Use of Padé approximants in the evaluation of $\alpha$ and $\delta$ for one-dimensional maps

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Abstract. Recently an analytic algorithm for evaluating the Feigenbaum indices of one-dimensional maps was developed using a perturbative expansion. We find that the use of Padé approximants in the resulting asymptotic series, significantly improves the technique.

Keywords. Feigenbaum scenario; universal constants; asymptotic series; Padé approximants.
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1. Introduction

It is well known that most nonlinear dissipative dynamical systems follow the period doubling route to irregular or chaotic behaviour. Such systems can be modelled by one-dimensional maps of the form,

$$x_{n+1} = f_a(x_n) = 1 - a|x_n|^2$$

in the interval $(-1, 1)$. Their transition to chaos is characterized by two universal indices $\alpha$ and $\delta$ (Feigenbaum 1980); The scale factor $\alpha$ is defined as

$$\alpha = \lim_{n \to \infty} \frac{\Delta x_n}{\Delta x_{n+1}},$$

where $\Delta x_n$ is half the separation between the fixed points in the $2^n$-cycle and the rate of accumulation of bifurcations

$$\delta = \lim_{n \to \infty} \frac{a_{n+1} - a_n}{a_{n+2} - a_{n+1}}.$$ 

At $a = a_c$, $f^{2^n}(0)$ approaches zero geometrically as $1/\alpha^n$ where $f^{2^n}$ represents $2^n$ iterations of $f$. It has been established by Feigenbaum that $f^{2^n}$ in the neighbourhood of $x = 0$ which is the extremum of the map, asymptotically approaches a universal function $g(x)$ (Feigenbaum 1983). The function $g(x)$ is the fixed point function of the doubling and rescaling operator defined by the relation (Feigenbaum 1978),

$$g(x) = -\alpha g(g(x|x)).$$

$$501$$
Considering small perturbations in the neighbourhood of \( g(x) \) we write

\[
g(x) = g(x) + \varepsilon h(x).
\]  

(5)

The function \( h(x) \) satisfies the equation (Feigenbaum 1979)

\[
- \alpha [g'(g(x)) h(x) + h(g(x))] = \delta h(x).
\]  

(6)

\( \delta > 1 \) represents the only unstable direction of \( g(x) \). The two functional renormalization group equations (4) and (6) can be solved simultaneously for \( \alpha, \delta, g(x) \) and \( h(x) \). For this an analytic algorithm based on a perturbative scheme has been developed recently (Virendra Singh 1985; Ambika and Babu Joseph 1986). The essence of the method is briefly discussed in §2.

In this paper we discuss the use of Padé approximants in improving the convergence of the perturbative series. The series that occur in the expression for \( \alpha \) as well as the equation for \( \delta \) are asymptotic in nature. So there is a definite advantage in replacing them by their Padé approximants of the required order. The details of these calculations are given in §3. We take up three specific examples with \( z = 2, 4 \) and 6. In all the three cases the values of \( \alpha \) and \( \delta \) are evaluated using Padé approximants of different orders. It is interesting to note how fast the calculated values converge as higher and higher orders are considered. Our concluding remarks are given in §4.

2. Perturbative scheme

The universal function \( g(x) \) can be written as a power series

\[
g(x) = 1 + \sum_{n=1}^{\infty} P_n |x|^n
\]  

(7)

with the normalization \( g(0) = 1 \). In the neighbourhood of the extremum at \( x = 0 \), \( g(x) \) is positive for any \( z \) and so \( g(g(x)) \) can also be expanded into a similar power series

\[
g(g(x)) = 1 + \sum_{r=1}^{\infty} P_r + \left( P_1 \sum_{r=1}^{\infty} rz P_r \right) |x|^z
\]  

\[
+ \left( P_2 \sum_{r=1}^{\infty} rz P_r + P_1^2 \sum_{r=1}^{\infty} \frac{rz(rz - 1)}{2!} P_r \right) |x|^{2z} + ....
\]  

(8)

We find that the perturbative scheme depends on proper redefinition of the coefficients \( P_n \) occurring in (7) and (8) as

\[
P_n \alpha^n = S_n |x|^n.
\]  

(9)

Using (7), (8) and (9) in (4) and equating coefficients of \( |x|^n \) on both sides, we get

\[
\frac{1}{\alpha} + 1 + |x|^z \sum_{r=1}^{\infty} \frac{S_r}{\alpha^r} = 0,
\]  

(10)

\[
\frac{1}{\alpha} + \sum_{r=1}^{\infty} rS_r \alpha^{r-1} = 0,
\]  

(11)
Padé approximants to evaluate $\alpha$ and $\delta$

\[
S_n \left[ 1 - \frac{1}{|\alpha|^{(n-1)}} \right] + \sum_{l=2}^{n} \sum_{r=1}^{\infty} \left( rz^l \right) \frac{S_r}{\alpha^{l-1}}
\times \sum_{m_1 > 1, m_2 > 1 \ldots, m_l > 1} \frac{S_{m_1}S_{m_2}\ldots S_{m_l} \delta_{m_1 + m_2 + \ldots + m_l, n}}{|\alpha|^{(n-l)}}
= 0 \quad \text{(for } n = 2, 3, 4 \ldots \text{).} \quad (12)
\]

To solve this set of coupled nonlinear equations, the coefficients $S_n$ are expanded in inverse powers of $\alpha$

\[
S_n(\alpha) = \sum_{m=0}^{\infty} \frac{S_{nm}}{\alpha^m}. \quad (13)
\]

When (13) is put back in (11) and (12) we get a hierarchy of equations to be solved successively for the coefficients $S_{nm}$. Using these in (10), we get the equation for $\alpha$ as

\[
\frac{1}{\alpha} + 1 + |\alpha|^2 \sum_{r=1}^{\infty} \sum_{m=0}^{\infty} \frac{S_{rm}}{\alpha^{r+m}} = 0. \quad (14)
\]

The function $h(x)$ can also be expanded into a power series:

\[
h(x) = 1 + \sum_{n=1}^{\infty} h_n |x|^n. \quad (15)
\]

Using (15), (7), (9) and (13) in (6) and equating the coefficients of $|x|^n$, we get the system of equations to be solved for $\delta$.

\[
-\alpha \left[ 1 + \sum_{r=1}^{\infty} h_r + |\alpha|^2 \sum_{r=1}^{\infty} \frac{rzS_r}{\alpha^r} \right] = \delta, \quad (16)
\]

\[
-\alpha \left[ \sum_{r=1}^{\infty} \left( \frac{rz}{1} \right) h_r \frac{S_1}{\alpha} + \sum_{r=1}^{\infty} 2 \left( \frac{rz}{2} \right) \frac{|\alpha|^{r-1} S_r S_1}{\alpha^r} \right.
+ h_1 \sum_{r=1}^{\infty} \left( \frac{rz}{1} \right) \frac{S_r}{\alpha^r} = \delta h_1, \quad \text{(for } n = 1 \text{)} \quad (17)
\]

and

\[
-\alpha \left[ \sum_{l=1}^{n} \sum_{r=1}^{\infty} \left( \frac{rz}{l} \right) h_r \frac{S_{l+1}}{\alpha^{l+1}} + (l+1) \left( \frac{rz}{l+1} \right) \frac{S_r}{\alpha^{l+1-r}} \right]
\times \sum_{m_1 > 1, m_2 > 1 \ldots, m_n > 1} \frac{S_{m_1}S_{m_2}\ldots S_{m_n} \delta_{m_1 + m_2 + \ldots + m_n, n}}{|\alpha|^{(n-l)}}
+ \sum_{n' > 1} \sum_{l' > 1} \sum_{r' > 1} h_{n'} (l+1) \left( \frac{rz}{l+1} \right) \frac{S_{r'}}{|\alpha|^{n'-r'}}
\times \sum_{m_1 > 1, m_2 > 1 \ldots, m_{n'} > 1} \frac{S_{m_1}S_{m_2}\ldots S_{m_{n'}} \delta_{m_1 + m_2 + \ldots + m_{n'}, n'}}{|\alpha|^{(n'-l')}}
+ h_n \sum_{r=1}^{\infty} \frac{rzS_r}{\alpha^{r+(n-1)e}} = \delta h_n \quad \text{(for } n = 2, 3, 4 \ldots \text{).} \quad (18)
\]
This set could be written in the form of a matrix eigenvalue equation:

\[ Dh = \delta h. \]  

(19)

Using the expansion in (9) and the already determined coefficients \( S_{nm} \), the elements \( D_{ij} \) can be computed. The detailed forms of the matrix elements \( D_{ij} \) for a given \( z \) are given in the Appendix. The largest real eigenvalue of \( D \) furnishes the relevant \( \delta \).

The \( S_{nm} \) coefficients as well as the constant \( \alpha \) for a quadratic map were calculated by Singh (1985). The equations for \( \delta \) were obtained as described above by Ambika and Babu Joseph (1986) who evaluated \( \alpha \) and \( \delta \) for a quartic map.

3. Use of Padé approximants

It has been mentioned (Ambika and Babu Joseph 1988) that the series in \( 1/\alpha \) appearing in the equation for \( \alpha \) in (14) as well as those in the matrix elements \( D_{ij} \) are asymptotic. Hence the truncation of the series is very crucial in deciding the accuracy of the result. We find that including higher powers of \( 1/\alpha \) results in fluctuations of the value about the numerical value. The accuracy as well as the convergence can be improved by replacing the series by its Padé approximant \([L|M]\) (Baker 1975). Thus the equation for \( \alpha \) is written as

\[ \frac{1}{\alpha} + \sum_{j=0}^{L} P_{j} \left( \frac{1}{\alpha} \right)^{j} = 0 \]  

where

\[ [L|M] = \frac{\sum_{j=0}^{L} P_{j} \left( \frac{1}{\alpha} \right)^{j}}{\sum_{j=0}^{M} Q_{j} \left( \frac{1}{\alpha} \right)^{j}} \]  

(20)

with \( Q_{0} = 1 \). Equating the series in (14) to (21), the coefficients \( P_{j} \) and \( Q_{j} \) are evaluated to any required order. Then (20) is solved using the Newton-Raphson scheme. It is clear from Appendix A that elements \( D_{ij} \) can be written as a power series in \( 1/\alpha \) and the corresponding Padé approximants can be calculated to different orders in their places. Thus,

\[ D_{11} = -\alpha - |\alpha|^{z+1}[L|M]_{11}, \]

\[ D_{21} = -|\alpha|^{z-1}[L|M]_{21}, \]  

etc.  

(22)

Using the resulting matrix \( D \), the eigenvalue \( \delta \) is calculated.

We illustrate the efficiency of this scheme by considering three specific cases corresponding to \( z = 2, 4 \) and 6. The \( S_{nm} \) coefficients of the three maps are given in tables 1 to 3 while the values of \( \alpha \) calculated using (20) for different orders of Padé approximants are given in table 1.

The series in \( 1/\alpha \) occurring in \( D_{ij} \) are replaced by the corresponding \([L|M]\) Padé approximants for \( z = 2, 4 \) and 6. We consider terms up to \( D_{44} \) and the largest eigenvalue of the resulting \( 4 \times 4 \) matrix is computed. This is done for different orders of \([L|M]\) and the results are given in table 5. The convergence of \( \delta \) to the numerical value is evident.
Padé approximants to evaluate $\alpha$ and $\delta$

### Table 1. $S_{mn}$ coefficients for $z = 2.$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.5</td>
<td>0.25</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.125</td>
<td>0.781</td>
</tr>
<tr>
<td>2</td>
<td>0.125</td>
<td>0</td>
<td>0.03125</td>
<td>0</td>
<td>0.359375</td>
<td>0.265625</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>0.0000000</td>
<td>0.0937500</td>
<td>0.2187500</td>
<td>0.1796875</td>
<td>4</td>
<td>0.0000000</td>
<td>0.0078125</td>
<td>0.0156250</td>
</tr>
<tr>
<td>4</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>5</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>5</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>7</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
</tbody>
</table>

### Table 2. $S_{mn}$ coefficients for $z = 4.$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.25</td>
<td>0.1875</td>
<td>0.046875</td>
<td>0.0039062</td>
<td>0.0000000</td>
<td>0.1875000</td>
<td>0.0298461</td>
</tr>
<tr>
<td>2</td>
<td>0.0937500</td>
<td>0.0468750</td>
<td>0.0527343</td>
<td>0.1171875</td>
<td>0.0911865</td>
<td>0.1835625</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>0.0156250</td>
<td>0.0351562</td>
<td>0.1074218</td>
<td>0.1855468</td>
<td>0.2403564</td>
<td>4</td>
<td>0.0009765</td>
<td>0.0219765</td>
</tr>
<tr>
<td>4</td>
<td>0.0009765</td>
<td>0.0219765</td>
<td>0.0531005</td>
<td>0.1071166</td>
<td>5</td>
<td>0.0000000</td>
<td>0.0051269</td>
<td>0.0097045</td>
</tr>
<tr>
<td>5</td>
<td>0.0000000</td>
<td>0.0051269</td>
<td>0.0097045</td>
<td>7</td>
<td>0.0000000</td>
<td>0.0006408</td>
<td>8</td>
<td>9</td>
</tr>
</tbody>
</table>

### Table 3. $S_{mn}$ coefficients for $z = 6.$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1666667</td>
<td>0.1388888</td>
<td>0.0462962</td>
<td>0.0077160</td>
<td>0.0006430</td>
<td>0.5390303</td>
<td>0.4311199</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0694444</td>
<td>0.4629200</td>
<td>0.0289351</td>
<td>0.0887345</td>
<td>0.2055469</td>
<td>0.0277543</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>0.0154320</td>
<td>0.0192901</td>
<td>0.0482335</td>
<td>0.0180755</td>
<td>0.0777722</td>
<td>4</td>
<td>0.0019290</td>
<td>0.0184863</td>
</tr>
<tr>
<td>4</td>
<td>0.0019290</td>
<td>0.0184863</td>
<td>0.0535836</td>
<td>0.0777722</td>
<td>5</td>
<td>0.0001260</td>
<td>0.0064296</td>
<td>0.0271824</td>
</tr>
<tr>
<td>5</td>
<td>0.0000035</td>
<td>0.0013544</td>
<td>0.0000000</td>
<td>7</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>8</td>
</tr>
</tbody>
</table>

However, to get an idea about the size of the region in which $P_M^L$ is a good approximation to the series, we compute the singularities of $P_M^L$ in the complex plane, for diagonal Padé approximants. The distance of the nearest pole from the origin corresponding to $z = 2, 4$ and $6$ are given in Table 6.

### 4. Concluding remarks

We find that the convergence of the $\alpha$ and $\delta$ values improves significantly when the asymptotic series are replaced by the corresponding Padé approximants. The
Table 4. $a$ values for $z = 2, 4$ and 6 using Padé approximants. The numerical values are also given for comparison (Mendes 1981).

<table>
<thead>
<tr>
<th>Padé approximants</th>
<th>$z$ values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[L/M]$</td>
<td>2</td>
</tr>
<tr>
<td>[1 1 1]</td>
<td>2.538017</td>
</tr>
<tr>
<td>[2 1 1]</td>
<td>2.923075</td>
</tr>
<tr>
<td>[2 2 1]</td>
<td>2.490066</td>
</tr>
<tr>
<td>[3 1 1]</td>
<td>2.318146</td>
</tr>
<tr>
<td>[3 2 1]</td>
<td>2.461020</td>
</tr>
<tr>
<td>[3 3 1]</td>
<td>2.523401</td>
</tr>
<tr>
<td>[4 4 1]</td>
<td>2.492727</td>
</tr>
<tr>
<td>[5 5 1]</td>
<td>2.494229</td>
</tr>
<tr>
<td>Numerical values</td>
<td>2.502</td>
</tr>
</tbody>
</table>

Table 5. $b$ values for $z = 2, 4$ and 6 using Padé approximants. The numerical values are given for comparison (Mendes 1981).

<table>
<thead>
<tr>
<th>Padé approximants</th>
<th>$z$ values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[L/M]$</td>
<td>2</td>
</tr>
<tr>
<td>[1 1 1]</td>
<td>4.640784</td>
</tr>
<tr>
<td>[2 1 1]</td>
<td>4.833893</td>
</tr>
<tr>
<td>[2 2 1]</td>
<td>4.773389</td>
</tr>
<tr>
<td>[3 1 1]</td>
<td>4.261563</td>
</tr>
<tr>
<td>[3 2 1]</td>
<td>4.581424</td>
</tr>
<tr>
<td>[3 3 1]</td>
<td>4.669775</td>
</tr>
<tr>
<td>[4 4 1]</td>
<td>4.664557</td>
</tr>
<tr>
<td>Numerical value</td>
<td>4.669</td>
</tr>
</tbody>
</table>

Table 6. Distance to nearest pole from the origin $R_\mu^M$ in the complex plane of diagonal Padé approximants $P_\mu^M$ corresponding to $z = 2, 4$ and 6.

<table>
<thead>
<tr>
<th>$P_\mu^M$</th>
<th>$z = 2$</th>
<th>$z = 4$</th>
<th>$z = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1 1 1]</td>
<td>No poles</td>
<td>0.166667</td>
<td>0.222217</td>
</tr>
<tr>
<td>[2 2 1]</td>
<td>1.118034</td>
<td>0.361249</td>
<td>0.319311</td>
</tr>
<tr>
<td>[3 3 1]</td>
<td>1.226683</td>
<td>0.811874</td>
<td>0.815977</td>
</tr>
<tr>
<td>[4 4 1]</td>
<td>1.326701</td>
<td>1.086010</td>
<td>1.068047</td>
</tr>
<tr>
<td>[5 5 1]</td>
<td>1.337311</td>
<td>1.155285</td>
<td>1.098317</td>
</tr>
</tbody>
</table>
Padé approximants to evaluate $\alpha$ and $\delta$

Calculations are continued only up to $[5|5]$ approximant, but even at this stage, the convergence is really good. It is clear from Table 6 that the poles are shifting as higher and higher orders are considered. Moreover, we find that the root of the numerator of $P_M^L$ is cancelled by the root of the denominator for $L = 5$ and $M = 5$. This shows the corresponding Padé approximant is a good replacement for the series. Thus the perturbative scheme together with the Padé method, forms a reliable analytic tool for computing the $\alpha$ and $\delta$ values for one-dimensional maps.

Acknowledgement

We wish to express our sincere thanks to Prof. K Babu Joseph for his valuable guidance and fruitful discussions.

Appendix

For a given $z$ the elements of matrix $D$ computed using the set of equations (16), (17) and (18) and expansion given in (13) are

$$
D_{11} = - \left[ \alpha + z\alpha^2 \left( S_{10} + \frac{1}{\alpha} (S_{11} + 2S_{20}) + \frac{1}{\alpha^2} (S_{12} + 2S_{21} + 3S_{30}) 
+ \frac{1}{\alpha^3} (S_{13} + 2S_{22} + 3S_{31} + 4S_{40}) + \frac{1}{\alpha^4} (S_{14} + 2S_{23} + 3S_{32} 
+ 4S_{41} + 5S_{50}) + \frac{1}{\alpha^5} (S_{15} + 2S_{24} + 3S_{33} + 4S_{42} + 5S_{51} + 6S_{60}) 
+ \frac{1}{\alpha^6} (S_{16} + 2S_{25} + 3S_{34} + 4S_{43} + 5S_{52} + 6S_{61} + 7S_{70}) \right) \right]
$$

$$
D_{12} = - \alpha,
$$

$$
D_{13} = - \alpha,
$$

$$
D_{14} = - \alpha,
$$

$$
D_{21} = - z\alpha^{2-1} \left[ (z - 1)S_{10}^2 + \frac{1}{\alpha} ((z - 1)2S_{10}S_{11} + (2z - 1) 
\times 2S_{20}S_{10} + \frac{1}{\alpha^2} ((z - 1)(2S_{10}S_{12} + S_{11}^2) + (2z - 1) 
\times (2S_{20}S_{11} + 2S_{10}S_{21}) + (3z - 1)3S_{30}S_{10}) 
+ \frac{1}{\alpha^3} ((z - 1)(2S_{10}S_{13} + 2S_{11}S_{12}) + (2z - 1) 
\times (2S_{12}S_{20} + 2S_{21}S_{11} + 2S_{22}S_{10}) + (3z - 1) 
\times (3S_{30}S_{11} + 3S_{31}S_{10}) + (4z - 1)4S_{40}S_{10} \right) \right] \right]
$$
\[
D_{22} = -z\left[2S_{10} + \frac{1}{x}(2S_{11} + 2S_{20}) + \frac{1}{x^2}(2S_{12} + 2S_{21} + 3S_{30})\right],
\]
\[
D_{23} = -2z\left[S_{10} + \frac{S_{11}}{x} + \frac{S_{12}}{x^2}\right],
\]
\[
D_{24} = -3z\left[S_{10} + \frac{S_{11}}{x} + \frac{S_{12}}{x^2}\right],
\]
\[
D_{31} = -\frac{z(z-1)}{x^2}S_{20}S_{10} - z^2x^{-2}\left[\left(\frac{(z-1)(z-2)}{2!}S_{10}^{3} + \frac{1}{x}\right)S_{10}^{3} + \frac{1}{x}\right]
\times \left\{\frac{(z-1)(z-2)}{2!}3S_{10}^{2}S_{11} + \frac{(2z-1)(2z-2)}{2!}2S_{20}S_{10}^{2}\right\}
\times 3S_{10}S_{13} + 6S_{10}S_{11}S_{12} + S_{11}^{3}) + \frac{(z-1)(z-2)}{2!}
\times 2 \times (2S_{20}S_{10}S_{12} + S_{20}S_{11} + 2S_{21}S_{10}S_{11} + S_{22}S_{10}^{2})
\times 3 \times (2S_{30}S_{10}S_{11} + S_{31}S_{10}^{2})
\times \left\{\frac{(z-1)(z-2)}{2!}4S_{40}S_{10}^{2}\right\} + \frac{1}{x^3}\left\{\frac{(z-1)(z-2)}{2!}\right\}
\times 3S_{10}S_{14} + 6S_{10}S_{11}S_{13} + 3S_{11}^{2}S_{12} + 3S_{10}S_{12}^{2})
\times \left\{\frac{(2z-1)(2z-2)}{2!}\right\} \times 2 \times (2S_{20}S_{10}S_{13} + 2S_{20}S_{11}S_{12}
+ 2S_{21}S_{10}S_{12} + 2S_{21}S_{11}^{2} + 2S_{22}S_{10}S_{11} + S_{23}S_{10}^{2})
\times \left\{\frac{(3z-1)(3z-2)}{2!}\right\} \times 3 \times (2S_{30}S_{10}S_{12} + S_{30}S_{12}^{2})
\]
Padé approximants to evaluate $\alpha$ and $\delta$

\[ + 2S_{31}S_{10}S_{11} + S_{32}S_{10}^2 + \frac{(4z - 1)(4z - 2)}{2!} \times 4 \]

\[ \times (2S_{40}S_{10}S_{11} + S_{41}S_{10}^2) + \frac{(5z - 1)(5z - 2)}{2!} 5S_{50}S_{10}^2 \] \] \],

\[ D_{32} = - \left[ \frac{z(z - 1)}{2!} S_{10}^2 + \frac{1}{\alpha^2} \{ z(z - 1)zS_{10}^2 \} + \frac{1}{\alpha^2} \{ z(z - 1)z \} 2S_{10}S_{11} \]

\[ + z(z - 1)2S_{10}S_{11} + 2z(2z - 1)S_{20}S_{10} \] \],

\[ D_{33} = - \left[ \frac{2z(2z - 1)}{2!} S_{10}^2 + \frac{1}{\alpha^2} \{ 2z(2z - 1) \} 2S_{10}S_{11} \] \],

\[ D_{34} = - \left[ \frac{3z(3z - 1)}{2!} \left( S_{10}^2 + \frac{2S_{10}S_{11}}{\alpha^2} \right) \right] \],

\[ D_{41} = - az^2 \left[ \frac{(z - 1)(z - 2)(z - 3)}{3!} S_{10}^3 \alpha + \frac{1}{\alpha^2} \left\{ 4S_{10}S_{11} \right\} \]

\[ \times \frac{(z - 1)(z - 2)(z - 3) + (z - 1)(2z - 2)(z - 3)(3z - 1)}{3!} - 2S_{20}S_{10}^3 \]

\[ + \frac{1}{\alpha^2} \left\{ (z - 1)(z - 2)(z - 3) \right\} - 2(4S_{10}S_{12} + 6S_{10}^2S_{11}) \]

\[ + \frac{(2z - 1)(2z - 2)(z - 3)}{3!} \times 2 \times [3S_{20}S_{10}^2S_{11} + S_{21}S_{10}^3] \]

\[ + (3z - 1)(3z - 2)(3z - 3) \times 3S_{30}S_{10}^3 \]

\[ + \frac{1}{\alpha^2} \left\{ (z - 1)(z - 2)(z - 3) \right\} \times 3! \]

\[ \times (4S_{10}S_{13} + 12S_{10}^2S_{11}S_{12} + 4S_{10}S_{11}^3) \]

\[ + \frac{(2z - 1)(2z - 2)(z - 3)}{3!} \times 2 \times (3S_{20}S_{10}^2S_{12} + 3S_{20}S_{10}S_{11}^3) \]

\[ + 3S_{21}S_{10}^2S_{11} + S_{22}S_{10}^3 \]

\[ + \frac{(3z - 1)(3z - 2)(3z - 3)}{3!} \times 3 \]

\[ \times (3S_{30}S_{10}^2S_{11} + S_{31}S_{10}^3) + \frac{(4z - 1)(4z - 2)(4z - 3)}{3!} \]

\[ \times 4S_{40}S_{10}^3 \] \],

\[ D_{42} = - \left[ \frac{z(z - 1)(z - 2)}{3!} S_{10}^3 + \frac{z(z - 1)(z - 2)}{2!} S_{10} S_{11} \right] \],

\[ D_{43} = - \left[ \frac{2z(2z - 1)(2z - 2)}{3!} S_{10}^3 \alpha \right] \],

\[ D_{44} = - \left[ \frac{3z(3z - 1)(3z - 2)}{3!} S_{10}^3 \right] \].
References

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