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Transition to chaos in a driven pendulum with nonlinear dissipation

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Abstract. The Melnikov-Holmes method is used to study the onset of chaos in a driven pendulum with nonlinear dissipation. Detailed numerical studies reveal many interesting features like a chaotic attractor at low frequencies, band formation near escape from the potential well and a sequence of subharmonic bifurcations inside the band that accumulates at the homoclinic bifurcation point.

Keywords. Driven pendulum; nonlinear dissipation; Melnikov threshold; subharmonic bifurcations.

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1. Introduction

A forced pendulum with damping which models a large variety of physical situations has been an important topic of study for a long time. It has been established that such a system exhibits chaotic behaviour in certain regions of the parameter space (D'Humieres *et al* 1982; Bohr *et al* 1984; Kadanoff 1985). In all these studies, the damping term is a simple linear function of phase velocity. In this paper we choose a nonlinear, x-dependent damping of the van der Pol type. This kind of damping can intrinsically lead to the formation of a band in phase space and can strongly influence the response of the pendulum to a periodic driving force. At present we are unable to cite an example where the van der Pol pendulum directly plays a role. However, we would like to mention that the corresponding oscillator is one of the most extensively studied nonlinear systems, which can model self-excited oscillations in nonlinear circuits, lasers etc. The oscillator is known to have mode-locking and period doubling cascades (Parlitz and Lauterborn 1987) and has been studied for different forms of x-dependent damping (Robinson 1987).

The lowest threshold for onset of chaos in the system considered is derived using the Melnikov-Holmes method (Melnikov 1963; Greenspan and Holmes 1981; Lichtenberg and Lieberman 1983; Cusumano and Holmes 1987; Ling 1987; Ling and Bao 1987). This is included in § 2. The Melnikov criterion does not imply that the trajectories will be asymptotically chaotic. It is possible to have transient chaos followed by asymptotically periodic trajectories. So we carry out a detailed numerical analysis including essentially studies of phase portraits, Poincaré maps, power spectra and computation of maximum Lyapunov exponents. Our main observations are summarized in § 3 and the concluding remarks in § 4.

2. Application of Melnikov-Holmes method

The equation of motion for the van der Pol pendulum can be written as,

$$\dot{x} = v$$

$$\dot{v} = -\sin x + \varepsilon[\beta(1 - x^2)\dot{x} + A\sin\omega t], \tag{1}$$

where A and ω are the amplitude and frequency of the driving term and β is the damping constant. ε is a small parameter. The unperturbed system corresponding to $\varepsilon = 0$ is integrable with the hamiltonian

$$H_0 = \tfrac{1}{2}v^2 - \cos x. \tag{2}$$

The homoclinic orbits are given by (Guckenheimer and Holmes 1983)

$$x_0(t) = 2\tan^{-1}(\sinh t), \tag{3}$$

$$v_0(t) = 2\,\mathrm{sech}\, t. \tag{4}$$

When the perturbation is switched on, the stable and unstable manifolds of the separatrix or homoclinic orbit do not join smoothly. The separation of the manifolds in a surface of section is measured by the Melnikov function $M(t_0)$ (Lichtenberg and Lieberman 1983) which for (1) is

$$M(t_0) = \int_{-\infty}^{\infty} [\beta(1 - x_0^2(t - t_0))v_0^2(t - t_0) + Av_0(t - t_0)\sin\omega t]\,\mathrm{d}t. \tag{5}$$

Changing the variable to $\tau = t - t_0$ and using the explicit forms for x_0 and v_0 given in (3) and (4) we get,

$$\begin{aligned} M(t_0) = &-4\beta\int_{-\infty}^{\infty} \mathrm{sech}^2\tau\,\mathrm{d}\tau \\ &+ 16\beta\int_{-\infty}^{\infty} \mathrm{sech}^2\tau[\tan^{-1}(\sinh\tau)]^2\,\mathrm{d}\tau \\ &- 2A\cos\omega t_0\int_{-\infty}^{\infty} \mathrm{sech}\,\tau\sin\omega\tau\,\mathrm{d}\tau \\ &+ 2A\sin\omega t_0\int_{-\infty}^{\infty} \mathrm{sech}\,\tau\cos\omega\tau\,\mathrm{d}\tau. \end{aligned} \tag{6}$$

This can be integrated to give

$$M(t_0) = -(72 - 8\pi^2)\beta + 2\pi A\,\mathrm{sech}\,(\pi\omega/2)\sin\omega t_0. \tag{7}$$

The Melnikov threshold is thus

$$A_c = \frac{(8\pi^2 - 72)}{2\pi}\beta\cos h(\pi\omega/2). \tag{8}$$

For $A > A_c$, $M(t_0)$ oscillates between positive and negative values indicating that the stable and unstable manifolds intersect producing local chaos. At $A = A_c$, $M(t_0, A_c)$ has a quadratic zero in the surface of section i.e.

$$M(t_0, A_c) = \partial M(t_0, A_c)/\partial t_0 = 0,$$

while

$$\left.\frac{\partial M}{\partial A}\right|_{A_c} \neq 0. \tag{9}$$

Thus A_c corresponds to a bifurcation value at which homoclinic tangency occurs. It has been shown by Greenspan and Holmes (1981) that the homoclinic bifurcation is the limit of a countable sequence of subharmonic bifurcations that take place inside the separatrix.

3. Numerical study

For investigating the behaviour of the system considered in §2 numerically, we integrated (1) using fourth order Runge-Kutta-Gills Scheme. There are three control

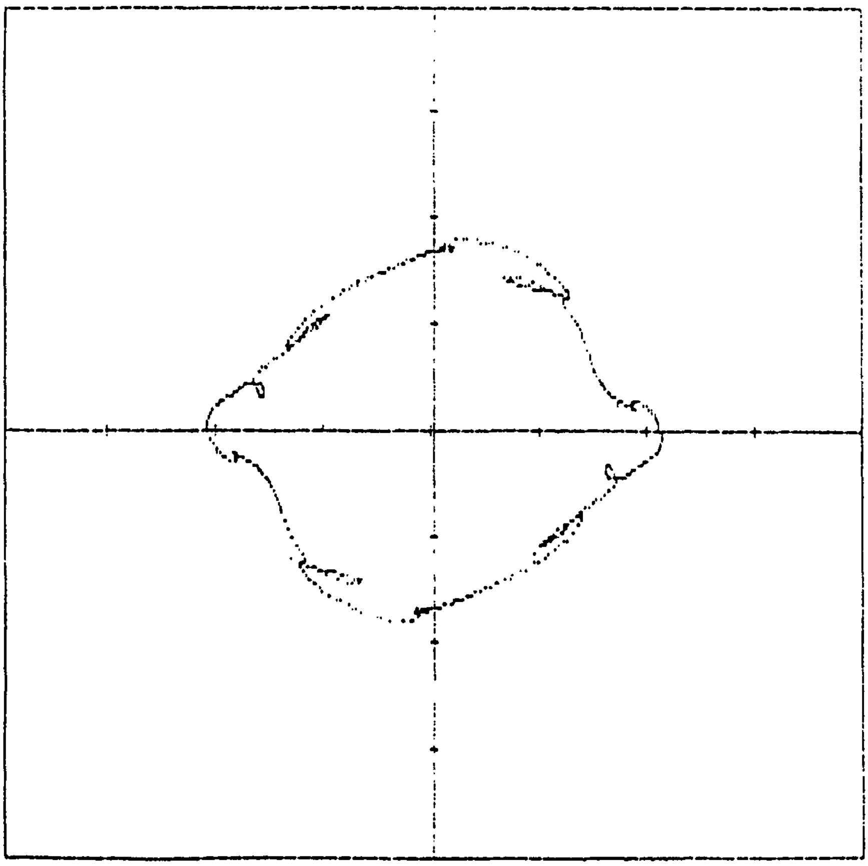

Figure 1. Phase portrait of the system in equation (1) corresponding to $\omega = 0{\cdot}04$, $\beta = 0{\cdot}2$ and $A = 0{\cdot}1$.

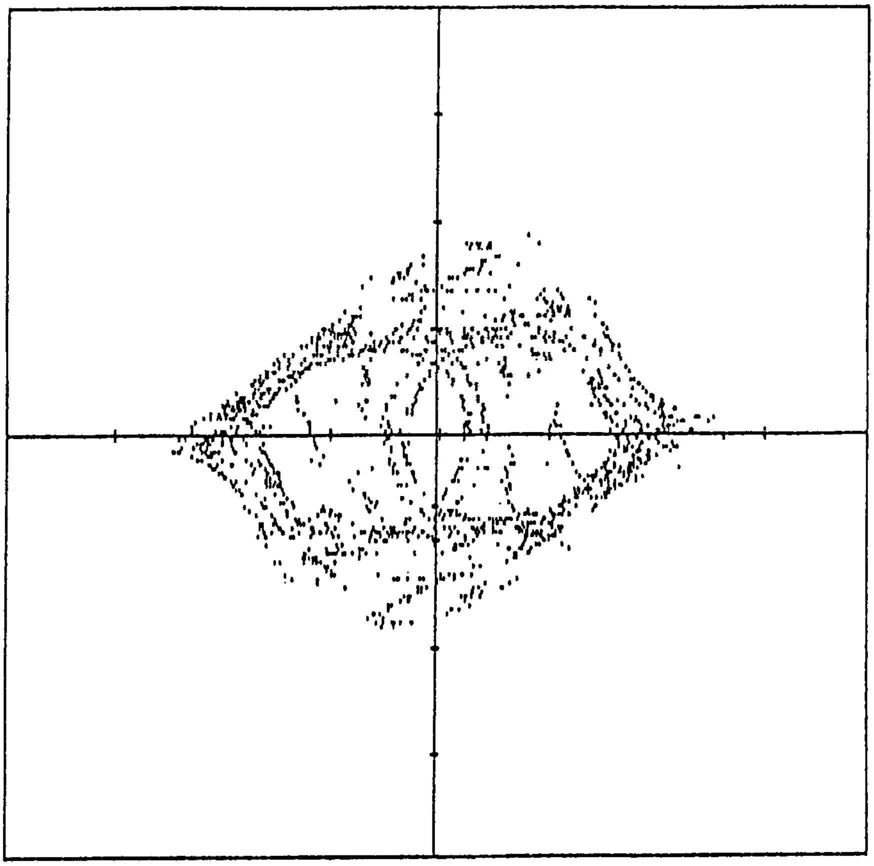

Figure 2a. Phase portrait of the chaotic attractor existing for $\omega = 0{\cdot}04$, $\beta = 0{\cdot}2$ and $A = 0{\cdot}6$.

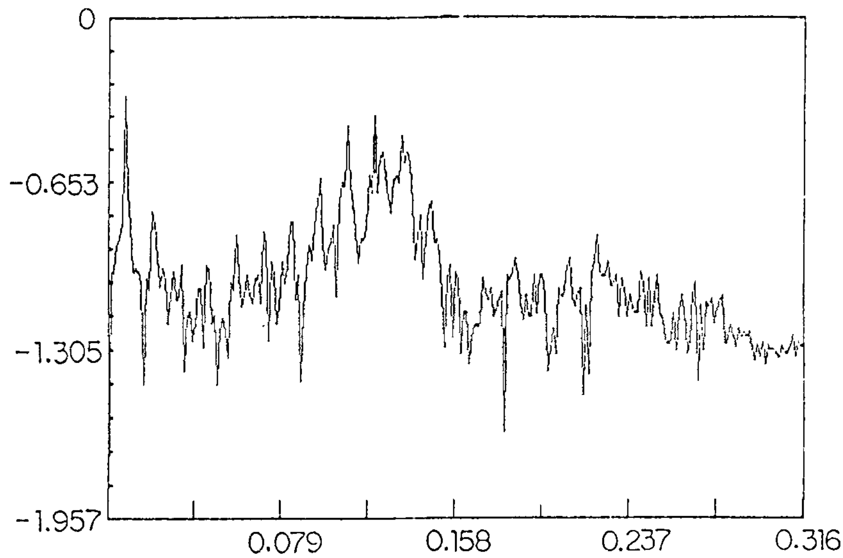

Figure 2b. Power spectrum using FFT of the chaotic attractor given in figure 2a. Log(power) is plotted along the y-axis with frequency along the x-axis.

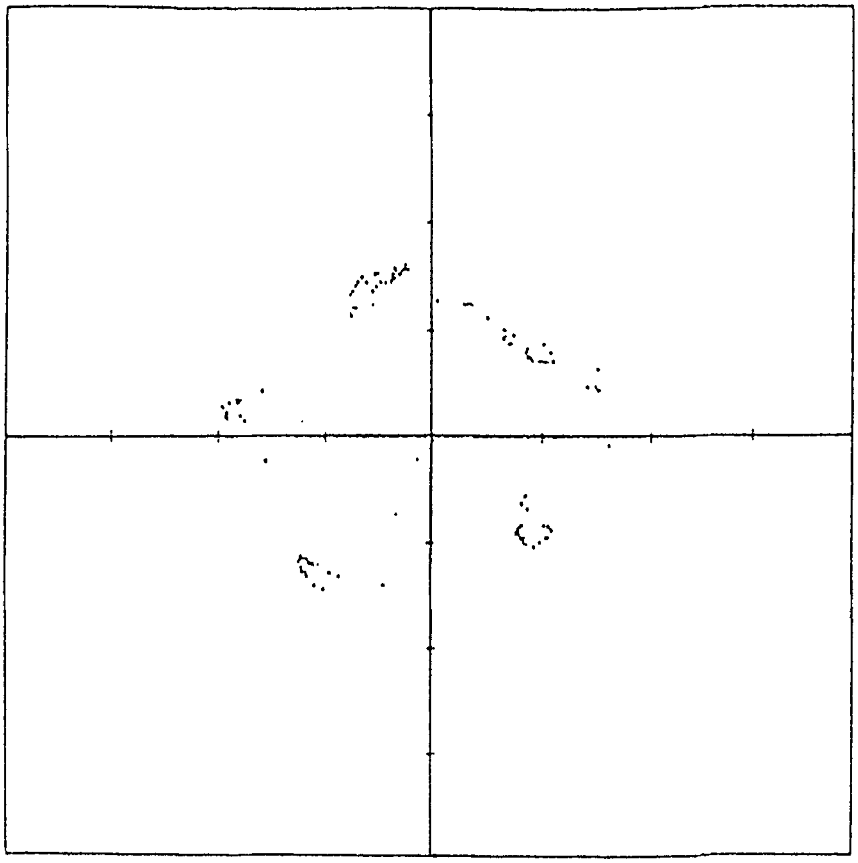

Figure 2c. The Poincaré map of the chaotic attractor in figure 2a.

parameters, namely, the damping constant β, the driving amplitude A and the driving frequency ω. In our studies we kept β mostly at 0·2 while A and ω are varied, over a wide range. The prominent features of our investigations are summarized below.

Because of the nonlinear dissipative term, the system would always enter into a limit cycle behaviour, which makes it difficult to trace the region near the separatrix. However for small values of ω, the phase portraits below A_c show periodic orbits near the separatrix. Thus when $\omega = 0{\cdot}04$, $\beta = 0{\cdot}2$, $A_c \approx 0{\cdot}22$. Figure 1 shows the phase portrait for $A = 0{\cdot}1$. The Poincaré map corresponding to this reveals a periodic 5-cycle. The system becomes chaotic for $A > 0{\cdot}25$. The chaotic attractor for $A = 0{\cdot}6$ and the corresponding power spectrum using fast fourier transform (FFT) are shown in figure 2. The Poincaré map at this value is a set of points shown in figure 2c. The maximum Lyapunov exponent σ_{max} is computed for these values of A using the Benettin Scheme (Benettin *et al* 1976). It is found that $\sigma_{max} = -2{\cdot}5 \times 10^{-2}$ for $A = 0{\cdot}1$ while it is $9{\cdot}8 \times 10^{-2}$ for $A = 0{\cdot}6$. This confirms the existence of a chaotic attractor for $A = 0{\cdot}6$. Qualitatively the same type of behaviour exists for frequencies upto 0·06.

For values of ω lying in the range, $0{\cdot}08 < \omega < 1$, we find that the system shows a tendency towards the formation of a thick or band-like limit cycle. For a pendulum with a dissipation of the usual type i.e. $\beta\dot{x}$, the limit cycle forms the stable attractor of the system. However, for the system considered here, since the dissipation has a quadratic

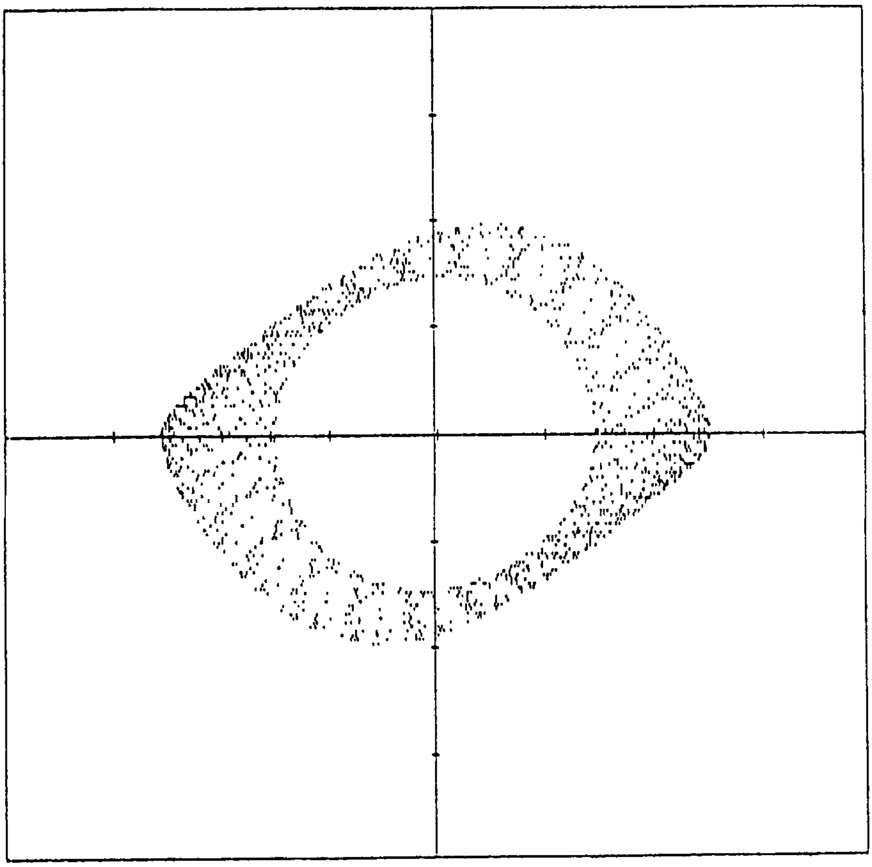

Figure 3. Band-like limit cycle for $\omega = 0{\cdot}4$, $\beta = 0{\cdot}2$ and $A = 0{\cdot}2$.

dependence on x, the system has to adjust continuously as x changes along the trajectory and this repeated attempt to approach a limit cycle leads to a thick band-like cycle. We observe that this occurs below the Melnikov criterion, but near escape from the potential well. Thus for $\omega = 0{\cdot}4$ the band exists for A values from 0·1 to 0·28 figure 3 shows the band in the phase portrait at $A = 0{\cdot}2$.

The FFT analysis for these values reveals some hidden periodicity inside the band. For $A = 0{\cdot}2$, the FFT shows only four fundamental modes while at $A = 0{\cdot}21$, a period-doubling takes place producing peaks on either side of the original modes. Two more period-doubling are observed for $A = 0{\cdot}217$ and 0·218. This sequence accumulates near $A \approx 0{\cdot}27$. This accumulation point should correspond to homoclinic bifurcation value. Using (8) this works out to be $A_c \approx 0{\cdot}2667$, very close to the value obtained using the FFT analysis.

For slightly higher values of A i.e. $A > 0{\cdot}27$, the trajectory extends from $-\pi$ to $+\pi$ spending most of the time near $\pm\pi$. When A increases beyond 0·29, the system either jumps to a nearby fixed point or diverges to ∞. For higher values of ω, the same type of behaviour is generally observed. However at $\omega = 1$, the band which exists upto $A = 0{\cdot}2$ splits into periodic trajectories. Thus at $A = 0{\cdot}3$, we observe asymptotically a periodic

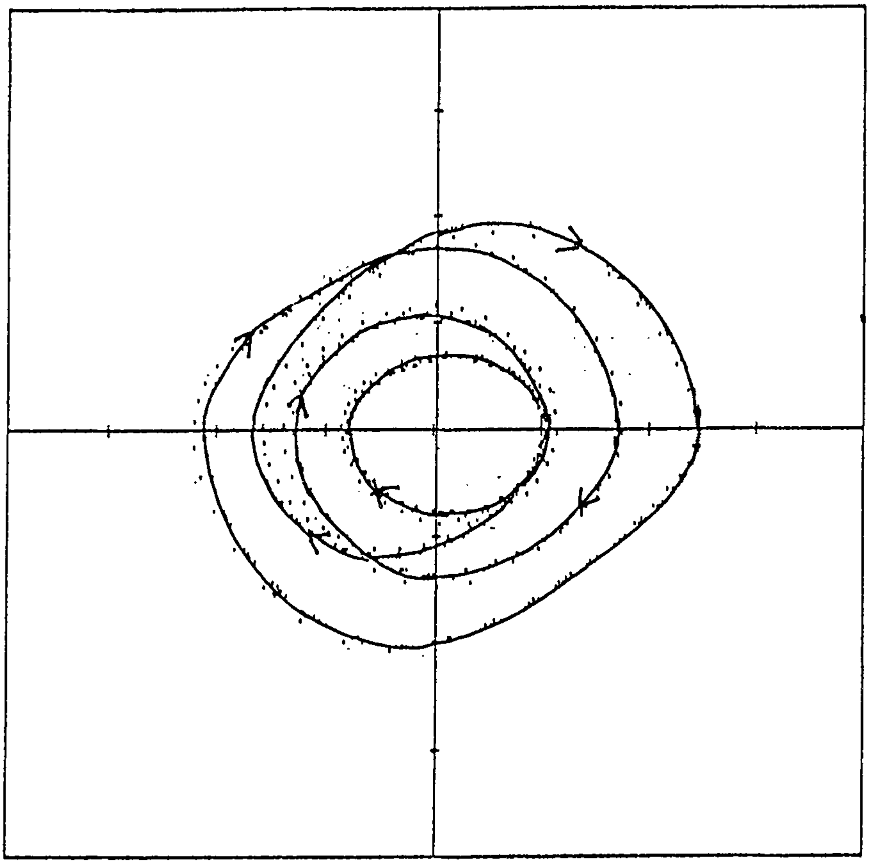

Figure 4. Asymptotic four-cycle observed at $\beta = 0{\cdot}2$, $\omega = 1$ and $A = 0{\cdot}3$.

four cycle. This is clear from figure 4. When A is increased further, the system tends to a limit cycle with the same periodicity as the external force. We observe some transient chaos in this region before the trajectory settles down to the one cycle.

4. Concluding remarks

The van der Pol pendulum described in this paper is found to exhibit a variety of interesting dynamical phenomena. There exists a chaotic attractor above the Melnikov threshold at low frequencies but the system shows a strong tendency towards band formation at higher frequencies. Before escaping from the potential well, a sequence of period-doublings takes place inside the band.

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