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An approximate analysis of the interaction between two tearing modes of the same helicity

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Abstract. The interaction between tearing modes with the mode numbers $m = 1$, $n = 1$ and $m = 2$, $n = 2$ is investigated for different initial amplitudes of the modes, using a single helicity approximation, a step-current profile and a time-independent resistivity. Also included are the results on the temporal behaviour of the amplitude of the uncoupled mode $m = 2$, $n = 2$ in the situation with identical equilibrium parameters as ours.

Keywords. Tearing modes; helicity; magnetic island; magnetohydrodynamics; fundamental mode.

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1. Introduction

One of the major impediments to confinement in thermonuclear devices is the phenomenon of disruption. Previous works (Waddell *et al* 1979; Carreras *et al* 1980; Hicks *et al* 1982, 1984; Turner and Wesson 1982) have suggested that nonlinear interaction between tearing modes (Furth *et al* 1963, 1973) of different helicities can initiate disruptions in tokamak plasmas. It has earlier been demonstrated (Rutherford 1973; White *et al* 1977; White 1983; Priest 1985) that as tearing mode develops and enters the nonlinear regime, the width of the island with $m \geqslant 2$ (m and n used in the text are the poloidal and the toroidal mode numbers given by $k_y = m/r$ and $k_z = -n/R$, where r and R are the minor and the major radii of the torus and k_y, k_z, the y and the z components of the wave number respectively) first linearly increases with time and eventually the process culminates in the saturation of the mode. However, it has been shown by explicit analytical and numerical calculations (Waddell *et al* 1979; Carreras *et al* 1980; Hicks *et al* 1982, 1984; Turner and Wesson 1982) that if two islands overlap during simultaneous development of two modes, the modes exhibit an enhanced growth rate owing to a much greater efficacy of nonlinear interaction, the time scale of the phenomenon being of the same order of magnitude as that of disruption. Turner and Wesson (1982) showed that nonlinear interaction between modes with mode numbers $m = 1$, $n = 1$ and $m = 2$, $n = 1$ can lead to the growth of $m = 2$ mode on a time scale comparable to that of disruption and thus affect confinement in thermonuclear devices. A qualitatively similar result was obtained by Kadomtsev (1977) who showed that coupling between modes $m = 1$, $n = 1$ and $m = 2$, $n = 1$ can reveal the distinctive signatures of disruption.

In the above scenario one is led naturally to think that the coupling would increase appreciably if the modes overlap from the very outset. This can be achieved if the modes have identical helicity so that their mode rational surfaces are situated precisely at the same position. To this end we have addressed ourselves to the problem of investigating the interaction between two modes of identical helicity. Here we deal with the interaction between two tearing modes of mode numbers $m = 1, n = 1$ and $m = 2, n = 2$, that have developed to such an extent before coupling sets in that their widths compare fairly well with the shear lengths of the equilibrium field. The mode with $m = 1, n = 1$ is already known to be responsible for internal disruptions while disruptions have been observed in PLT (Sauthoff *et al* 1978), where along with $m = 1$, $n = 1$ mode the oscillations with $m = 2$, $n = 2$ have also been seen at the identical location at which $q = m/n = 1$. Though the helical symmetry is maintained intact, the evolution of the mode with $m = 1, n = 1$ will be influenced by such a coupling with the mode $m = 2, n = 2$ thereby affecting the phenomenon of sawtooth oscillations. The development of the mode $m = 2$, $n = 2$ too will be consequently influenced. A detailed study of the interaction between the modes is therefore essential to an understanding of the processes inside the system.

In § 2 we discuss briefly the derivation of the relevant MHD equations. Section 3 describes the current model used here and the evolution of the fundamental mode, i.e., the mode with $m = 1$, $n = 1$. In § 4 we deduce the equation giving the temporal development of the mode with $m = n = 2$. In § 6 we discuss the numerical results of § 5. Finally in the appendix we give a short derivation of the essential results used in the paper.

2. MHD equations

We consider a cylindrically symmetric system and adopt a single helicity approximation so that the perturbed variables are functions of $r, m\theta + k_z z$ and t, the symbols having their usual meanings. The magnetic field $\mathbf{B}$ is written as,

$$\mathbf{B} = \nabla\psi \times \mathbf{e}_z - (k_z r/m) B_z \mathbf{e}_\theta + B_z \mathbf{e}_z, \tag{1}$$

where ψ is the poloidal flux function. Assuming a large toroidal component of the magnetic field we can assume incompressibility in the plane perpendicular to z-axis and write the velocity $\mathbf{v}$ as,

$$\mathbf{v} = \nabla\phi \times \mathbf{e}_z, \tag{2}$$

where ϕ is any function of r, $m\theta + k_z z$ and t. We also follow the usual tokamak ordering (Rosenbluth *et al* 1976). The relevant MHD equations are (to the lowest order in $k_z r/m$),

$$\rho_m \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = (\nabla \times \mathbf{B}) \times \mathbf{B}/4\pi - \nabla p, \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\bigg/ c, \tag{4}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B}/c = \eta \mathbf{J}, \tag{5}$$

$$\nabla \times \mathbf{B} = 4\pi \mathbf{J}/c, \tag{6}$$

where η is the Spitzer resistivity (Spitzer 1956) and the other symbols have their usual meanings. Combining these equations with (1) and (2) we get,

$$\rho_m \frac{\mathrm{d}}{\mathrm{d}t}(\nabla_\perp^2 \phi) = -\mathbf{e}_z \cdot [\nabla_\perp \psi \times \nabla_\perp(\nabla_\perp^2 \psi)]/4\pi, \tag{7}$$

$$\frac{\partial \psi}{\partial t} + \mathbf{v}\cdot\nabla\psi = -c\eta J_z + E_0, \tag{8}$$

$$J_z = -c(\nabla_\perp^2 \psi + 2k_z B_z/m)/4\pi, \tag{9}$$

where E_0 is the externally imposed electric field at the wall of the tokamak (which is later put equal to zero). Here we can write,

$$\psi = \psi_0(r) + \sum_{i=1}^{2} \psi_1^{(i)}(r,t)\cos(m_i\theta + k_{zi}z),$$

where $m_{1,2} = 1,2$; $k_{z1,2} = -(1,2/R)$ and ψ_0, $\psi_1^{(i)}$ refer to the equilibrium and the first order perturbed flux functions of various harmonics respectively. $\psi_1^{(i)}(r,t)$ inside the island is approximated by $\psi_1^{(i)}(r,t) = \psi_1^{(i)}(r_x,t)(1+s_i x)$, where r_x is the radius vector of X-point, $x = r - r_x$, the radius vector relative to the X-point and $s_i = [\psi_1^{(i)\prime}/\psi_1^{(i)}]_{r=r_x}$. The mode with $m = n = 1$ having a much higher linear growth rate than the other one will start from a large amplitude in the nonlinear regime and can be assumed to remain more dominant compared to the mode with $m = n = 2$ throughout our analysis. The equation of the separatrix of the island is under this approximation (White *et al* 1977),

$$\cos\theta_s = -1 + [x^2/\delta(1+s_1 x)],$$

where

$$\delta = -2[\psi_1^{(1)}/\psi_0'']_{r=r_x},$$
$$s_1 = [\psi_1^{(i)\prime}/\psi_1^{(1)}]_{r=r_x}. \tag{9a}$$

3. Evolution of the fundamental mode

We adopt the step-current model of the equilibrium current density (that is a good approximation to many realistic current profiles), i.e., $J_{z,\mathrm{eq}} = J_z^{(0)}$ for $r \leqslant a$ and $=0$ for $r > a$. Expressing the total current density outside the island as $J_z = J_{z0} + J_{z1} + \cdots$, where J_{z0} and J_{z1} are the contributions to the current density to various orders in the perturbation, we obtain

$$J_{z1} = \left(\frac{\mathrm{d}J_z^{(0)}}{\mathrm{d}\psi}\right)_{\psi=\psi_0} \psi_1 = 0 \tag{9b}$$

for this step current profile. Thus for $r \leqslant a$ as well as $r > a$, the current density J_z can be represented by $J_z = J_{z0}(\psi)$. Moreover, it is known that inside the island the current density is expressed by a function of r and θ different from that outside the island (White *et al* 1977; White 1983; Priest 1985). From the continuity of J_z at $\psi = \psi_s$ for all r, θ we can write the expression of J_z inside the island in general (even when the form $J_z = J_z(\psi)$

for the current density inside the island does not hold good, e.g., for the mode with $m = 1$) as,

$$J_z = J_b(\psi) + \sum_n a_n(r, \theta)(\psi - \psi_s)^n,$$

where $a_n(r, \theta)$ and $J_b(\psi)$ are unspecified functions of r, θ and ψ respectively. The current density inside the island is fourier-analysed as,

$$J_z = \sum_i J_z^{(i)}(r, t) \cos m_i\theta,$$

where

$$J_z^{(i)} = (2/\pi) \int_0^{\theta_s} \left[J_b(\psi) + \sum_n a_n(r, \theta)(\psi - \psi_s)^n \right] \cos m_i\theta \, \mathrm{d}\theta$$

and θ_s is the magnitude of θ on the separatrix. We expand $J_b(\psi)$ in the form of a Taylor series about the equilibrium value $\psi = \psi_0$ and retain only the first two terms to obtain $J_b(\psi) = \alpha + \beta\psi$ (White *et al* 1977; White 1983; Priest 1985), where α, β are constants. We get,

$$\begin{aligned} J_z^{(1)} = (2/\pi)[(\alpha + \beta\psi_0)\sin\theta_s + \beta\{\psi_1^{(1)}(\sin 2\theta_s/2 + \theta_s) + (\psi_1^{(2)}/2) \\ \times (\sin\theta_s + \sin 3\theta_s/3)\}] + \sum_n \bar{a}_n(r, \theta_s)(\psi - \psi_s)^n, \end{aligned} \quad (10)$$

where $\bar{a}_n(r, \theta_s) = (2/\pi)\int_0^{\theta_s} a_n(r, \theta)\cos m_i\theta \, \mathrm{d}\theta$. The equation $\nabla_\perp^2 \psi_1^{(1)} = -4\pi J_z^{(1)}/c$ gives on integration along the separatrix $\psi = \psi_s$,

$$\begin{aligned} \Delta_1^{(1)'}\psi_1^{(1)} &= [\psi_1^{(1)'}]_{r=r_2} - [\psi_1^{(1)'}]_{r=r_1} \\ &= -(8/c)\int_{r_1}^{r_2} \mathrm{d}r \left[(\alpha + \beta\psi_0)\sin\theta_s \right. \\ &\quad \left. + \beta\left\{ \frac{\psi_1^{(1)}}{2}\left(\theta_s + \frac{\sin 2\theta_s}{2}\right) + \frac{\psi_1^{(2)}}{2}\left(\sin\theta_s + \frac{\sin 3\theta_s}{3}\right) \right\} \right], \end{aligned} \quad (11)$$

where $r = r_1$ and $r = r_2$ are the radial co-ordinates of the extremities of the island. Along the separatrix

$$\cos\theta_s = -1 + x^2/[\delta(1 + s_1 x)]$$

we have,

$$x = s_1\delta\cos^2(\theta_s/2) + \sqrt{2\delta}\cos(\theta_s/2) \quad \text{for } x > 0$$

and

$$= s_1\delta\cos^2(\theta_s/2) - \sqrt{2\delta}\cos(\theta_s/2) \quad \text{for } x < 0.$$

Utilising these expressions we can finally write (11) in the form,

$$\begin{aligned} \Delta_1^{(1)'}\psi_1^{(1)} = -(64\sqrt{\delta/2}/c)[\{\alpha + \beta\psi_0(r_x)\}/3 \\ + 2\beta s_1\delta\psi_0'(r_x)/5 + 2\beta\delta\psi_0''(r_x)/15 + 7\beta\psi_1^{(1)}(r_x)/15]. \end{aligned} \quad (12)$$

Similarly we get by integrating the equation

$$\nabla_\perp{}^2\psi_1{}^{(2)} = -4\pi J_z{}^{(2)}/c$$

that

$$(\text{here } \Delta_1{}^{(2)'}\psi_1{}^{(2)} = [\psi_1{}^{(2)'}]_{r=r_2} - [\psi_1{}^{(2)'}]_{r=r_1})$$

$$\Delta_1{}^{(2)'}\psi_1{}^{(2)} = -(64\sqrt{\delta/2}/c)[\{\alpha + \beta\psi_0(r_x)\}/15 - 2\beta s_1\delta\psi_0'(r_x)/35 - 2\beta\delta\psi_0''(r_x)/105 - 19\beta\psi_1{}^{(1)}(r_x)/105]. \tag{13}$$

We obtain from (12) and (13)

$$\beta = -35c[\Delta_1{}^{(1)'}\psi_1{}^{(1)} + 5\Delta_1{}^{(2)'}\psi_1{}^{(2)}]/[1024\sqrt{2\delta}\psi_1{}^{(1)}(r_x)] \tag{14}$$

and

$$\alpha + \beta\psi_0(r_x) = 35c[3\Delta_1{}^{(2)'}\psi_1{}^{(2)} - 15\Delta_1{}^{(1)'}\psi_1{}^{(1)}/7]/1024\sqrt{2\delta}. \tag{15}$$

Expressing (8) at $r = r_s$, where

$$(\psi_0')_{r=r_s} = 0, \quad \theta_s = \pi - (\delta s_1{}^2/2)^{1/2},$$

we obtain by neglecting $\psi_1{}^{(2)}$ and E_0 on the right hand side,

$$\frac{\partial\psi_1{}^{(1)}}{\partial t} = 35c^2\eta(\Delta_1{}^{(1)'}\psi_1{}^{(1)})\psi_1{}^{(1)}/[1024\sqrt{2\delta}(\psi_1{}^{(1)})_{r=r_x}]. \tag{16}$$

Now since $J_z = 0$ outside the island we get from (8), $\phi_1{}^{(1)} = r\dot{\psi}_1{}^{(1)}/\psi_0'$, where we have used

$$\phi_1 = \sum_{m_i=1}^{2} \phi_1{}^{(i)}(r,t)\cos m_i\theta, \quad \dot{\psi}_1{}^{(1)} = \partial\psi_1{}^{(1)}/\partial t, \quad \psi_0' = \mathrm{d}\psi_0/\mathrm{d}r. \tag{17}$$

With our assumed model of the equilibrium current density it readily follows that

$$\begin{aligned}\psi_0' &= -(2\pi J_{z0}a^2/cr + krB_{0z}/m) \quad \text{for } a < r < b,\\ &= -r(2\pi J_{z0}/c + kB_{0z}/m) \quad \text{for } 0 < r < a,\end{aligned} \tag{18}$$

where b is the radius of the cylinder or equivalently the minor radius of the tokamak. We obtain by combining (7), (17) and (18),

$$\frac{\partial}{\partial t}(\nabla^2\phi_1{}^{(1)}) = \frac{\partial}{\partial t}[\nabla^2(r\dot{\psi}_1{}^{(1)}/\psi_0')] = 0. \tag{19}$$

Since $\psi_1{}^{(1)}$ satisfies simultaneously (19) and also the condition that $\nabla^2\psi_1{}^{(1)} = 0$ outside the island we can write,

$$\begin{aligned}\psi_1{}^{(1)} &= a_1 r(\gamma_1 t + \delta_1) \quad \text{for } 0 < r < a,\\ &= (c_2 r + c_3/r)(\gamma_1 t + \delta_1) \quad \text{for } a < r < r_1,\end{aligned}$$

and

$$= c_1(r - b^2/r)(\gamma_1 t + \delta_1) \quad \text{for } r_2 < r < b, \tag{20}$$

where $a_1, c_1, c_2, c_3, \gamma_1, \delta_1$ are constants to be determined later (see Appendix) and we have used the boundary condition that $\psi_1^{(1)}$ is finite at $r=0$ and zero at $r=b$.

The Lagrangian displacement $\xi_1^{(1)}$ defined by $\xi_1^{(1)} = \psi_1^{(1)}/\psi_0'$ is continuous at $r=a$ because of the continuity of both $\psi_1^{(1)}$ and ψ_0' there.

Now

$$\xi_1^{(1)\prime} = \psi_1^{(1)\prime}/\psi_0' - \psi_1^{(1)}\psi_0''/(\psi_0')^2;$$

$$\nabla_\perp^2 \psi_1^{(1)} = -4\pi J_z^{(1)}/c = -4\pi \frac{\mathrm{d}J_{z0}}{\mathrm{d}r} \psi_1^{(1)}/c\psi_0'. \tag{21}$$

Integrating (21) we obtain that the discontinuity of $\psi_1^{(1)\prime}$ at $r=a = 4\pi J_{z0}(\psi_1^{(1)}/\psi_0')_{r=a}/c$ for our step-current model. Again since

$$[(\psi_0'')_{r=a+\varepsilon} - (\psi_0'')_{r=a-\varepsilon}]_{\varepsilon\to 0} = 4\pi J_{z0}/c$$

from (18), we obtain from (21) that $\xi_1^{(1)\prime}$ is also continuous at $r=a$. Matching $\xi_1^{(1)}$ and $\xi_1^{(1)\prime}$ at $r=a$ we get from (20), $c_2 = a_1/(1-r_s^2/a^2)$, $c_3 = -c_2 r_s^2$. Thus we can write,

$$\Delta_1^{(1)\prime}\psi_1^{(1)} = (\Delta_1^{(1)\prime}\psi_1^{(1)})_0(\gamma_1 t + \delta_1),$$

where

$$(\Delta_1^{(1)\prime}\psi_1^{(1)})_0 = c_1(1 + b^2/r_2^2) - \frac{a_1(1 + r_s^2/r_1^2)}{(1 - r_s^2/a^2)}$$

and $r=r_s$ represents the location of the mode rational surface.

Substituting for $\Delta_1^{(1)\prime}\psi_1^{(1)}$ in (16) we obtain,

$$\psi_1^{(1)} = \psi_1^{(1)}(r_x)(1 + s_1 x)(3\alpha\delta_1 t/2 + \beta)^{2/3},$$

where

$$\alpha\delta_1/\beta = (35c^2\eta/2048)[-\psi_0''(r_x)/(\psi_1^{(1)}(r_x))^3]^{1/2}(\Delta_1^{(1)\prime}\psi_1^{(1)})_0. \tag{22}$$

The pattern of evolution of the fundamental mode as given by (22) thus depends on the parameters of the problem, namely, the width of the current layer a, the minor radius of the tokamak b and the position of the mode rational surface $r=r_s$.

4. Evolution of the mode with $m=2$, $n=2$

The equation determining the evolution of the mode with $m=n=2$ (i.e., $\psi_1^{(2)}$) at $r=r_s$ is given from (8) by

$$\frac{\partial \psi_1^{(2)}}{\partial t} + \mathbf{v}_1^{(1)}\cdot\nabla\psi_1^{(1)} = -c\eta J_z^{(2)}, \tag{23}$$

where $\mathbf{v}_1^{(1)}$ is the first harmonic component of the velocity perturbation and $J_1^{(2)}$ the second harmonic component of the perturbation in current density given by

$$J_z^{(2)} = (2/\pi)\int_0^{\theta_s} (\alpha + \beta\psi)\cos 2\theta \, \mathrm{d}\theta.$$

Substituting from (14) and (15) we obtain,

$$J_z^{(2)} = [35c(\Delta_1^{(1)'}\psi_1^{(1)})_0/512\pi(2\delta)^{1/2}][15(2x^2/\delta)^{1/2}/7(1+s_1x)^{1/2} - \{\psi_1^{(1)}(2x^2/\delta)^{1/2}/(1+s_1x)^{1/2} + \psi_1^{(2)}\pi/2\}/(\psi_1^{(1)})_{r=r_x}]. \tag{24}$$

In the region inside the island, equation (7) gives, when combined with (22),

$$\rho_m \frac{\mathrm{d}}{\mathrm{d}t}\nabla_\perp^2\phi_1^{(1)} = -\psi_0'(1-s_1r_x)\psi_1^{(1)}(r_x)T(t)/4\pi r^3,$$

where

$$T(t) = (3\alpha\delta_1 t/2 + \beta)^{2/3}. \tag{25}$$

Or

$$\nabla_\perp^2\phi_1^{(1)} = -(\psi_0'/10\pi\rho_m r^3\alpha\delta_1)(1-s_1r_x)\psi_1^{(1)}(r_x)(3\alpha\delta_1 t/2+\beta)^{5/3}. \tag{26}$$

Let $\phi_1^{(1)} = \Phi(r)T_1(t)$, where $\Phi(r)$ and $T_1(t)$ are hitherto unspecified functions of r and t respectively. We get,

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2} + \frac{\mathrm{d}\Phi}{\mathrm{d}r}/r - \Phi/r^2 = -\psi_0'(1-s_1r_x)\psi_1^{(1)}(r_x)/4\pi\rho_m r^3;$$

$$T_1 = (2/5\alpha\delta_1)(3\alpha\delta_1 t/2+\beta)^{5/3}. \tag{27}$$

Solving this equation by the method of the variation of parameters we obtain,

$$\Phi = [c_{10}r + c_{20}/r - \psi_1^{(1)}(r_x)(1-s_1r_x)k_zB_{0z} \times (1 + r_s^2/3r^2)/4\pi\rho_m]\sin\theta, \tag{28}$$

where c_{10}, c_{20} are constants to be found out later (see Appendix (A)).
Substituting for $\phi_1^{(1)}$ in the relation $\mathbf{v}_1^{(1)} = \nabla\phi_1^{(1)} \times \mathbf{e}_z$ we obtain,

$$\mathbf{v}_{1r}^{(1)} = [c_{10} + c_{20}/r^2 - \psi_1^{(1)}(r_x)(1-s_1r_x)k_zB_{0z} \times (1 + r_s^2/3r^2)/4\pi\rho_m r]\,T(t)\cos\theta, \tag{29a}$$

$$\mathbf{v}_{1\theta}^{(1)} = -[c_{10} - c_{20}/r^2 + \psi_1^{(1)}(r_x)(1-s_1r_x)k_zB_{0z}r_s^2/6\pi\rho_m r^3] \times T(t)\sin\theta. \tag{29b}$$

Putting the expressions of $\mathbf{v}_1^{(1)}\cdot\nabla\psi_1^{(1)}$ and $J_z^{(2)}$ in (23) we get,

$$\frac{\partial\psi_1^{(2)}}{\partial t} + 2\Phi(r_s)(3\alpha\delta_1 t/2+\beta)^{7/3}/5\alpha\delta_1 = \alpha_3(\gamma_1 t + \delta_1)/T + \alpha(\gamma_1 t+\delta_1)\psi_1^{(2)}/T^{3/2}, \tag{30}$$

where

$$\alpha_3 = -(35c^2\eta x_s/1024\pi)(\psi_0''/\psi_1^{(1)})_{r=r_x}(\Delta_1^{(1)'}\psi_1^{(1)})_0 \times (s_1x_s - 8/7)/(1+s_1x_s)^{1/2},$$

$$\Phi(r_s) = -[\{\psi_1^{(1)}(r_x)\}^2(1 - s_1r_x)kB_{0z}\{2(1 + s_1x)r_s^2/r^3 + 3s_1 \times (1 + r_s^2/3r_1^2)\}/24\pi\rho_m r_s + \psi_1^{(1)}(r_x)\{c_{10}(1 - s_1r_x) - c_{20} \times (1 + 2s_1r_s - s_1r_x)/r_s^2\}], \quad x_s = r_s - r_x.$$

Finally we obtain by solving the differential equation (30),

$$(\psi_1^{(2)})_{r=r_s} = (2/3\alpha\delta_1)(9\alpha\delta_1^2/4\gamma_1)^{1/3} z_1^{p-2/3}\exp(z_1)\Big[\alpha_3\Big\{(3\delta_1/2) \times \int_{z_0}^{z_1} z^{1-p}\exp(-z)\,\mathrm{d}z + (\delta_1 - 2\beta\gamma_1/3\alpha\delta_1)\int_{z_0}^{z_1} z^{-p} \times \exp(-z)\,\mathrm{d}z\Big\} - (2/5\alpha\delta_1)\Phi(r_s)(9\alpha\delta_1^2/4\gamma_1)^3\int_{z0}^{z} z^{3-p} \times \exp(-z)\,\mathrm{d}z\Big] + (\psi_1^{(2)})_0 f(z_0)/f(z_1), \tag{31}$$

where

$$p = 4(\delta_1 - 2\beta\gamma_1/3\alpha\delta_1)/3\delta_1, \quad z_0 = 4\beta\gamma_1/9\alpha\delta_1^2, \quad (\psi_1^{(2)})_0 = (\psi_1^{(2)})_{t=0},$$
$$z_1 = (3\alpha\delta_1 t/2) + \beta, \quad f(z) = (9\alpha\delta_1^2 z/4\gamma_1)^{2/3-p}\exp(-z).$$

5. Numerical results

The amplitude of the mode $\psi_1^{(2)}$ is found out by numerical integration of the integrals appearing in equation (31) and then plotted as a function of time for various initial amplitudes of the modes $\psi_1^{(1)}$ and $\psi_1^{(2)}$ respectively (figures 1 and 2). Also shown graphically is the temporal behaviour of the mode $\psi_1^{(2)}$, if present alone with the set of equilibrium parameters identical with ours (figure 3). The equilibrium parameters chosen are relevant to the DITE experiment (Turner and Wesson 1982), i.e., $B_{0z} = 1{\cdot}34\,T$, $J_{0z} = 1{\cdot}5$ MA, $n = 3 \times 10^{13}$/c.c., $R = 1{\cdot}7$ m, $b = 26{\cdot}0$ cm. a has been chosen to be $6{\cdot}5$ cm and correspondingly r_s found out to be $18{\cdot}655$ cm. η has been taken equal to the Spitzer resistivity at the electron temperature $T_e = 100$ eV. The derivations of some essential results needed for computation are furnished in the appendix. A few typical order of magnitude estimates of the growth rates obtained from figure 1 are summarized in table 1. Similar calculations performed from figure 2 with two different initial widths of the mode $\psi_1^{(2)}$ show that the growth rate of the mode $m = n = 2$ still remains practically the same as in the previous case (of the order of $10^5\,\mathrm{sec}^{-1}$) for the equilibrium parameters we have chosen. This shows that the evolution of the mode $\psi_1^{(2)}$ is determined predominantly by the initial amplitude of the mode $\psi_1^{(1)}$ which is chosen to be the same in all these cases.

6. Discussion

The present results show that although the amplitude of the mode $m = n = 2$, if left alone, decreases with time in the nonlinear regime for our choice of equilibrium parameters (as is evident from figure 3), it grows due to interaction of two $m = 1, n = 1$

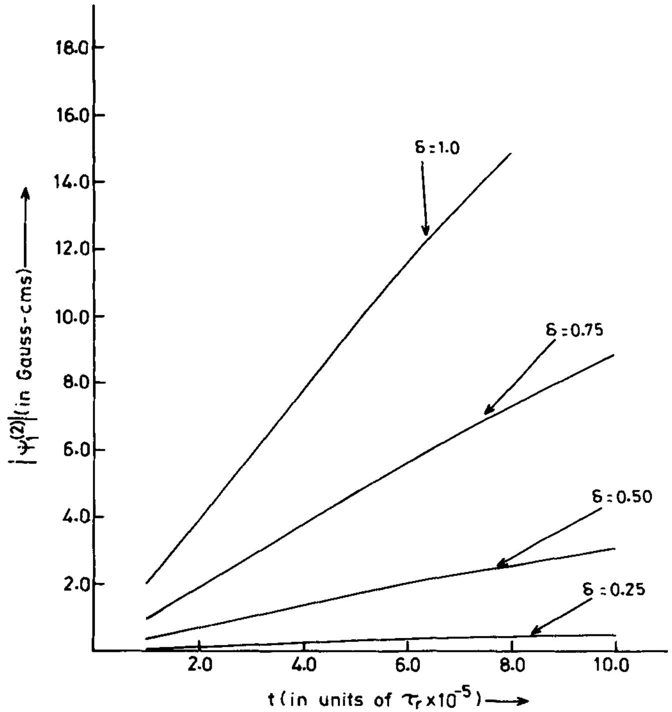

Figure 1. $\psi_1^{(2)}$ as a function of time expressed in units of $\tau_r \times 10^{-5}$, where $\tau_r = 4\pi b^2/c^2\eta$ is the resistive time. Initial $\psi_1^{(2)} = 0$. b, T_e, η and δ have the same meanings as in the text.

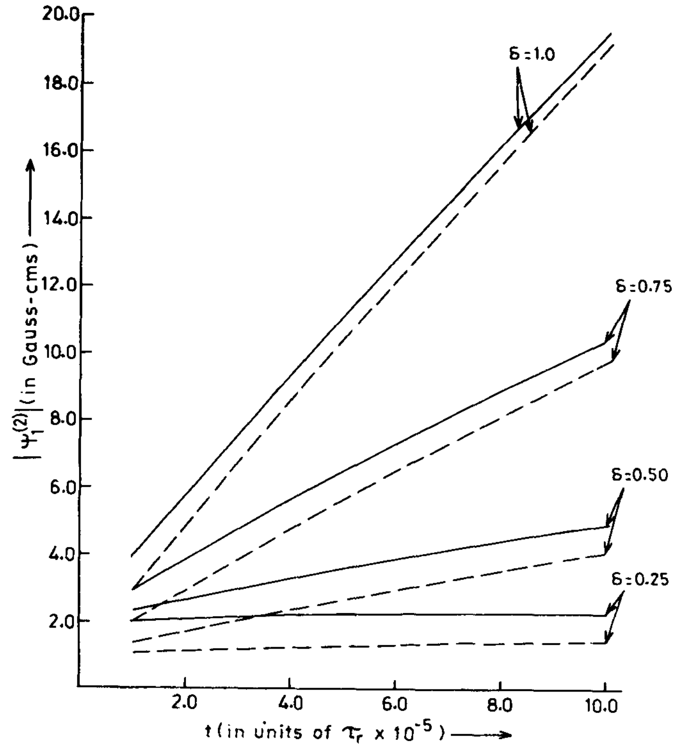

Figure 2. $\psi_1^{(2)}$ as a function of time expressed in the same units as before. The dotted and the continuous curves correspond to initial $\psi_1^{(2)}$ equal to 1·0 and 2·0 respectively. b, T_e, η and δ are the same as above.

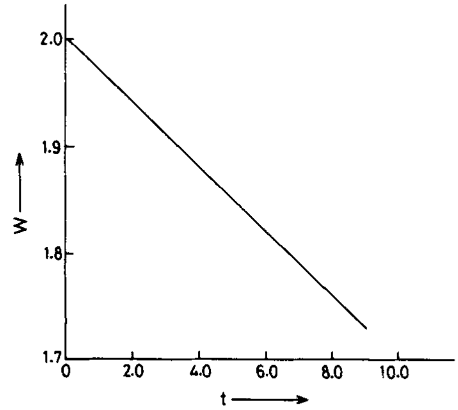

Figure 3. The uncoupled mode $\psi_1^{(2)}$ against time expressed in units of $\tau_r \times 10^{-4}$. The initial width of the island has been taken to be 2·0. All the parameters remain the same as before.

Table 1. Growth rates of the mode $\psi_1^{(2)}$ ($m = n = 2$) for various initial widths of $m = 1$, $n = 1$ island (initial amplitude of $\psi_1^{(2)} = 0$).

δ (in cm^2) (initial width of $m = 1$, $n = 1$ island $= 2(2\delta)^{1/2}$)	$\frac{d\psi_1^{(2)}}{dt} \Big/ \psi_1^{(2)}$ (in s^{-1}) (growth rate)
1·0	2×10^5
0·75	$1·5 \times 10^5$
0·50	10^5
0·25	10^5

modes (as can be seen from figures 1 and 2). Moreover, a comparison of the pattern of temporal development of the modes in our case with that obtained by others (Waddell *et al* 1979; Carreras *et al* 1980) reveals that while in the former the modes overlap *ab initio* giving rise to an almost constant growth rate of the flux perturbation, in the latter the growth rate builds up from a low magnitude to a large one as the islands overlap each other.

Our calculations have, however, been performed under various simplifying assumptions. Unlike in the present case, where the mode with $m = n = 1$ has been assumed to be more dominant compared to that with $m = 2$, $n = 2$, an exact analysis of the interaction will need a rigorous nonlinear treatment and involve more than two modes (e.g., the (2, 2) mode, after it has grown to a sufficiently large magnitude can interact with the $(-1, -1)$ mode and the coupling will modify the (1, 1) mode which further affects the destabilization of the (2, 2) mode). Nevertheless, our approximate analysis describes correctly the build-up of the process for a reasonable period of time and gives us a feel for the efficacy of the interaction. Secondly, our calculations have been done with a time-independent resistivity and a step current profile. A detailed nonlinear calculation including a self-consistently calculated time-dependent resistivity and a realistic current profile is now being performed.

Appendix

A. Determination of a_1/c_1

The radial and the azimuthal components of the first order velocity perturbation are given outside the island by

$$v_{1r}^{(1)} = -\dot{\psi}_1^{(1)}/\psi_0' \quad \text{and} \quad v_{1\theta}^{(1)} = \frac{\partial}{\partial r}(r\dot{\psi}_1^{(1)}/\psi_0') \text{ respectively.} \tag{A.1}$$

Inside the island these are given by equations (29a) and (29b) respectively, namely,

$$v_{1r}^{(1)} = [c_{10}r + c_{20}/r - \psi_1^{(1)}(r_x)(1 - s_1 r_x)k_z B_{0z}(1 + r_s^2/3r^2)/4\pi\rho_m] \times T_1(t)\text{`}\cos\theta/r, \tag{29a}$$

$$v_{1\theta}^{(1)} = -[c_{10} - c_{20}/r^2 + \psi_1^{(1)}(r_x)(1 - s_1 r_x)k_z B_{0z} r_s^2/6\pi\rho_m r^3] \times T_1(t)\sin\theta, \tag{29b}$$

where

$$T_1(t) = 2(3\alpha\delta_1 t/2 + \beta)^{5/3}/5\alpha\delta_1.$$

Matching the two expressions for $v_{1r}^{(1)}$ at $r = r_1$ ($r = r_1$ and $r = r_2$ are the extremities of the island), we get,

$$c_{10}r_1 + c_{20}/r_1 = \alpha_{1c}(\gamma_1/\delta_1)(a_1\delta_1/\beta^{2/3}) \times \left[c_1\delta_1(1 + b^2/r_2^2) - \frac{a_1\delta_1(1 + r_s^2/r_1^2)}{(1 - r_s^2/a^2)}\right] - \beta_{1c},$$

where

$$\alpha_{1c} = -[5r_1/2(\psi_0')_{r_1}](35c^2\eta/2048)\left[-\frac{\psi_0''(r_x)}{\{\psi_1^{(1)}(r_x)\}^3}\right]^{1/2} \times (r_1 - r_s^2/r_1)/\beta(1 - r_s^2/a^2),$$
$$\beta_{1c} = -\psi_1^{(1)}(r_x)(1 - s_1 r_x)k_z B_{0z}(1 + r_s^2/3r_1^2)/4\pi\rho_m. \tag{A.2}$$

Similarly by matching the expressions for $v_{1\theta}^{(1)}$ at $r = r_1$ we obtain,

$$c_{10}r_1 - c_{20}/r_1 = \alpha_{2c}(\gamma_1/\delta_1)(a_1\delta_1/\beta^{2/3}) \times \left[c_1\delta_1(1 + b^2/r_2^2) - \frac{a_1\delta_1(1 + r_s^2/r_1^2)}{(1 - r_s^2/a^2)}\right] + \beta_{2c}, \tag{A.3}$$

where

$$\alpha_{2c} = -(5r_1/2)\cdot(35c^2\eta/2048)[-\psi_0''(r_x)/(\psi_1^{(1)}(r_x))^3]^{1/2} \times \left\{\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{ra_1(r - r_s^2/r)}{\psi_0'(1 - r_s^2/a^2)}\right]\right\}_{r=r_1},$$

$$\beta_{2c} = -\psi_1^{(1)}(r_x)(1 - s_1 r_x)k_z B_{0z} r_s^2/6\pi\rho_m r_1^2.$$

Similarly, we get by matching the two expressions for $v_{1r}^{(1)}$ and $v_{1\theta}^{(1)}$ respectively at $r=r_2$,

$$c_{10}r_2 + c_{20}/r_2 = \alpha_{3c}(\gamma_1/\delta_1)(c_1\delta_1/\beta^{2/3})\left[c_1\delta_1(1+b^2/r_2^2) - \frac{a_1\delta_1(1+r_s^2/r_1^2)}{(1-r_s^2/a^2)}\right] - \beta_{3c}, \tag{A.4}$$

where

$$\alpha_{3c} = -[5r_2/2(\psi_0')_{r_2}](r_2 - b^2/r_2)(35c^2\eta/2048) \times [-\psi_0''(r_x)/(\psi_1^{(1)}(r_x))^3]^{1/2},$$

$$\beta_{3c} = -\psi_1^{(1)}(r_x)(1-s_1r_x)k_zB_{0z}(1+r_s^2/3r_2^2)/4\pi\rho_m$$

and

$$c_{10}r_2 - c_{20}/r_2 = \alpha_{4c}(\gamma_1/\delta_1)(c_1\delta_1/\beta^{2/3})\left[c_1\delta_1(1+b^2/r_2^2) - \frac{a_1\delta_1(1+r_s^2/a^2)}{(1-r_s^2/a^2)}\right] + \beta_{4c}, \tag{A.5}$$

where

$$\alpha_{4c} = -(5r_2/2)(35c^2\eta/2048)\left[-\frac{\psi_0''(r_x)}{\{\psi_1^{(1)}(r_x)\}^3}\right]^{1/2} \times \left[\frac{\mathrm{d}}{\mathrm{d}r}\{r(r-b^2/r)/\psi_0'\}\right]_{r=r_2},$$

$$\beta_{4c} = -\psi_1^{(1)}(r_x)(1-s_1r_x)k_zB_{0z}r_s^2/6\pi\rho_m r_2^3.$$

From equations (A.2) and (A.3) we get,

$$c_{10} = (1/2r_1)\left[(\alpha_{1c}+\alpha_{2c})(\gamma_1/\delta_1)(a_1\delta_1/\beta^{2/3})\left\{c_1\delta_1(1+b^2/r_2^2) - \frac{a_1\delta_1(1+r_s^2/r_1^2)}{(1-r_s^2/a^2)}\right\} + \beta_{2c} - \beta_{1c}\right],$$

$$c_{20} = (r_1/2)\left[(\alpha_{1c}-\alpha_{2c})(\gamma_1/\delta_1)(a_1\delta_1/\beta^{2/3})\left\{c_1\delta_1(1+b^2/r_2^2) - \frac{a_1\delta_1(1+r_s^2/r_1^2)}{(1-r_s^2/a^2)}\right\} - (\beta_{3c}+\beta_{4c})\right]. \tag{A.6}$$

Similarly from (A.4) and (A.5) we get,

$$c_{10} = (1/2r_2)\left[(\alpha_{3c}+\alpha_{4c})(\gamma_1/\delta_1)(c_1\delta_1/\beta^{2/3})\left\{c_1\delta_1(1+b^2/r_2^2) - \frac{a_1\delta_1(1+r_s^2/r_1^2)}{(1-r_s^2/a^2)}\right\} + \beta_{4c} - \beta_{3c}\right],$$

$$c_{20} = (r_2/2)\left[(\alpha_{3c} - \alpha_{4c})(\gamma_1/\delta_1)(c_1\delta_1/\beta^{2/3})\left\{c_1\delta_1(1 + b^2/r_2{}^2) - \frac{a_1\delta_1(1 + r_s{}^2/r_1{}^2)}{(1 - r_s{}^2/a^2)}\right\} - (\beta_{3c} + \beta_{4c})\right]. \tag{A.7}$$

Combining the pair of equations (A.6) and (A.7) we get by matching the two couples of expressions of c_{10} and c_{20},

$$a_1/c_1 = [(\alpha_{3c} + \alpha_{4c})/r_2 + 7r_2(\alpha_{3c} - \alpha_{4c})/3r_s{}^2]/ \times [(\alpha_{1c} + \alpha_{2c})/r_1 + 7r_1(\alpha_{1c} - \alpha_{2c})/3r_s{}^2].$$

Putting the expressions for $\alpha_{1c} \cdots \alpha_{4c}$ we obtain finally,

$$a_1/c_1 = (3/20)(1 - r_s{}^2/a^2) \times [\{10(r_s{}^2 - b^2)/3r_s - 8r_s/3\}/(2\delta)^{1/2} - 2(b^2 - r_s{}^2)/3\delta]. \tag{A.8}$$

B. Calculation of γ_1/δ_1

From the two equations (A.6) and (A.7) we get by equating the two expressions for c_{10} (or c_{20}),

$$(\gamma_1/\delta_1)[(1 + b^2/r_2{}^2)/R_1 - (1 + r_s{}^2/r_1{}^2)/(1 - r_s{}^2/a^2)] \times (a_1\delta_1)^2/\beta^{2/3} \cdot [(\alpha_{1c} + \alpha_{2c})/r_1 - (\alpha_{3c} + \alpha_{4c})/r_2R_1] = [(\beta_{4c} - \beta_{3c})/r_2 - (\beta_{2c} - \beta_{1c})/r_1], \quad \text{where } R_1 = a_1/c_1. \tag{B.1}$$

Putting the expressions for $\alpha_{1c} \cdots \beta_{4c}$ from (A.2) and (A.5) we obtain,

$$\gamma_1/\delta_1 = 7\psi_1{}^{(1)}(r_x)(1 - s_1r_x)k_zB_{0z}(2\delta)^{3/2}\beta^{2/3}/[12\pi\rho_m r_s{}^4(a_1\delta_1)^2X] \times 1/[(1 + b^2/r_2{}^2)/R_1 - (1 + r_s{}^2/r_1{}^2)/(1 - r_s{}^2/a^2)],$$

where

$$X = [(\alpha_{1c} + \alpha_{2c})/r_1 - (\alpha_{3c} + \alpha_{4c})/r_2R_1]/2. \tag{B.2}$$

C. Calculation of $a_1\delta_1$ and s_1

Matching the expressions for $\psi_1{}^{(1)}$ inside and outside the island at $r = r_1$ and $r = r_2$ respectively, we obtain,

$$\psi_1{}^{(1)}(r_x)(1 + s_1x_1)\beta^{2/3} = a_1\delta_1(r_1 - r_s{}^2/r_1)/(1 - r_s{}^2/a^2) \tag{C.1}$$

and

$$\psi_1{}^{(1)}(r_x)(1 + s_1x_2)\beta^{2/3} = c_1\delta_1(r_2 - b^2/r_2). \tag{C.2}$$

From the above two equations, we get,

$$s_1 = \frac{c_1(r_s^2 - b^2)(1 - r_s^2/a^2) + 4[-(\psi_0'')_{r=r_x}]^{1/2}\beta^{1/3}a_1 r_s}{2\beta^{1/3}[-\psi_1^{(1)}(r_x)/\psi_0''(r_x)]^{1/2}\{c_1(r_s^2 - b^2)(1 - r_s^2/a^2) - 4[(-\psi_1^{(1)}/\psi_0'')^{1/2}]\beta^{1/3}a_1 r_s\}}. \tag{C.3}$$

We obtain by coupling equations (C.1) and (C.3)

$$Az^2 + B_1 z + C = 0, \quad \text{where } A = 8\psi_0'' r_s a_1,$$
$$B_1 = -16 r_s a_1^2 \delta_1/(1 - r_s^2/a^2), \quad C = 4c_1 a_1 \delta_1 (r_s^2 - b^2),$$
$$z = \beta^{1/3}[-(\psi_1^{(1)}/\psi_0'')_{r=r_x}]^{1/2}.$$

Putting $c_1\delta_1 = a_1\delta_1/R_1$ we get finally,

$$a_1\delta_1 = \frac{\psi_0'' r_s \delta}{\left[\dfrac{4r_s}{(1 - r_s^2/a^2)}(\delta/2)^{1/2} + (b^2 - r_s^2)/R_1\right]}. \tag{C.4}$$

D. Determination of the behaviour of the mode $\psi_1^{(2)}$ ($m = 2$, $n = 2$) if left alone.

Since $\nabla^2\psi_1^{(2)} = 0$ outside the island for the step-current model we get

$$R_1^{(2)} = A_1 r^2 \quad \text{for } 0 < r < a,$$
$$= A_2 r^2 + A_3/r^2 \quad \text{for } a < r < r_1, = A_4(r^2 - b^4/r^2) \tag{D.1}$$

for $r_2 < r < b$, where $R_1^{(2)}$ represents the radially-dependent part of $\psi_1^{(2)}$, $A_1 \cdots A_4$ are arbitrary constants of integration and r_1, r_2 the radial co-ordinates of the extremities of the island. Matching the Lagrangian displacement $\xi_1^{(2)} = (-\psi_1^{(2)}/\psi_0')$ and its derivative $d\xi_1^{(2)}/dr$ at $r = a$ we get,

$$A_2 = A_1(r_s^2 - 2a^2)/2(r_s^2 - a^2), \quad A_3 = \frac{A_1}{2}\cdot\frac{r_s^2 a^4}{(r_s^2 - a^2)}. \tag{D.2}$$

Equating the two expressions for $\psi_1^{(2)}$ at $r = r_s$ we obtain,

$$A_1 = 2(r_s^2 - b^4/r_s^2)A_4/(r_s^2 - a^2). \tag{D.3}$$

Logarithmic discontinuity across the island is given by,

$$\Delta' = [(\psi_1^{(2)'})_{r_2} - (\psi_1^{(2)'})_{r_1}]/(\psi_1^{(2)})_{r=r_s}$$
$$= 2[A_1(r_2 + b^4/r_2^3) - (A_2 r_1 - A_3/r_1^3)]/A_4(r_s^2 - b^4/r_s^2).$$

Putting the expressions for $A_1 \cdots A_4$ from the above we get,

$$\Delta'(w) = 2\left[(r_2 + b^4/r_2^3) - \frac{r_1(r_s^2 - b^4/r_s^2)}{(r_s^2 - a^2)^2} \times \{(r_s^2 - 2a^2) - r_s^2 a^4/r_1^4\}\right]\Big/(r_s^2 - b^4/r_s^2). \tag{D.4}$$

Here $r_{1,2}=r_s \mp w/2$, where w represents the width of the island and equals $4[-\psi_1^{(1)}/\psi_0'']^{1/2}$. Putting in the equation $\mathrm{d}w/\mathrm{d}t = 1{\cdot}66\eta(r_s)\Delta'(w)c^2/4\pi$ (White *et al* 1977), where $\eta(r_s)$ is the expression for the Spitzer resistivity (Spitzer 1956) we can find out the evolution of the width of the island with $m=2$, $n=2$ by numerical calculation. The behaviour of the width of the island as a function of time is exhibited graphically in figure 3.

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