to form an infinite defect-free perfect Penrose tiling (PPT)" by use of *local rules alone*. However, their claim has been contested by Jaric and Ronchetti (1988) who have pointed out that in the growth rules of Onada *et al* *nonlocality is certainly implied*. Specifically, since in the procedure of Onada *et al* attaching a tile at a site requires an *examination of the entire surface* Jaric and Ronchetti assert that their growth algorithm in ultimately (and inevitably) non-local.

Interestingly, in their algorithm, Onada *et al* (1988) have imposed the condition that in the growth of any cluster only a restricted set of "eight vertex configurations" are allowed to occur. There is, however, no discussion regarding why only these eight configurations arise.

Here we will show that if on placement of Penrose's rhombi we add another condition to the Penrose's original "matching edge rules" which ensures indefinite continuity, then we naturally end up with the eight configurations of Onada *et al* (1988). The added condition on the placement of Penrose's rhombi around a point is as follows. When we fit all the rhombi around a point O, none of the terminal points P_1, P_2,... etc. of all the bonds OP_1, OP_2,... etc. (connected to O) should end up as a "dead end". (By a "dead end" we mean a site where one cannot affix a rhombus in a defect-free manner so further growth at that site is arrested.) In other words, we exclude configurations involving even a single "dead end" amongst P_1, P_2,.... The proof of our proposition is presented in § 2 and the conclusions in § 3.

2. Proof

To prove our proposition we first consider the two basic building blocks viz. the two Penrose rhombi illustrated in figure 1. The single arrowed edges and the double arrowed edges will be designated by X and Y respectively. According to Penrose's original rules when two rhombi are placed together, their common edge has to be of the same type and so also their arrow directions (de Bruijn 1981).

To explore all possible ways of placing the rhombi together around a point, we observe that the internal angles of the rhombi are α, 2α, 3α and 4α (where $\alpha = 36°$). Since the total angle around a point is 360° our problem reduces to seeking partitions of 10α in terms of these four angles (α, 2α, 3α and 4α) i.e. of 10 in terms of 1, 2, 3 and 4. Table 1 lists all these partitions in terms of the number of terms in them. One can easily see that we need a minimum of three terms (because the highest number occurring is 4) and a maximum of ten. We will presently examine permutations of different terms in a given partition and study the implications. To proceed further, we adopt the convention that the angles will always be defined going anticlockwise. The two bonds defining any angle will be specified using the letters X, Y, and the arrow directions on the bonds will be specified using the symbols $>$ and $<$. (By $>$ we mean arrow points away and by $<$ we mean arrow points toward the vertex in the angle). With these conventions, the bonds associated with different angles are as in table 2.

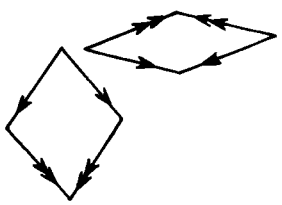

Figure 1. Fat and skinny rhombic tiles.

Table 1. Partitions of 10 in terms of 1, 2, 3, 4.

Total no. of terms in the partitions	Actual partitions		
3	(4, 4, 2)	(4, 3, 3)	
4	(4, 4, 1, 1)	(4, 3, 2, 1)	(4, 2, 2, 2)
	(3, 3, 2, 2)	(3, 3, 3, 1)	
5	(4, 3, 1, 1, 1)	(4, 2, 2, 1, 1)	(2, 2, 2, 2, 2)
	(3, 2, 2, 2, 1)	(3, 3, 2, 1, 1)	
6	(4, 2, 1, 1, 1, 1)	(3, 3, 1, 1, 1, 1)	(3, 2, 2, 1, 1, 1)
	(2, 2, 2, 2, 1, 1)		
7	(2, 2, 2, 1, 1, 1, 1)	(3, 2, 1, 1, 1, 1, 1)	
	(4, 1, 1, 1, 1, 1, 1)		
8	(2, 2, 1, 1, 1, 1, 1, 1)	(3, 1, 1, 1, 1, 1, 1, 1)	
9	(2, 1, 1, 1, 1, 1, 1, 1, 1)		
10	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)		

Table 2. Different angles and their associated bonds (for explanation of the symbols, see text).

Angle	Bonds	Short form
α	$(Y>, X>)$ or $(X>, Y>)$	(11, 21) or (21, 11)
2α	$(Y<, Y<)$ or $(X>, X>)$	(10, 10) or (21, 21)
3α	$(Y>, X<)$ or $(X<, Y>)$	(11, 20) or (20, 11)
4α	$(Y<, Y<)$ or $(X<, X<)$	(10, 10) or (20, 20)

$(Y< \equiv 10;\ Y> \equiv 11;\ X< \equiv 20;\ X> \equiv 21)$.

Using table 2, one can check out whether any two angles can adjoin or not and also the bond sequence to permit it. Thus it is easy to see that the following pairs of angles are not adjoinable (α, 4α); (4α, α); (2α, 3α) and (3α, 2α). In all other cases ($m\alpha$, $n\alpha$) combination is admissible if the bond sequence is appropriate.

Let us next take each partition listed in table 1. We will explore whether or not a consistent set of bond sequence exists for that partition. For example, for (4, 4, 2) a consistent bond sequence {(10, 10), (10, 10), (10, 10)} exists in that second bond of an angle is compatible with the first bond of its succeeding angle etc. The configuration associated with the (4, 4, 2) is displayed in figure 2(a). Any permutation of this partition actually will lead to the same figure (albeit rotated as a whole). For (4, 3, 3) a consistent

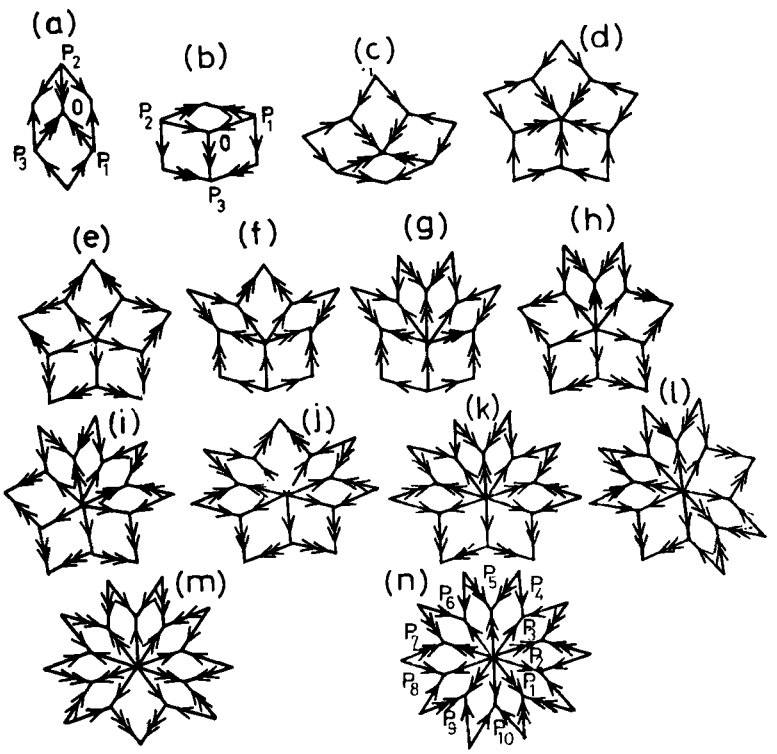

Figure 2. All possible vertex configurations allowed by matching rules (for details refer text).

bond sequence {(20, 20), (20, 11), (11, 20)} exists and the corresponding configuration is displayed in figure 2(b).

Turning to partitions with four terms, since (4α, α) and (α, 4α) are incompatible, the partition (4, 4, 1, 1) does not have any consistent bond sequence. One can verify that (4, 3, 2, 1) (or its permutations) also has no consistent bond sequence. Actually since (α, 4α) and (2α, 3α) pairs are incompatible the only combinations one need to pay attention to are (1, 2, 4, 3) [≡(4, 3, 1, 2)] and (4, 2, 1, 3) [≡(1, 3, 4, 2)]. And for none of these consistent bond sequence exists. For (4, 2, 2, 2) the consistent sequence {(10, 10), (10, 10), (10, 10), (10, 10)} exists and the corresponding configuration is shown in figure 2(c). (3, 3, 2, 2) is obviously ruled out and (3, 3, 3, 1) also can be seen not to allow the admissible bond sequence. In the five term partitions (4, 3, 1, 1, 1) and (3, 2, 2, 2, 1) are prima facie ruled out due to (α, 4α) and (2α, 3α) incompatibility. Amongst others, (4, 2, 2, 1, 1) is not allowed, but (2, 2, 2, 2, 2) actually allows two sets of bond sequence {(10, 10), (10, 10), (10, 10), (10, 10), (10, 10)} and {(21, 21), (21, 21), (21, 21), (21, 21), (21, 21)}. The corresponding configurations are shown in figures 2(d) and 2(e). Also (3, 3, 2, 1, 1) for the permutation (3, 3, 1, 2, 1) allows the bond sequence {(11, 20), (20, 11), (11, 21), (21, 21), (21, 11)}. The corresponding configuration is shown in figure 2(f).

We now turn to six term partitions. The (α, 4α) incompatibility rules out (4, 2, 1, 1, 1, 1). One can also verify that (3, 2, 2, 1, 1, 1) (and its permutations also) do not allow a consistent bond sequence. The other two partitions (3, 3, 1, 1, 1, 1) and (2, 2, 2, 2, 1, 1) allow the bond sequence {(11, 20), (20, 11), (11, 21), (21, 11), (11, 21), (21, 11)} and {(21, 21), (21, 21), (21, 21), (21, 21), (21, 11), (11, 21)} respectively. The corresponding configurations are shown in figures 2(g) and 2(h) respectively. From seven term partitions only (2, 2, 2, 1, 1, 1, 1) allows two consistent bond sequences for the two permutations: {(21, 21), (21, 21), (21, 21), (21, 11), (11, 21), (21, 11), (11, 21)} for (2, 2, 2, 1, 1, 1, 1) and {(21, 21), (21, 21), (21, 11), (11, 21), (21, 21), (21, 11), (11, 21)} for (2, 2, 1, 1, 2, 1, 1). The corresponding configurations are shown in figures 2(i) and 2(j). From the remaining partitions (2, 2, 1, 1, 1, 1, 1) and its permutation (2, 1, 1, 2, 1, 1, 1, 1) admit the configurations shown in figures 2(k) and 2(l) respectively. The partitions (2, 1, 1, 1, 1, 1, 1, 1, 1) and (1, 1, 1, 1, 1, 1, 1, 1, 1, 1) admit configurations shown in figures 2(m) and 2(n).

Figures (2a through 2n) admit some consistent set of bond sequence or the other around the point O. We will now examine if there is any dead end at any of the terminal points P_1, P_2,... etc. of the bonds OP_1, OP_2,... etc. in these configurations. For this we observe that when two angles $m\alpha$, $n\alpha$ adjoint at O, then at the terminal point P_i of their common bond, the adjoining angles are $(5-n)\alpha$ and $(5-m)\alpha$. Further the arrow direction with respect to P_i is opposite to that with respect to O.

Let us now examine the terminal points P_i in figure 2(n). Since here $m = n = 1$, the adjoining angles $(5-n)\alpha$ and $(5-m)\alpha$ at the terminal points P_i are 4α and 4α. By virtue of the already established result that only (4, 4, 2) is an admissible configuration, it follows that at each of the P_i the two skiny rhombi (involving 144°) must adjoin with 2α ($= 72°$) of the fat rhombus. It is easy to verify, however, that if there is a succession of angle α three times and the corresponding terminal points are P_1, P_2, P_3 and P_4, then a "dead end" is inevitable. This is because with respect to O the two common bonds OP_2 and OP_3 have to be labelled as 21 and 11 (or vice versa). However, with respect to P_2 and P_3 their labels would be 20 and 10 respectively. A reference to table 2 then clearly shows that whereas at P_3 the 2α vertex of a fat rhombus can be fitted, at P_2 the same cannot be fitted consistent with the required bond sequence for 2α. Thus P_2 is a "dead end". One can easily use this rule, viz. "the inevitability of a dead end whenever there is a

succession of angle α three times", and discard the configuration of figures 2(g), 2(i), 2(k), 2(l), 2(m) and 2(n). For all the remaining eight figures none of the $\{P_i\}$ involves a "dead end". These are the "eight vertex configurations" of Onada *et al* (1988).

3. Conclusions

The growth algorithm of Onada *et al* (1988) involves the following two steps. On the surface of a cluster first they identify whether there are any "forced vertices". (By a forced vertex one means a site where an additional rhombus can be attached in only one way.) If there are forced vertices then first affix rhombi at those sites. After all of these have been exhausted then take any of the remaining sites and continue the growth by affixing the fat (3α) rhombus making sure that one always stays within the eight configurations discussed above. The identification of forced vertices is quite simple if one uses the information on partitions listed in table 1. However, as mentioned in the introduction, Jaric and Ronchetti (1988) have pointed out that the examination *of entire surface* before attaching an unforced tile makes Onada *et al*'s algorithm intrinsically non-local!

In conclusion we would like to point out that extension of our ideas to other quasi periodic structures in two dimensions (i.e. involving other than five fold rotation, for example, twelve-fold rotation) is straightforward. The work on extending these to three dimensions is contemplated.

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