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Renormalization of a gauge theory in a nonlinear gauge

SATISH D JOGLEKAR

Department of Physics, Indian Institute of Technology, Kanpur 208016, India

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Abstract. We discuss renormalization of an O(3) gauge model with the gauge fixing term given by $\mathscr{L}_{\text{g.f.}} = -1/\zeta|(\partial_\mu - igA_\mu^3)W^{+\mu}|^2 - (1/2\alpha)(\partial A^3)^2$. We utilize earlier results on the general theory of renormalization of gauge theories in quadratic gauges to prove multiplicative renormalizability of the theory together with a subtractive renormalization of gauge fixing and ghost terms. We show that this model has a double BRS invariance and that it is preserved under renormalization.

Keywords. Renormalization; gauge theory; nonlinear gauge.

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1. Introduction

Calculations in spontaneously broken theories are generally done in linear gauges Feynman rules and the renormalization procedure (Abers and Lee 1973; Fujikawa *et al* 1973; Zinn-Justin 1974; Lee 1975) in these gauges are simpler than in more complicated nonlinear gauges. As such nonlinear gauges have been more or less only of an academic interest (Joglekar 1974; Das 1981, 1982).

Recently, however, many calculations in spontaneously broken theories have been performed (Deshpande and Nazerimonford 1983) in a particular nonlinear R_ξ gauge. In the context of an SU(2) × U(1) model described by the Lagrange density, (in the notations of Fugikawa *et al* 1973, Appendix C)

$$\mathscr{L}_0 = \left|\partial_\mu s^+ + iM_w W_\mu^+ + \frac{i}{\sqrt{2}} g W_\mu^+ s^0 + i\left(-eA_\mu + \frac{G\cos 2\theta}{2} Z_\mu\right)s^+\right|^2 + \left|\partial_\mu s^0 - i\frac{M_z}{\sqrt{2}} Z_\mu - i\frac{GZ_\mu}{2} s^0 + \frac{i}{\sqrt{2}} g W_\mu^- s^+\right|^2,$$

where

$$\phi = \begin{pmatrix} s^+ \\ \dfrac{v}{\sqrt{2}} + s^0 \end{pmatrix} \equiv \begin{pmatrix} s^+ \\ \dfrac{v + \eta + i\chi}{\sqrt{2}} \end{pmatrix}$$

is a complex doublet. In these notations $\mathscr{L}_{\text{g.f.}}$ in the nonlinear R_ξ gauges is given by

$$-\mathscr{L}_{\text{g.f.}} = \frac{1}{2\alpha}(\partial\cdot A)^2 + \frac{1}{2\eta}(\partial_\mu Z^\mu + \eta M_z \chi)^2 + \frac{1}{\xi}\left|(\partial_\mu - igA_\mu^3) W^{+\mu} - i\xi M_w s^+\right|^2.$$

This choice for the gauge fixing has been made because it simplifies the Feynman rules of such theories and has in addition simpler electromagnetic WT identities. As such, renormalization of gauge theories in such gauges becomes an interesting problem.

In an earlier paper (Joglekar 1988), we had developed the general theory of renormalization in nonlinear gauges. In this work we shall apply the results of this theory to the discussion of renormalization of an unbroken O(3) model in a nonlinear R_ξ-like gauge. This model in this gauge has an interesting property that it has a double BRS invariance as shown in § 2. We shall show that the theory can be renormalized to preserve the double BRS invariance and that the coupling constant g that appears in the gauge fixing term is renormalized the same way as the coupling constant that appears in the invariant action.

2. The model

We shall consider the O(3) Yang-Mills theory given by the Lagrange density

$$\mathscr{L}_0[A] = -\tfrac{1}{4}F^\alpha_{\mu\nu}(x)F^{\alpha\mu\nu}(x) \tag{1}$$

with

$$F^\alpha_{\mu\nu}(x) = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + g f^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu,$$
$$f^{\alpha\beta\gamma} \equiv \varepsilon_{\alpha\beta\gamma}. \tag{2}$$

We shall quantize the theory in a gauge whose gauge-fixing term is given by

$$-\mathscr{L}_{\text{g.f.}} = \frac{1}{2\alpha}(\partial^\mu A^3_\mu)^2 + \frac{1}{\xi}\left|(\partial^\mu - igA^{3\mu})W^+_\mu\right|^2$$
$$\equiv \sum_\alpha \tfrac{1}{2} f_\alpha[A]^2, \tag{3}$$

where

$$W^\pm = (A_1 \pm iA_2)/\sqrt{2}.$$

$\mathscr{L}_0[A]$ of (1) is invariant under the gauge transformations

$$\delta A^\alpha_\mu = \partial_\mu \Lambda^\alpha - g f^{\alpha\beta\gamma}A^\beta_\mu \Lambda^\gamma,$$

i.e.

$$\delta W^\pm_\mu = \partial_\mu \Lambda^\pm \mp igA^3_\mu \Lambda^\pm \pm ig\Lambda^3 W^\pm_\mu,$$
$$\delta A^3_\mu = \partial_\mu \Lambda^3 - ig(W^+\Lambda^- - W^-\Lambda^+),$$
$$\Lambda^\pm = (\Lambda^1 \pm i\Lambda^2)/\sqrt{2}. \tag{4}$$

With the help of this form for the gauge transformations, one can write the ghost Lagrangian viz.:

$$\mathscr{L}_g = \bar{c}_+\{(\partial_\mu - igA^3_\mu)[(\partial_\mu - igA^3_\mu)c_+ + igW^+_\mu c_3]$$
$$- igW^+_\mu[\partial_\mu c_3 - ig(W^+_\mu c_- - W^-_\mu c_+)]\} + \text{h.c.}$$
$$+ \bar{c}_3\{\partial^2 c_3 - ig\partial_\mu(W^+_\mu c_- - W^-_\mu c_+)\} \tag{5}$$

with

$$c_+ = \frac{c_1 \pm ic_2}{\sqrt{2}}, \quad \bar{c}_\pm = \frac{\bar{c}_1 \pm i\bar{c}_2}{\sqrt{2}}.$$

The total effective action is

$$\mathscr{L}_{\text{eff}}[A,c,\bar{c}] = \mathscr{L}_0[A] + \mathscr{L}_{\text{g.f.}}[A] + \mathscr{L}_{\text{g}}[A,c,\bar{c}]. \tag{6}$$

This effective action has an extra invariance. Omitting the term $-\frac{1}{2}(\partial^\mu A_\mu^3)^2$ in the $\mathscr{L}_{\text{g.f.}}[A]$, but keeping the rest of the terms in $\mathscr{L}_{\text{g.f.}}$ and $\mathscr{L}_{\text{g.}}$, this $\mathscr{L}_{\text{eff}}$ is invariant under the local "electromagnetic" gauge transformations viz:

$$\begin{aligned}
&A_\mu^3 \to A_\mu^{3\prime} = A_\mu^3 + \partial_\mu \alpha(x \\
&W_\mu^\pm \to W_\mu^{\pm\prime} = \exp[\pm i\alpha(x)] W_\mu^\pm, \\
&c_\pm \to c'_\pm = \exp[\pm i\alpha(x)] c_\pm, \\
&\bar{c}_\pm \to \bar{c}'_\pm = \exp[\mp i\alpha(x)] \bar{c}_\pm, \\
&c_3 \to c'_3 = c_3, \\
&\bar{c}_3 \to \bar{c}'_3 = \bar{c}_3,
\end{aligned} \tag{7}$$

as can be seen from the form of $\mathscr{L}_{\text{g.f.}}$ and $\mathscr{L}_{\text{g}}$ of (3) and (5).

We shall express the two invariances differently. We add a pair of new ghost and antighost free fields C and $\bar{C}$ to the action. They do not couple to any other fields. Their use is in being able to express the two invariances as the BRS invariances of the single effective Lagrangian,

$$\mathscr{L}_{\text{eff}}[A,c,\bar{c},C,\bar{C}] = \mathscr{L}_{\text{eff}}[A,c,\bar{c}] + \bar{C}\partial^2 C. \tag{8}$$

Then $\mathscr{L}_{\text{eff}}$ of (8) is invariant under the following two BRS transformations:

(i)
$$\begin{aligned}
&\delta A_\mu^\alpha = D_\mu^{\alpha\beta} c_\beta \lambda, \\
&\delta c_\alpha = -\tfrac{1}{2} g f^{\alpha\beta\gamma} c_\beta c_\gamma \lambda, \\
&\delta \bar{c}_\alpha = -\eta^{-1/2} f_\alpha[A]\lambda \quad \eta = \xi \quad \text{for} \quad \alpha = 1,2 \\
&\qquad\qquad\qquad\qquad\qquad\quad = \alpha \qquad\qquad\quad = 3, \\
&\delta C = \delta \bar{C} = 0.
\end{aligned} \tag{9}$$

(ii)
$$\begin{aligned}
&\delta A_\mu^3 = \partial_\mu C \lambda, \\
&\delta W_\mu^\pm = \pm igCW_\mu^\pm \lambda, \\
&\delta c_\pm = \pm igC\lambda c_\pm, \\
&\delta \bar{c}_\pm = \mp igC\lambda \bar{c}_\pm, \\
&\delta c_3 = 0 = \delta \bar{c}_3 = \delta C, \\
&\delta \bar{C} = -\frac{1}{\alpha}\partial^\mu A_\mu^3 \lambda.
\end{aligned} \tag{10}$$

Our aim is to renormalize the theory preserving both kinds of BRS invariances. We would also like to see if the coupling constant g that appears in gauge-fixing term is renormalized the same way as that appearing in the gauge-invariant Lagrangian.

3. Results on the renormalization in quadratic gauges

In Joglekar (1988) we discussed the renormalization of gauge theories in quadratic gauges when the gauge fixing term of the form:

$$\mathscr{L}_{\text{g.f.}} = -\sum_{\alpha} \frac{1}{2\eta_\alpha}(\partial^\mu A^\alpha_\mu + \zeta^\alpha_{\beta\gamma} A^\beta_\mu A^{\gamma\mu})^2, \tag{11}$$

where the term $\zeta^\alpha_{\beta\gamma} A^\beta_\mu A^{\gamma\mu}$ preserves the global invariance of the theory. It was shown there that the gauge fixing and the ghost terms are subtractively renormalized; so that it is convenient to start from the beginning with

$$\mathscr{L}'_{\text{g.f.}} = -\sum_{\alpha} \frac{1}{2\eta_\alpha}(\partial^\mu A^\alpha_\mu + \zeta^\alpha_{\beta\gamma} A^\beta_\mu A^{\gamma\mu} + \tau^{\alpha\beta}_\gamma \bar{c}_\beta c_\gamma)^2 = -\sum_\alpha f'^2_\alpha[A, c, \bar{c}] \tag{12}$$

with $\tau^{\alpha\beta}_\gamma = -\tau^{\beta\alpha}_\gamma \cdot \tau^{\alpha\beta}_\gamma$ depends on the loop expansion parameter a and are zero in the tree approximation.

Then the ghost term must be modified as follows:

$$\mathscr{L}'_{\text{g}} = \mathscr{L}_{\text{g}} - \tfrac{1}{4} g f^{\alpha\beta\gamma} c_\beta c_\gamma \tau^{\kappa\sigma}_\alpha \bar{c}_\kappa \bar{c}_\sigma. \tag{13}$$

Let

$$S'_{\text{eff}} = \int [\mathscr{L}_0[A] + \mathscr{L}'_{\text{g.f.}} + \mathscr{L}'_{\text{g}}] \, \mathrm{d}^n x. \tag{14}$$

The structure of counterterms was established in Joglekar (1988). To state this structure, it is necessary to introduce sources for composite operators that enter BRS transformations. We consider,

$$S'_{\text{eff}}[A, c, \bar{c}, K, L, R] = S_0[A] - \sum_{\alpha=1}^{3} \tfrac{1}{2}(f'_\alpha - \eta_\alpha^{-1/2} R_\alpha)^2 + S'_g + K_i D^\alpha_i c_\alpha + \tfrac{1}{2} g f^{\alpha\beta\gamma} c_\beta c_\gamma L_\alpha. \tag{15}$$

The structure of the counterterms is expressed conveniently in terms of the nilpotent operator

$$\mathscr{G} = \frac{\delta S}{\delta A_i} \frac{\delta}{\delta K_i} + \frac{\delta S}{\delta c_\alpha} \frac{\delta}{\delta L_\alpha} + \frac{\delta S}{\delta \bar{c}_\alpha} \frac{\delta}{\delta R_\alpha} \tag{16}$$

and is given in terms of the divergent part of the generating functional for proper vertices in any given loop approximation as,

$$\{\Gamma\}^{\text{div}} = \alpha_1(\varepsilon) S_0[A] + \mathscr{G} X[A, c, \bar{c}, K, L, R; \varepsilon], \tag{17}$$

where X is a Lorentz invariant local functional of dimension 3 and ghost number (Joglekar and Lee 1976) minus one. From this it follows that X has the most general structure given below:

$$X = \sum_{i=1}^{3} \alpha_2^{(i)}(\varepsilon) K_i A_i + \sum_{\beta=1}^{3} \alpha_3^{(\beta)}(\varepsilon) L_\beta c_\beta + \sum_{\beta=1}^{3} \alpha_4^\beta(\varepsilon) R_\beta \bar{c}_\beta + \sum_{\beta=1}^{3} \alpha_5^\beta(\varepsilon) \partial^\beta_i \bar{c}_\beta A_i + \sum_{(\alpha ij)} \alpha_6^{(\alpha ij)}(\varepsilon) \tfrac{1}{2} \bar{c}_\alpha \lambda^\alpha_{ij} A_i A_j + \sum_{\beta\gamma\delta} \alpha_7^{(\beta\gamma\delta)}(\varepsilon) \tfrac{1}{2} \rho^{\beta\gamma}_\delta \bar{c}_\beta \bar{c}_\gamma. \tag{18}$$

Then in terms of S'_{eff}, the following counterterms are needed

(i) $$S_0[A] \equiv \int \mathscr{L}_0[A]\,\mathrm{d}^n x,$$

(ii) $$\int \frac{\delta S'_{\mathrm{eff}}}{\delta A^\alpha_\mu(x)} A^\alpha_\mu(x)\,\mathrm{d}^n x\ \alpha = 1,2,3\ \alpha \text{ not summed},$$

(iii) $$\int \frac{\delta S'_{\mathrm{eff}}}{\delta c_\alpha(x)} c_\alpha(x)\,\mathrm{d}^n x\ \alpha = 1,2,3\ \alpha \text{ not summed},$$

(iv) $$\int \mathrm{d}^n x \{\partial^\mu \bar{c}_\alpha D^{\alpha\beta}_\mu c_\beta + \eta_\alpha^{-1/2} f'_\alpha \partial\cdot A^\alpha\}\ \alpha \text{ not summed},$$

(v) $$\int [\tfrac{1}{2}\eta_\alpha^{-1/2} f'_\alpha \lambda^\alpha_{\beta\gamma} A^\beta_\mu A^{\gamma\mu} - \bar{c}_\alpha \lambda^\alpha_{\beta\gamma} A^\gamma_\mu D^{\beta\delta}_\mu c_\beta]\,\mathrm{d}^n x\ \alpha \text{ not summed},$$

(vi) $$\int \mathrm{d}^n x [\eta_\alpha^{-1/2} f'_\alpha \rho^{\alpha\beta}_\gamma \bar{c}_\beta c_\gamma + \tfrac{1}{4} g \rho^{\lambda\beta}_\alpha \bar{c}_\lambda \bar{c}_\beta f^{\alpha\delta\varepsilon} c_\delta c_\varepsilon]\ \alpha \text{ not summed}. \tag{19}$$

Further, our action has extra U(1) local invariance of (7). This has certain consequences on the structure of divergences. These will be derived in the next section in the form of an extra WT identity. This WT identity must be imposed on the structure of counterterms in (19) above and only those counterterms that satisfy this WT identity can be allowed counterterms. This, as we shall see restricts the counterterms such that the g in $\mathscr{L}_{\mathrm{g.f.}}$ is renormalized the same way as that in $\mathscr{L}_0[A]$.

4. Consequences of the U(1) invariance

To obtain the consequences of U(1) invariance of (7) in the form of the WT identity for the generating functional for proper vertices, we introduce the generating functionals and expectation values as usual

$$W[J,\xi,\bar{\xi}] = \int [\mathrm{d}A\,\mathrm{d}c\,\mathrm{d}\bar{c}] \exp i \int \{\mathscr{L}'_{\mathrm{eff}}[A,c,\bar{c}] + J^\alpha_\mu(x) A^{\alpha\mu}(x) + \bar{\xi}_\alpha c_\alpha + \bar{c}_\alpha \xi_\alpha\}\,\mathrm{d}^n x, \tag{20}$$

$$Z = -i \ln W, \tag{21}$$

$$\langle A^\alpha_\mu \rangle = \delta Z/\delta J^\alpha_\mu;\quad \langle c_\alpha \rangle = \delta Z/\delta \bar{\xi}_\alpha;\quad \langle \bar{c}_\alpha \rangle = -\delta Z/\delta \xi_\alpha. \tag{22}$$

Dropping brackets $\langle\ \ \rangle$ around expectation values,

$$\Gamma[A,c,\bar{c}] = Z - \int [J^\alpha_\mu A^{\alpha\mu} + \bar{\xi}_\alpha c_\alpha + \bar{c}_\alpha \xi_\alpha]\,\mathrm{d}^n x. \tag{23}$$

Now we consider transformations of (7) on the integration variables of (20) and, equating the change to zero, obtain:

$$0 = \int [\mathrm{d}A\,\mathrm{d}c\,\mathrm{d}\bar{c}] \int \mathrm{d}^n x [-\partial^\mu J^3_\mu + iJ^-_\mu W^{+\mu} - iJ^+_\mu W^{-\mu} + i\bar{\xi}_- c_+ - i\xi_+ \bar{c} + i\bar{c}_- \xi_+ - i\bar{c}_+ \xi_- - \frac{1}{\alpha}\partial^2(\partial\cdot A^3)] \exp i\{\cdots\},$$

$$J^\pm \equiv (J^1 \pm iJ^2)/\sqrt{2}.$$

Translating in terms of Γ by the use of

$$\delta\Gamma/\delta A_\mu^\alpha = -J_\mu^\alpha; \quad \delta\Gamma/\delta\bar{c}_\alpha = -\xi_\alpha; \quad \delta\Gamma/\delta c_\alpha = \bar{\xi}_\alpha \tag{24}$$

and using

$$\delta\Gamma_0/\delta A_\mu^3 \equiv \frac{\delta}{\delta A_\mu^3}\left[\Gamma + \frac{1}{2\alpha}\int \mathrm{d}^n x(\partial\cdot A^3)^2\right] = -J_\mu^3 - \frac{1}{\alpha}\partial^2 A_\mu^3, \tag{25}$$

we obtain,

$$\int \mathrm{d}^n x\left\{\partial^\mu \frac{\delta\Gamma_0}{\delta A_\mu^3} - iW_\mu^+\frac{\delta\Gamma_0}{\delta W_\mu^+} + iW_\mu^-\frac{\delta\Gamma_0}{\delta W_\mu^-} + i\frac{\delta\Gamma_0}{\delta c_+}c_+ \right.$$
$$\left. - i\frac{\delta\Gamma_0}{\delta c_-}c_- - i\bar{c}_-\frac{\delta\Gamma_0}{\delta\bar{c}_-}\right\} \equiv \mathscr{G}_1\Gamma_0 = 0. \tag{26}$$

Now suppose that the theory has been renormalized upto $(n-1)$ loops, then the divergence in nth loop approximation is a local polynomial satisfying

$$\mathscr{G}_1\{\Gamma_0\}_n^{\mathrm{div}} = 0. \tag{27}$$

The content of the (27) is precisely that $\{\Gamma\}_n^{\mathrm{div}}$ and hence the counterterms needed are invariant under U(1) transformations of (7).

Hence only those counterterms in (15) that are invariant under U(1) transformations of (7) are allowed. It is easy to see from the form of the counterterms number 4 and 5 in (15) that only a particular combination of counterterms 4 and 5 is allowed by U(1) invariance.

As done in Joglekar (1988), it is easy to show that the remaining counterterms correspond to the following renormalizations

$$A_\mu^3 = Z_3^{1/2}A_\mu^{3(r)}; \quad W_\mu^\pm = Z^{1/2}W_\mu^{\pm(r)}; \quad c_\pm = \tilde{Z}^{1/2}c_\pm^{(r)},$$
$$\bar{c}_\pm = \tilde{Z}^{1/2}\bar{c}_\pm^{(r)}; \quad c_3 = \tilde{Z}_3^{1/2}c_3^{(r)}; \quad \bar{c}_3 = \tilde{Z}_3^{1/2}c_3^{(r)},$$
$$g = Xg^{(r)}; \quad \alpha = Z_3\alpha^{(r)}; \quad \xi = W\xi^{(r)},$$
$$\tau_\gamma^{\alpha\beta} = Y^{(\alpha\beta\gamma)}\tau_\gamma^{\alpha\beta(r)}. \tag{28}$$

That these renormalizations can be implemented and theory made finite can be proved by induction exactly as was done in Joglekar (1988). The following identities help in verifying the relations between the form of the counterterms due to renormalizations of (28) and the structure of divergence of (19).

$$2S_0[A] = \left\{-g\frac{\partial}{\partial g} + 2\eta\frac{\partial}{\partial\eta} + \tau\frac{\partial}{\partial\tau} + \int \mathrm{d}^n x A_\mu^\alpha(x)\frac{\delta}{\delta A_\mu^\alpha(x)}\right\}S'_{\mathrm{eff}}.$$

See also equations (6.15), (6.16) and (6.17), of Joglekar (1988). The final form of renormalized $\mathscr{L}'_{\mathrm{g.f.}}$ is

$$\mathscr{L}'_{\mathrm{g.f.}} = -\frac{1}{2\alpha}(\partial\cdot A^3)^2 - \frac{1}{\xi}\left|(\partial^\mu - igA^{3\mu})W_\mu^+ + \tau_1\bar{c}_-c_3 + \tau_2\bar{c}_3c_+\right|^2$$

and $\mathscr{L}'_{\mathrm{g}}$ can be obtained from (13).

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