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Circular orbits of an electron around a proton in Schwarzschild geometry

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Abstract. The energy of an electron in circular orbits around a proton in Schwarzschild geometry has been investigated and is found to be red shifted. The electrical dipole moment of such a system is also estimated.

Keywords. Charged particle orbits; general relativity; Schwarzschild geometry.

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1. Introduction

Charged particle orbits in the field of a static charge near a Schwarzschild black hole have been recently discussed by Sonar *et al* (1985) and Chellathurai *et al* (1986). They found that the charged particles can execute circular orbits around the axis joining the static charge and the black hole. In this paper we apply the same formalism to a system of a static proton and an electron-executing circular orbits in the gravitational field described by Schwarzschild metric.

2. Calculation

The background geometry is given by

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)c^2\,\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\varphi^2, \qquad (1)$$

wherein $m = MG/c^2$, M being the mass of the black hole and other symbols having usual meaning. The electric field of the proton stationary at $r = b, \theta = 0$ can be obtained from the four-potential

$$A_i = (A_t, 0, 0, 0), \qquad (2)$$

where

$$A_t = \frac{e[(r-m)(b-m) - m^2\cos\theta]}{br[(r-m)^2 + (b-m)^2 - m^2 - 2(r-m)(b-m)\cos\theta + m^2\cos^2\theta]^{1/2}} + \frac{em}{br}, \qquad (3)$$

where e is the charge of the proton.

The relevant equations of motion (Chellathurai *et al* 1986) describing the orbits of the electron are

$$\rho^2 \sin^2\theta\dot{\varphi} = L, \tag{4}$$

$$(1-2/\rho)\dot{\tau} = E - \lambda A_\tau, \tag{5}$$

$$\rho \sin^2\theta\dot{\varphi}^2 - \dot{\tau}^2/\rho^2 = \lambda A_{\tau,\rho}\dot{\tau}, \tag{6}$$

$$\sin\theta\cos\theta\dot{\varphi}^2 = \frac{\lambda}{\rho^2}A_{\tau,\theta}\dot{\tau}, \tag{7}$$

$$1 = (1-2/\rho)\dot{\tau}^2 - \rho^2\sin^2\theta\dot{\varphi}^2, \tag{8}$$

where we have used the dimensionless parameters $\rho = r/m, \sigma = s/m, \tau = ct/m, \beta = b/m$, $L = l/mm_0c, E = E'/m_0c^2, \lambda = -e^2/mm_0c^2$ and $A_\tau = mA_t/e$ with m_0, l, E' are rest mass, angular momentum and the energy of the electron in c.g.s. units. The dots overhead denote differentiation with respect to σ. In dimensionless from A_τ is given by

$$A_\tau = \frac{B}{\beta\rho A^{1/2}} + \frac{1}{\beta\rho}, \tag{9}$$

where

$$A = (\rho-1)^2 + (\beta-1)^2 - 1 - 2(\rho-1)(\beta-1)\cos\theta + \cos^2\theta,$$

$$B = (\rho-1)(\beta-1) - \cos\theta. \tag{10}$$

We use the set of algebraic equations (4)–(8) to calculate $\dot{\varphi}, \dot{\tau}, \rho, \theta$ and E of the electron in the circular orbit when β, λ and L are prescribed. This approach is different from that used by earlier workers who assigned the values of β, λ, L and E and then used the above set of equations to obtain a constraint equation connecting ρ and θ.

Using (4), (5) and (8) we get

$$E = \pm\left(1 + \frac{L^2}{\rho^2\sin^2\theta}\right)^{1/2}(1-2/\rho)^{1/2} + \lambda A_\tau. \tag{11}$$

The two terms in the above expression for E correspond respectively to the kinetic energy and the electrostatic potential energy, in the limit when the gravitational field is absent and the velocity of the electron is non-relativistic. As the kinetic energy has to be positive we choose the positive sign with the first term in (11). By using (9) in (6) and (7) we obtain

$$\beta A^{3/2}\frac{L^2\cos\theta}{\rho\sin^4\theta} = \lambda D\left(1 + \frac{L^2}{\rho^2\sin^2\theta}\right)^{1/2}(1-2/\rho)^{-1/2} \tag{12}$$

and

$$\frac{D\sin^2\theta}{\cos\theta} - \frac{D\rho\sin^4\theta}{L^2\cos\theta}\left(1 + \frac{L^2}{\rho^2\sin^2\theta}\right)\left(1 - \frac{2}{\rho}\right)^{-1}$$

$$= A[\rho(\beta-1) - B - A^{1/2}] - \rho BF, \tag{13}$$

where

$$D = (\rho-1)^2 + (\beta-1)^2 - 1 - (\rho-1)^2(\beta-1)^2 \tag{14}$$

and

$$F = (\rho-1) - (\beta-1)\cos\theta. \tag{15}$$

In principle, (12) and (13) can be solved for ρ and θ in terms of β, λ and L and therefore A_τ and E can be calculated. However the solutions of (12) and (13) for any arbitrary values of β, λ and L seem to be quite formidable.

We, therefore, first examine the same system of equations in the flat space time limit, as obtained by 'switching off' the gravitational field. In this limit, the exact solutions of (12) and (13) are

$$\rho\cos\theta = \beta, \tag{16}$$

$$\rho\sin\theta = -\frac{L^2}{\lambda}\left(1-\frac{\lambda^2}{L^2}\right)^{1/2} \tag{17}$$

and the energy is given by

$$E_{\text{flat-space}} = \left[1+\frac{\lambda^2}{L^2}\left(1-\frac{\lambda^2}{L^2}\right)^{-1}\right]^{1/2} - \frac{\lambda^2}{L^2}\left(1-\frac{\lambda^2}{L^2}\right)^{-1/2}. \tag{18}$$

Equation (16) implies that the proton lies in the plane of the orbit of the electron while (17) gives the radius ρ_1 of the circular orbit.

Although the results (16)–(18) are true (in flat space time) for any arbitrary β, λ and L, we choose, for our further discussion, $m = 1{\cdot}5 \times 10^5$ and $l = n\hbar$ where $\hbar$ is the Plancks constant and n is an integer $1, 2, \ldots$. Therefore $L = n \times 2{\cdot}56 \times 10^{-16}$ and $\lambda = -1{\cdot}9 \times 10^{-18}$ which are very small compared to β which is taken to be > 2. For such a choice of β, λ and L we can approximate (18) as

$$E_{\text{flat-space}} \sim \left(1+\frac{\lambda^2}{L^2}\right)^{1/2} - \frac{\lambda^2}{L^2}. \tag{19}$$

We now consider (12) and (13). The gravitational field will modify the solutions (16) and (17). However the modifications will be quite negligible as can be seen by comparing the electrostatic force and the gravitational force (in terms of Newtonian theory). The former is $e^2/r_1^2 \sim 10^{-2}$ where $r_1 \sim L^2/\lambda$ while the latter is $\sim m_0 c^2/m \sim 10^{-13}$. Following the flat space time solutions we take the solutions of (12) and (13) of the form $\rho = \beta(1+\alpha)$ and $\sin\theta = \delta$. Both δ and α are expected to be very small compared to unity. Substituting these in (12) and (13) we obtain

$$\delta \sim -L^2/\lambda\beta \tag{20}$$

and

$$\alpha \sim -L^6/(\lambda^4\beta^3) \tag{21}$$

neglecting the terms of smaller orders of magnitudes. We now calculate A_τ and finally obtain

$$E \sim \left[\left(1+\frac{\lambda^2}{L^2}\right)^{1/2} - \frac{\lambda^2}{L^2}\right](1-2/\beta)^{1/2}. \tag{22}$$

Also, comparing (20) and (21) with the corresponding solutions in flat spacetime, we find that the value of $\sin\theta$ remains essentially unaltered, while

$$\beta - \rho\cos\theta \sim (L^6/\beta^2\lambda^4) \sim (10^{-24}/\beta^2) \tag{23}$$

which implies that the plane of the orbit of the electron is displaced towards the gravitating source. As we have invoked Newtonian gravitation for the above conclusions, these are true at least when β is sufficiently large.

3. Discussion

The relationship (22) gives the energy of the electron in circular orbits as observed by the observer at infinity (Misner *et al* 1973). Now an electron in circular orbit around a proton with $l = n\hbar$ forms a hydrogen atom (Bohr's model), and therefore the transition between the energy levels, as obtained by assigning different values of n in (19) gives rise to the well-known hydrogen lines. If now, one agrees to apply the same rules of transitions even when the space time is curved, (22) gives the same lines red-shifted by the factor $(1 - 2/\beta)^{1/2}$, which is to be expected. The whole exercise can be regarded as an alternative derivation of the gravitational red shift, obtained by studying the dynamics of the system, in a special case of Schwarzschild gravitational field.

In the above derivation we have assumed that the orbits of the electron have not changed significantly as a result of the gravitational field. The cause for the difference in (22) and (19) can be attributed to the curvature of the space time as evidenced by (11) and also to the modified electromagnetic field as a result of gravity. A similar calculation as above with Newtonian gravitational field does not give us any shift of the energy levels. We find an important difference in the general relativistic and the Newtonian calculations. In the former case the energy E which is to be calculated, appears in (5) as a constant of motion and further this E is interpreted as the energy observed by the observer at infinity. In the Newtonian calculations the energy is calculated as the sum of kinetic and potential energies after obtaining the expressions for the velocity of the electron and its distances from the proton and the gravitating source. We may interpret this as the energy at the location of the electron. One can then apply the quantum theory of radiation and the weak principle of equivalence to get the red shift observed by the distant observer (Weinberg 1972).

The plane of the orbit of the electron is displaced slightly towards the gravitating source as indicated by (23). Such a displacement will give rise to a net dipole moment $\sim 10^{-19}/b^2$ (in c.g.s. units).

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References

Chellathurai V, Dhurandhar S V and Dadhich N 1986 *Pramana – J. Phys.* **27** 485
Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: W H Freeman) p. 657
Sonar V P, Dhurandhar S V and Dadhich N 1985 *Bull. Astron. Soc. India* **13** 372
Weinberg S 1972 *Gravitation and cosmology* (New York: John Wiley) p. 84