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A general law for quantum mechanical joint probabilities: generalisation of the Wigner formula and the collapse postulate for successive measurements of discrete as well as continuous observables

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Abstract. The fundamental prescriptions of quantum theory have so far remained incomplete in that there is no satisfactory prescription for the joint probabilities of successive observations of arbitrary sequence of observables. The joint probability formula derived by Wigner is based on the collapse postulate due to Von Neuman and Lüders and is applicable only to observables with purely discrete spectra. Earlier attempts to generalize the collapse postulate to observables with continuous spectra have been unsatisfactory as they lead to only finitely additive (and not σ-additive) joint probabilities in general. In this paper a suitable generalisation of the Wigner joint probability formula is proposed, which is completely satisfactory in the sense that it leads to σ-additive joint probabilities for successive observations of arbitrary sequence of observables, consistent with all the other basic prescriptions of quantum theory. This general law for quantum mechanical joint probabilities is arrived at by a reformulation of earlier results on expectation values in successive measurements. The generalized Wigner joint probability formula is also shown to be a consequence of a general collapse postulate, which allows for changes in state due to measurement from normal states to non-normal states also. As an illustration of our results, the probability distribution of the outcomes of a momentum measurement which immediately succeeds a position measurement is computed, and this seems to shed an entirely new light on the uncertainty principle.

Keywords. Successive measurements; joint probabilities; σ-additivity; collapse postulate; observables with continuous spectrum.

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1. Introduction

One of the outstanding problems of quantum theory, which has eluded a satisfactory solution so far, has been that of extending the conventional description of successive observations which is available only for observables with a purely discrete spectrum to the general case when an arbitrary sequence of observables (with continuous as well as discrete spectra) are measured. This has resulted in a profound 'incompleteness' of the theory at the level of its fundamental prescriptions, so that we have no way of discussing the statistics of successive observations of even the basic observables such as position, momentum, etc., which have a continuous spectrum. In this paper we show that the conventional prescription for the joint probabilities of successive observations, (the Wigner joint probability formula (Wigner 1963) for observables with a purely discrete spectrum) can be extended to a general law for quantum mechanical joint probabilities for successive observations of arbitrary observables leading to σ-additive joint probabilities, which are affine functions of the density operators and which reduce to the conventional prescription for the case of observables with a purely discrete spectrum, as well as for the case of observables which are mutually compatible. This general law for quantum mechanical probabilities is arrived at by a suitable reformulation of an earlier proposal (Srinivas 1980) for the evaluation of general expectation values in successive measurements. We also propose a general collapse postulate which is applicable to arbitrary observables, and which automatically leads to the above σ-additive joint probabilities for successive observations.

The present paper is organised as follows. In § 2 we recall the conventional prescriptions for joint probabilities in quantum mechanics, such as the generalized Born statistical formula which gives the joint probabilities for any set of mutually compatible observables, and also the Wigner formula for the joint probabilities for successive observations of observables with a purely discrete spectrum. In § 3 we briefly review the earlier attempts at extending the collapse postulate for observables with continuous spectra and indicate their limitations. In § 4 we show how by modifying an earlier proposal (Srinivas 1980) for evaluating expectation values, it is possible to arrive at a general law of quantum mechanical joint probabilities for successive observations of arbitrary sequences of observables (discrete as well as continuous). It is demonstrated that this law always leads to σ-additive joint probabilities which are affine functions of the density operator, and which reduce to the conventional prescriptions of quantum theory such as the generalized Born statistical formula and the Wigner formula. In § 5 we consider a more general framework of quantum theory where the set of all states is the set of positive norm-one linear functionals on the space of all bounded operators on a Hilbort space. We show that in this enlarged state space (which includes the so-called non-normal states also) a general collapse postulate can be formulated which is applicable for arbitrary observables and which leads to the σ-additive joint probabilities for successive measurements proposed in § 4. Finally in § 6 we illustrate our results by considering the example of a position measurement immediately followed by a momentum measurement.

2. The conventional prescriptions of quantum theory

In this section, we shall recall the conventional prescriptions of quantum theory for joint probabilities. First of all, the basic prescription for probabilities in quantum mechanics is that the probability $\mathrm{Pr}^{\rho}_{A}(\Delta)$ of finding the outcomes of the measurement of the observable (represented by the self-adjoint operator) A to lie in a Borel set $\Delta \in B(R)$ when a measurement is made on an ensemble of systems in a state (characterized by the density operator) ρ is given by the so-called Born statistical formula

$$\mathrm{Pr}^{\rho}_{A}(\Delta) = \mathrm{Tr}\,(\rho P^{A}(\Delta)), \tag{1}$$

where $\Delta \rightarrow P^{A}(\Delta)$ is the unique spectral measure associated with the self-adjoint operator A. Further, if the observables $A_1, \ldots, A_n$ are mutually compatible (i.e. their spectral projectors commute), then the joint probability

$$\mathrm{Pr}^{\rho}_{A_1, A_2, \ldots, A_n}(\Delta_1, \Delta_2, \ldots, \Delta_n)$$

that the outcomes of measurements of $A_1, A_2, \ldots, A_n$ lie in the Borel sets $\Delta_1, \Delta_2, \ldots, \Delta_n$ respectively, is given by the so-called generalized Born statistical formula

$$\Pr^{\rho}_{A_1, A_2, \ldots, A_n}(\Delta_1, \Delta_2, \ldots, \Delta_n) = \mathrm{Tr}\,[\rho P^{A_1}(\Delta_1) P^{A_2}(\Delta_2) \ldots P^{A_n}(\Delta_n)]. \tag{2}$$

Note that (2) provides meaningful joint probabilities only for a set of compatible observables, as otherwise the right hand side will take on complex values, in case the spectral projectors do not commute with each other.

The conventional prescription for joint probabilities for successive observations is derived by using (1) in conjunction with the so-called 'collapse postulate' which specifies the state of a system after a measurement has been performed. For this purpose, let us consider an observable A with a purely discrete spectrum and with the spectral resolution

$$A = \sum_i \lambda_i P^A(\lambda_i). \tag{3}$$

For such an observable, the collapse postulate due to von Neumann (1955) and Lüders (1951) prescribes that, if ρ is the state of an ensemble immediately prior to measurement of A, then the (sub-) ensemble of all those systems which yield an outcome in the Borel set, $\Delta \in B(R)$ is given by $I^A(\Delta)\rho/\mathrm{Tr}\,[I^A(\Delta)\rho]$ immediately after the measurement, where

$$I^A(\Delta)\rho = \sum_{\lambda_i \in \Delta} P^A(\lambda_i)\rho P^A(\lambda_i). \tag{4}$$

Using (4) and (1), Wigner (1963) obtained the joint probabilities for successive observations of observables with purely discrete spectra.

Here, and in what follows, we shall adopt the Heisenberg picture, where the burden of time-evolution between two measurements, is entirely carried by the observables, the states being constant. Consider a sequence of measurements of observables $A_1(t_1), A_2(t_2), \ldots, A_n(t_n)$ carried out at times $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ where each $A_i(t_i)$ $(i = 1, 2, \ldots, n)$ has a purely discrete spectrum and the spectral resolution

$$A_i(t_i) = \sum_{\lambda_i} \lambda_i P^{A_i(t_i)}(\lambda_i). \tag{5}$$

Then the Wigner formula for the joint probability that the sequence of measurements of $A_1(t_1), A_2(t_2), \ldots, A_n(t_n)$ yields the outcomes $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, is given by

$$\begin{aligned} &\Pr^{\rho}_{A_1(t_1), \ldots, A_n(t_n)}(\lambda_1, \ldots, \lambda_n) \\ &\quad = \mathrm{Tr}\,[P^{A_n(t_n)}(\lambda_n) \ldots P^{A_1(t_1)}(\lambda_1)\rho P^{A_1(t_1)}(\lambda_1), \ldots, P^{A_n(t_n)}(\lambda_n)]. \end{aligned} \tag{6}$$

The joint probabilities that the outcomes of $A_i(t_i)$ lie in arbitrary Borel sets $\Delta_i \in B(R)$ $(i = 1, \ldots, n)$ can be expressed elegently via the transformations $I^{A_i(t_i)}(\Delta_i)$ as follows:

$$\Pr^{\rho}_{A_1(t_1), \ldots, A_n(t_n)}(\Delta_1, \ldots, \Delta_n) = \mathrm{Tr}\,[I^{A_n(t_n)}(\Delta_n), \ldots, I^{A_1(t_1)}(\Delta_1)\rho] \tag{7a}$$

where

$$I^{A_i(t_i)}(\Delta_i) = \sum_{\lambda_i \in \Delta_i} P^{A_i(t_i)}(\lambda_i)\rho P^{A_i(t_i)}(\lambda_i). \tag{7b}$$

3. Earlier proposals for the generalization of the collapse postulate for continuous observables

In this section, we shall briefly outline the earlier work on extending the collapse postulate for observables with continuous spectrum, mainly with a view to place the present work in perspective. For technical details, the reader may refer to von Neumann (1955), Davies and Lewis (1969) Davies (1976), Srinivas (1980) and Ozawa (1984, 1985).

For the case of an observable A with a continuous spectrum, Davies and Lewis (1969) proposed that the change of state due to measurement should also be assumed to be of the form

$$\rho \to \frac{I^A(\Delta)\rho}{\mathrm{Tr}\,[I^A(\Delta)\rho]},$$

where $\Delta \to I^A(\Delta)$ is a so-called operation-valued measure on the space $T(\mathscr{H})$ of all the trace-class operators on the Hilbert space; i.e. for each $\Delta \in B(R)$, $I^A(\Delta)$ is a positive operator on $T(\mathscr{H})$ such that for all density operators $\rho \in T(\mathscr{H})$

$$\mathrm{Tr}\,[I^A(R)\rho] = \mathrm{Tr}\,[\rho], \tag{8}$$

and

$$I^A\left(\bigcup_i \Delta_i\right)\rho = \sum_i I^A(\Delta_i)\rho, \tag{9}$$

whenever $\{\Delta_i\}$ is a sequence of mutually disjoint Borel sets in $B(R)$ and the right hand side of (9) is assumed to converge in the trace-norm topology on $T(\mathscr{H})$. To obtain a general collapse postulate we still need to have a canonical association $A \to \{I^A(\Delta)\}$ between self-adjoint operators and such operation-valued measures. Some of the basic requirements that such a collapse postulate (or the rule of association $A \to \{I^A(\Delta)\}$) needs to satisfy are the following I–IV.

I. For each self adjoint operator A, there is associated an operation-valued measure $\Delta \to I^A(\Delta)$ such that

$$\mathrm{Tr}\,[I^A(\Delta)\rho] = \mathrm{Tr}\,[\rho P^A(\Delta)]. \tag{10}$$

(This expresses the requirement of consistency with the Born statistical formula (1)).

II. For $\Delta_1, \Delta_2 \in B(R)$

$$\mathrm{Tr}\,[I^A(\Delta_1)I^A(\Delta_2)\rho] = \mathrm{Tr}\,[I^A(\Delta_1 \cap \Delta_2)\rho] \tag{11}$$

(This expresses the so-called 'weak repeatability' condition).

III. If $A_1, A_2, \ldots, A_n$ are all mutually compatible, then for all $\Delta_1, \Delta_2, \ldots, \Delta_n \in B(R)$

$$\mathrm{Tr}\,[I^{A_n}(\Delta_n)\ldots I^{A_2}(\Delta_2)I^{A_1}(\Delta_1)\rho] = \mathrm{Tr}\,[\rho P^{A_1}(\Delta_1)P^{A_2}(\Delta_2)\ldots P^{A_n}(\Delta_n)]. \tag{12}$$

(This expresses the requirement of consistency with the generalized Born statistical formula (2) for compatible observables, and is stronger than the weak repeatability condition II.)

IV. The state change $\rho \to I^A(R)\rho$ is derivable as the reduced evolution of the state of the system in a model of the measurement process, where the system together with the measuring apparatus are assumed to undergo (unitary) Hamiltonian evolution.

The incisive nature of the problem of extending the collapse postulate to continuous observables became quite clear when several no-go theorems were established showing the non-existance of operation-valued measures $\Delta \to I^A(\Delta)$ satisfying the above (very natural) requirements. First it was shown (Srinivas 1980) that for an observable with continuous spectrum the requirements I and III are incompatible. Then Ozawa (1984) showed that the requirements I, III and IV are incompatible. Soon after, Ozawa (1985) proved the much stronger result that requirements I and II are themselves incompatible, thus establishing the conjecture of Davies and Lewis (1969) that there are no repeatable operation-valued measures associated with an observable with continuous spectrum.

The above results showed that there was no generalization of collapse postulate as a transformation defined on the set of density operators alone, which is consistent with the conventional prescription (2) (the generalized Born statistical formula) for joint probabilities of compatible observables, which is one of the basic prescriptions of quantum theory widely employed in all physical situations. It therefore became clear that, if a generalization of the collapse postulate for continuous observables is to be sought (and this is necessary if the conventional Wigner formula (6), (7), is to be generalized in the usual way for arbitrary observables), then we need to 'extend' the standard framework of quantum mechanics by allowing for more general state changes (due to measurement), where a density operator state can be transformed under a measurement to the so-called non-normal states also, which are characterized by normalized positive linear functionals on $B(\mathscr{H})$, the set of all bounded operators on the Hilbert space $\mathscr{H}$. One such proposal was made in an earlier paper (Srinivas 1980) where the collapse associated with an observable A was characterized by a so-called 'expectation-valued measure' $\Delta \to \mathscr{E}^A(\Delta)$ where each $\mathscr{E}^A(\Delta)$ is a positive linear map on $B(\mathscr{H})$ such that

$$\mathscr{E}^A(R)I = I, \tag{13}$$

where I is the identity operator. The change of state associated with measurement can be obtained as the adjoint map $\mathscr{E}^A(\Delta)^*; B(\mathscr{H})^* \to B(\mathscr{H})^*$ using the standard duality between $B(\mathscr{H})$ and its dual $B(\mathscr{H})^*$. As is well known, such an adjoint map, transforms a normal or density operator state into another density operator state, only when $\{\mathscr{E}^A(\Delta)\}$ are themselves 'normal' or continuous under the ultraweak topology on $B(\mathscr{H})$. Using such a framework, it was shown (Srinivas 1980) that the requirement III can be satisfied by taking recourse to the following collapse postulate.

Let η be an invariant mean (see for instance Greenleaf 1969) on the additive group R; i.e., η is a positive linear functional on the space $C_B(R)$ of all bounded functions on R such that

$$\eta(e) = 1, \tag{14}$$

where $e \in C_B(R)$ is such that $e(x) = 1$ for all $x \in R$; and for each $a \in R$,

$$\eta(f_a) = \eta(f), \tag{15}$$

where $f_a(x) = f(x+a)$ for all $a \in R$. We now define the positive linear operator $\mathscr{E}^A(\Delta)$: $B(\mathscr{H}) \to B(\mathscr{H})$ by the equation:

$$\mathscr{E}^A(\Delta)(B) = P^A(\Delta)\varepsilon_\eta^A(B) \tag{16}$$

for all $B \in B(\mathscr{H})$, where ε_η^A is given by

$$\mathrm{Tr}\,[\rho\varepsilon_\eta^A(B)] = \eta[\mathrm{Tr}\,\{\rho\exp(iAx)B\exp(-iAx)\}] \tag{17}$$

for all density operators ρ.

It can then be shown (Tomiyama 1957; Arveson 1967) that ε_η^A is a so-called 'conditional expectation' from $B(\mathscr{H})$ onto $\mathscr{U}'_A = \{P^A(\Delta)\}'$, the commutant of A. If A has a continuous spectrum, then it is well known (see for instance, Davies 1976) that any such conditional expectation has to be non-normal.

Using the above generalization of the collapse postulate, the following formula for the joint probabilities was proposed in Srinivas (1980).

$$\begin{aligned}
&\mathrm{Pr}^\rho_{A_1(t_1),\ldots,A_n(t_n)}(\Delta_1,\ldots,\Delta_n)\\
&\quad= \mathrm{Tr}\,[\rho\mathscr{E}^{A_1(t_1)}(\Delta_1)\ldots\mathscr{E}^{A_n(t_n)}(\Delta_n)I]\\
&\quad= \mathrm{Tr}\,[\rho P^{A_1(t_1)}(\Delta_1)\varepsilon_\eta^{A_1(t_1)}\\
&\qquad\times(\ldots\varepsilon_\eta^{A_{n-1}(t_{n-1})}(P^{A_n(t_n)}(\Delta_n)\ldots)].
\end{aligned} \tag{18}$$

While the above formula reduces to the conventional generalized Born statistical formula (2) for the case of compatible observables and the Wigner formula (6) for the case of observables with purely discrete spectra, it is quite unsatisfactory as the joint probabilities (18) are in general *only* finitely additive and not σ-additive when we consider the case of incompatible observables with continuous spectra. This arises due to the fact that in (18) the non-normal conditional expectations such as $\varepsilon_\eta^{A_1(t_1)}$ are acting on the spectral projectors such as $P^{A_2(t_2)}(\Delta_2)$ which are σ-additive only in the strong or ultra-weak topology and not in the norm topology.

The problem of (18) as yielding only finitely additive joint probabilities was also discussed in Srinivas (1980). There it was suggested that, in such situations, the statistics of successive observations (such as expectation values etc) needs to be perhaps postulated separately as we do not have access to the rich integration theory developed for σ-additive probabilities. It was therefore proposed that the following prescription may be adopted as an independent postulate for calculating expectation values in any successive observation.

If $f(\lambda_1,\lambda_2,\ldots,\lambda_n)$ is any bounded measurable function on R^n then the expectation or mean value of the corresponding function of the outcomes in any sequence of measurements of observables $A_1(t_1),\ldots,A_n(t_n)$ may be taken to be

$$\begin{aligned}
&\exp^\rho_{A_1(t_1),\ldots,A_n(t_n)}[f(A_1(t_1),\ldots,A_n(t_n)]\\
&\quad= \underset{x_1}{\eta}\,\underset{x_2}{\eta}\ldots\underset{x_{n-1}}{\eta}\int_{R^n}\ldots\int f(\lambda_1,\lambda_2,\ldots,\lambda_n)\\
&\qquad\times\mathrm{Tr}\,[\rho P^{A_1(t_1)}(\mathrm{d}\lambda_1)\exp[iA_1(t_1)x_1]P^{A_2(t_2)}(\mathrm{d}\lambda_2)\exp[iA_2(t_2)x_2]\ldots
\end{aligned}$$

$$\times \exp[iA_{n-1}(t_{n-1})x_{n-1}]P^{A_n(t_n)}(\mathrm{d}\lambda_n)$$
$$\times \exp[-iA_{n-1}(t_{n-1})x_{n-1}]\ldots\exp[-iA_1(t_1)x_1]]. \tag{19}$$

In particular, if the (bounded measurable) function $f(\lambda_1,\lambda_2,\ldots,\lambda_n)$ is of the form

$$f(\lambda_1,\lambda_2,\ldots,\lambda_n)=f_1(\lambda_1)f_2(\lambda_2)\ldots f_n(\lambda_n)$$

then (19) simplifies to

$$\begin{aligned}&\exp^{\rho}_{A_1(t_1),\ldots,A_n(t_n)}[f_1(A_1(t_1))\ldots f_n(A_n(t_n))]\\&\quad=\mathrm{Tr}[\rho f_1(A_1(t_1))\varepsilon_\eta^{A_1(t_1)}(f_2(A_2(t_2))\ldots\\&\qquad\times\varepsilon_\eta^{A_{n-1}(t_{n-1})}(f_n(A_n(t_n))\ldots)].\end{aligned} \tag{20}$$

Our objective in this paper is to arrive at *σ-additive joint probabilities* for successive observations of arbitrary observables, consistent with the generalized Born statistical formula (2) for compatible observables and of course with the Wigner formula (6) for observables with purely discrete spectra. As we stated, the joint probabilities (18) though consistent with the conventional prescriptions, are unsatisfactory because they yield only finitely additive probabilities in general. The same will be the problem with the expectation value formulae (19), (20) at least in the way formulated above, as one gets back (18) from (19) or (20) by choosing

$$f(\lambda_1,\lambda_2,\ldots,\lambda_n)=\chi_{\Delta_1}(\lambda_1)\chi_{\Delta_2}(\lambda_2)\ldots\chi_{\Delta_n}(\lambda_n),$$

where $\chi_\Delta(\lambda)$ is the indicator function

$$\chi_\Delta(\lambda)=\begin{cases}1 & \lambda\in\Delta\\ 0 & \lambda\notin\Delta.\end{cases}$$

But what we shall show in the next section is that we can *retain the expectation value formula (20) for a much smaller class of functions* (than the set of all bounded measurable functions) which is however still large enough to characterize a unique σ-additive joint probability, which we may identify as the joint probability of successive measurements. We shall also see that the joint probabilities obtained this way, while being σ-additive, also retain the important property that the finitely additive joint probabilities given by (18) had, namely that they reduce to the conventional prescriptions of quantum mechanics for the case of compatible observables and also for the case of purely discrete observables. Thus, by reformulating the expectation value formula (20), we can arrive at a general law of quantum mechanical joint probabilities for successive observations of an arbitrary sequence of observables. Later, in § 5 we shall propose an appropriate generalization of the collapse postulate which in a natural way leads to the above general law for quantum mechanical joint probabilities.

4. General law for quantum mechanical joint probabilities

In this section we show how the expectation value formula (20) can be suitably reformulated so as to yield a general law of quantum mechanical joint probabilities for successive measurements of arbitrary sequence of observables. For this purpose we consider the compatified real line $\bar{R} = R \cup \{-\infty, \infty\}$. Let $C(\bar{R}^n)$ denote the set of continuous functions on $\bar{R}^n$. It is well known that any $f \in C(\bar{R}^n)$ is also an element of $C_B(R^n)$ (set of all bounded continuous functions on R^n) with the additional condition that f has well-defined limits when each of its arguments approach $\pm\infty$. Now we shall state the basic result concerning the joint probabilities for successive observations.

Theorem 1: For each density operator ρ the equation

$$\begin{aligned}\int_{\bar{R}^n} \cdots \int f_1(\lambda_1) f_2(\lambda_2) \ldots f_n(\lambda_n) \operatorname{Pr}^{\rho}_{A_1(t_1), A_2(t_2), \ldots, A_n(t_n)}(\mathrm{d}\lambda_1, \mathrm{d}\lambda_2, \ldots, \mathrm{d}\lambda_n) \\ = \operatorname{Tr}[\rho f_1(A_1(t_1)) \varepsilon_\eta^{A_1(t_1)}(f_2(A_2(t_2)) \ldots \\ \times \varepsilon_\eta^{A_{n-1}(t_{n-1})}(f_n(A_n(t_n))) \ldots)]\end{aligned} \tag{21}$$

assumed for all $f_i \in C(\bar{R})$ $(i = 1, 2, \ldots, n)$ characterizes a unique, regular, σ-additive joint probability measure

$$\operatorname{Pr}^{\rho}_{A_1(t_1), \ldots, A_n(t_n)}(\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n) \quad \text{on} \quad (\bar{R}^n, B(\bar{R}^n)).$$

Further, this joint probability is an affine function of the density operator ρ and reduces to the joint probability given by the Wigner formula (6) for the case when the observables $\{A_i(t_i) | i = 1, 2, \ldots, n\}$ all have purely discrete spectra. It also reduces to the joint probability measure given by the generalised Born statistical formula (2), when the observables $\{A_i(t_i) | i = 1, 2, \ldots, n\}$ are all mutually compatible.

Proof: To prove the above result we first note that the right hand side of (21) defines a linear functional on $C(\bar{R}^n)$. Since for $f_1 \in C(\bar{R})$ and $A_i(t_i)$ a self-adjoint operator, $f_i(A_i(t_i))$ is a bounded operator, and $\varepsilon_\eta^{A_i(t_i)}$ is a norm-one projection map from $B(\mathscr{H})$ onto $\mathscr{U}'_A$ the right hand side is a well-defined and defines a bounded linear functional for all $f \in C(\bar{R}^n)$ of the form

$$f(\lambda_1, \lambda_2, \ldots, \lambda_n) = f_1(\lambda_1) f_2(\lambda_2) \ldots f_n(\lambda_n). \tag{22}$$

Since the set of all 'decomposable functions' f of the form (22) is dense in $C(\bar{R}^n)$ we can conclude that the right hand side defines a unique linear functional on $C(\bar{R}^n)$. Further, from the basic property that $\varepsilon_\eta^{A_i(t_i)}$ map the identity operator into itself, we have the property that this linear functional maps the unit function $f(\lambda_1, \lambda_2, \ldots, \lambda_n) = 1$ onto the number 1.

We shall now show that the linear functional on $C(\bar{R}^n)$ defined by the right hand side of (21) is actually a positive linear functional. For this purpose, let us assume that

$$f_i \geqslant 0 \quad i = 1, 2, \ldots, n. \tag{23}$$

Then it follows that

$$\begin{aligned}\operatorname{Tr}[\rho f_1(A_1(t_1)) \varepsilon_\eta^{A_1(t_1)}(f_2(A_2(t_2)) \ldots \\ \times \varepsilon_\eta^{A_{n-1}(t_{n-1})}(f_n(A_n(t_n)) \ldots)] \geqslant 0\end{aligned} \tag{24}$$

which follows from the fact that $f_i(A_i(t_i)) \geqslant 0$ and $\varepsilon_\eta^{A_i(t_i)}$ maps positive elements of $B(H)$

into positive elements of $\mathscr{U}'_{A_i(t_i)}$. Thus we have shown that each positive element $f \in C(\bar{R}^n)$, which is decomposable in the form (22) is mapped onto a non-negative number by the right hand side of (21). Now, from the basic properties of $C(\bar{R}^n)$ it can be shown that any positive element of $C(\bar{R}^n)$ can be expressed as a limit of linear combinations (with non-negative coefficients) of such decomposable positive elements*. Hence it follows that the right hand side of (21) defines a unique positive linear functional on $C(\bar{R}^n)$ which maps the unit function onto unity.

Now, since $\bar{R}^n$ is a compact space, it follows from the Reisz-Markov-Kakutani theorem (see for instance Yosida 1965) that (21) serves to characterize a unique regular σ-additive joint probability measure $\Pr^{\rho}_{A_1(t_1),\ldots,A_n(t_n)}$ $(\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_n)$ on $(\bar{R}^n, B(\bar{R}^n))$. Also since the right side of (21) is an affine function of the density operator ρ (with respect to formation of mixtures), the joint probabilities are also affine functions of the density operator ρ.

Now we consider the case when the observables $\{A_i(t_i) | i = 1, 2, \ldots, n\}$ all have purely discrete spectra and have spectral resolutions as in (5). Now, if we employ the relations

$$f_i(A_i(t_i)) = \sum_{\lambda_i} f_i(\lambda_i) P^{A_i(t_i)}(\lambda_i) \tag{25}$$

and the result obtained in Srinivas (1980) that

$$\varepsilon_\eta^{A_i(t_i)}(B) = \sum_{\lambda_i} P^{A_i(t_i)}(\lambda_i) B P^{A_i(t_i)}(\lambda_i) \tag{26}$$

for all $B \in B(\mathscr{H})$ and for any invariant mean η, then we can easily see that (21) simplifies to

$$\begin{aligned}\int_{\bar{R}^n} \cdots \int f_1(\lambda_1)\ldots f_n(\lambda_n) \Pr^{\rho}_{A_1(t_1),\ldots,A_n(t_n)}(d\lambda_1,\ldots,d\lambda_n) \\ = \sum_{\lambda_1}\sum_{\lambda_2}\cdots\sum_{\lambda_n} f_1(\lambda_1) f_2(\lambda_2)\ldots f_n(\lambda_n) \\ \times \mathrm{Tr}\,[\rho P^{A_1(t_1)}(\lambda_1) P^{A_2(t_2)}(\lambda_2)\ldots P^{A_n(t_n)}(\lambda_n)\ldots \\ \times P^{A_2(t_2)}(\lambda_2) P^{A_1(t_1)}(\lambda_1)]\end{aligned} \tag{27}$$

for all $f_i \in C(\bar{R})$. It immediately follows from (27) that the joint probabilities for observables with purely discrete spectra are precisely those given by the Wigner formula (6).

Finally let us consider the case when the observables $\{A_i(t_i) | i = 1, 2, \ldots, n\}$ are all mutually compatible. Then since each $\varepsilon_\eta^{A_i(t_i)}$ is a projection map onto $\mathscr{U}'_{A_i(t_i)}$ the action of all the conditional expectations $\varepsilon_\eta^{A_i(t_i)}$ in the right hand side of (21) become trivial and the equation reduces to

$$\begin{aligned}\int_{\bar{R}^n} \cdots \int f_1(\lambda_1)\ldots f_n(\lambda_n) \Pr^{\rho}_{A_1(t_1),\ldots,A_n(t_n)}(\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_n) \\ = \mathrm{Tr}\,[\rho f_1(A_1(t_1)) f_2(A_2(t_2))\ldots f_n(A_n(t_n))]\end{aligned}$$

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*The author is indebted to Professor K R Parthasarathy for explaining to him a simple proof of this result using an elegent probability-theoretic technique.

$$= \int_{R^n} \cdots \int f_1(\lambda_1) f_2(\lambda_2) \ldots f_n(\lambda_n)$$
$$\times \operatorname{Tr}[\rho P^{A_1(t_1)}(\mathrm{d}\lambda_1) P^{A_2(t_2)}(\mathrm{d}\lambda_2) \ldots P^{A_n(t_n)}(\mathrm{d}\lambda_n)], \tag{28}$$

from which it follows that the joint probability measure for a set of mutually compatible observables is precisely that given by the generalized Born statistical formula (2). Thus we have completed the proof of Theorem 1.

The above theorem shows that (21) serves to characterize σ-additive joint probabilities for successive observations of arbitrary sequence of observables, discrete as well as continuous—joint probabilities which are affine functions of the density operator and reduce to the conventional prescriptions for the joint probabilities of compatible observables (generalized Born statistical formula (2)) and for the joint probabilities of successive observations of observables with purely discrete spectra (Wigner formula). Hence (21) may be said to provide a general quantum mechanical law for joint probabilities, as it indeed constitutes a very general prescription for the statistics of successive observations in quantum theory.

Apart from serving to characterize the joint probabilities, (21) also provides a direct rule for computing the mean or expectation value of any function of the form $f_1(\lambda_1) f_2(\lambda_2) \ldots f_n(\lambda_n)$ (where $f_i \in C(\bar{R}), i = 1, 2, \ldots, n$) of the outcomes of a sequence of measurements of observables $A_1(t_1), A_2(t_2), \ldots, A_n(t_n)$.

We can thus rewrite (21) as a prescription for the expectation value

$$\exp^{\rho}_{A_1(t_1), A_2(t_2), \ldots, A_n(t_n)}[f_1(A_1(t_1)) f_2(A_2(t_2)) \ldots f_n(A_n(t_n))]$$
$$= \operatorname{Tr}[\rho f_1(A_1(t_1)) \varepsilon_\eta^{A_1(t_1)}(f_2(A_2(t_2))$$
$$\times \varepsilon_\eta^{A_{n-1}(t_{n-1})}(f_n(A_n(t_n)) \ldots)] \tag{29}$$

for all $f_i \in C(\bar{R})$, $i = 1, 2, \ldots, n$. We can in fact show that the probabilities

$$\Pr^{\rho}_{A_1(t_1), \ldots, A_n(t_n)}(\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n)$$

defined by (21) lead to the more general expectation value formula

$$\exp^{\rho}_{A_1(t_1), \ldots, A_n(t_n)}[f(A_1(t_1), \ldots, A_n(t_n))]$$
$$= \underset{x_1}{\eta} \underset{x_2}{\eta} \cdots \underset{x_{n-1}}{\eta} \int_{\bar{R}^n} \cdots \int f(\lambda_1, \lambda_2, \ldots, \lambda_n)$$
$$\times \operatorname{Tr}[\rho P^{A_1(t_1)}(\mathrm{d}\lambda_1) \exp[iA_1(t_1)x_1] P^{A_2(t_2)}(\mathrm{d}\lambda_2) \exp[iA_2(t_2)x_2] \ldots$$
$$\times \exp[iA_{n-1}(t_{n-1})x_{n-1}] P^{A_n(t_n)}(\mathrm{d}\lambda_n) \exp[-iA_{n-1}(t_{n-1})x_{n-1}] \ldots$$
$$\times \exp[-iA_2(t_2)x_2] \exp[-iA_1(t_1)x_1]] \tag{30}$$

for all $f \in C(\bar{R}^n)$. Equation (30) is in fact the linear functional defined by the right hand side of (21) for arbitrary $f \in C(\bar{R}^n)$.

We should note that (29) or (30) for the expectation values *cannot* be used to directly compute moments, or characteristic functions or probabilities, as the corresponding functions λ^n, $\exp[it\lambda]$, $\chi_\Delta(\lambda)$ are not elements of $C(\bar{R})$ and hence cannot be used in (29) or

(30).* To calculate moments, characteristic functions etc, we should first evaluate the joint probability measure as characterized by (22) and then use that probability measure to compute these quantities.

5. The general collapse postulate

In this section we shall propose a general collapse postulate which is applicable to arbitrary observables and which automatically leads to the σ-additive joint probabilities for successive observations, given by (21), as a particular case. The no-go theorems mentioned in § 3 show that, for observables with continuous spectra, it is not possible to formulate the collapse postulate as a change of state defined on the class of density operator states alone, in a manner consistent with the generalized Born statistical formula (2) for compatible observables, or even with the weak repeatability condition (11). To be able to generalize the collapse postulate so as to be applicable to arbitrary observables, we should first suitably extend the conventional framework of quantum theory so as to allow for a larger class of states than the conventional density operator states. Such states, known as the 'non-normal states', have been widely discussed earlier in the literature in the context of quantum field theory and statistical mechanics (Emch 1972).

We shall continue with the conventional framework in so far as taking the set of observables to be the set of all self-adjoint operators on a Hilbert space $\mathscr{H}$. To characterize to the possible states of a system, we proceed as follows. Let $B(\mathscr{H})$ be the set of all bounded operators on $\mathscr{H}$. By $B(\mathscr{H})^*$ we denote the set of all continuous linear functionals on $B(\mathscr{H})$ or the dual of $B(\mathscr{H})$. If $\varphi \in B(\mathscr{H})^*$ and $B \in B(\mathscr{H})$ we denote the complex number which results by the action of φ on B, by $\langle \varphi, B \rangle$. Now an element $\varphi \in B(\mathscr{H})^*$ is said to be a state if

(i) φ is positive

$$\langle \varphi, B \rangle \geqslant 0 \quad B \geqslant 0. \tag{31}$$

(ii) φ is of norm one

$$\langle \varphi, I \rangle = 1. \tag{32}$$

Now each density operator ρ defines a state φ_ρ via

$$\langle \varphi_\rho, B \rangle = \mathrm{Tr}\,[\rho B]. \tag{33}$$

Such states which are continuous in the ultraweak topology on $B(\mathscr{H})$ are called normal states. Other norm-one positive linear functionals on $B(\mathscr{H})$ which are not representable as in (33) are said to characterize non-normal states, as these functionals are continuous only in the norm topology on $B(\mathscr{H})$.

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*We can now clearly see how the expectation value formula (19) proposed earlier (Srinivas 1980) is defective in that instead of restricting to $f \in C(\bar{R}^n)$ (as in (30)) it allowed arbitrary bounded measurable functions.

Following a recent suggestion of Ozawa (1987b) we propose to characterize the collapse or change in the state of a system due to measurement by positive linear transformations on $B(\mathscr{H})^*$. For notational convenience, let us write $B(\mathscr{H})^* = V$. Since V is a complete base-normed space, the general operational framework (Davies and Lewis 1969) can be employed in this case also. So, the change of state in a measurement can be characterized by an 'operation-valued measure' $\Delta \to I(\Delta)$ satisfying the following (i)–(iii):

(i) For each $\Delta \in B(\bar{R})$, $I(\Delta) \in B^+(V)$ $I(\Delta)$ is a positive linear map on $V = B(\mathscr{H})^*$

(ii) If $\{\Delta_i\}$ is a sequence of mutually disjoint elements of $B(\bar{R})$ then

$$I(\cup_i \Delta_i)\varphi = \sum_i I(\Delta_i)\varphi \tag{34}$$

for all positive elements $\varphi \in V$ where the right hand side converges in the norm topology on V

(iii)
$$\langle I(\bar{R})\varphi, I \rangle = \langle \varphi, I \rangle \tag{35}$$

for all positive $\varphi \in V$.

Following closely the argument in Davies (1976) we can show that every $B^+(V)$-valued measure $\Delta \to I(\Delta)$ satisfying (34), (35), is uniquely characterized by a family $\{I(f) | f \in C(\bar{R})\}$ satisfying the following (i)–(iii):

(i) For each $f \in C(\bar{R})$,

$$I(f) \in B(V)$$

(ii)
$$I(f) \geqslant 0 \tag{36}$$

whenever $f \geqslant 0$.

(iii)
$$\langle I(e)\varphi, I \rangle = \langle \varphi, I \rangle \tag{37}$$

for all states φ where $e(x) = 1$ for all $x \in \bar{R}$.

The correspondence between any given $B^+(V)$-valued measure $\Delta \to I(\Delta)$ and the associated family $\{I(f) | f \in C(\bar{R})\}$ is provided by the relation

$$\langle I(f), B \rangle = \int_{\bar{R}} f(\lambda) \langle I(\mathrm{d}\lambda)\varphi, B \rangle \tag{38}$$

for all $f \in C(\bar{R})$, $\varphi \in V$ and $B \in B(\mathscr{H})$.

Now, in order to generalize the collapse postulate, we need to present a canonical correspondence $A \to \{I^A(\Delta)\}$ between self-adjoint operators and associated operation-valued measures. Once such an association is given, we can formulate the appropriate generalization of the Wigner formula (7) in the form

$$\Pr^{\rho}_{A_1(t_1), \ldots, A_n(t_n)}(\Delta_1, \ldots, \Delta_n) = \langle I^{A_n(t_n)}(\Delta_n) \ldots I^{A_1(t_1)}(\Delta_1)\varphi, I \rangle. \tag{39}$$

Again, we only need to follow the standard argument outlined in Davies (1976) to show that (39) characterizes a unique regular σ-additive joint probability measure on $(\bar{R}^n, B(\bar{R}^n))$.

We may here note that the association $A \to \{I^A(\Delta)\}$ that was proposed in Srinivas (1980) does *not* conform to the framework presented here. The proposal there was to define $\{I^A(\Delta)\}$ by equation (16) or equivalently by

$$\langle I^A(\Delta)\varphi, B\rangle = \langle \varphi, P^A(\Delta)\varepsilon_\eta^A(B)\rangle \tag{40}$$

for all $B \in B(\mathscr{H})$ and $\varphi \in V$. As we noted earlier the map $\Delta \to I^A(\Delta)$ as defined by (40) is only finitely additive and not σ-additive in general. We now present a suitable reformulation of (40) which serves to provide a completely satisfactory generalization of the collapse postulate.

The general collapse postulate: To each self-adjoint operator A is associated a $B^+(V)$-valued measure $\Delta \to I^A(\Delta)$ which describes the changes in the state in any measurement of A, and is defined by the equation:

$$\langle I^A(f)\varphi, B\rangle = \langle \varphi, f(A)\varepsilon_\eta^A(B)\rangle \tag{41}$$

for all $f \in C(\bar{R})$, $\varphi \in V$ and $B \in B(\mathscr{H})$.

To show that this is indeed the appropriate generalization of the collapse postulate we prove the following.

Theorem 2: The collapse postulate as stated in equation (41) *does associate with each self-adjoint operator A a $B^+(V)$-valued measure $\Delta \to I^A(\Delta)$. The joint probabilities for successive observations as derived from the above collapse postulate, following* (39), *are affine functions of the state φ and reduce to the joint probabilities defined by* (21) *when the starting state of the ensemble is a normal or density operator state. The measurement transformations $\{I^A(\Delta)\}$ associated with an observable A satisfy the repeatability property*

$$I^A(\Delta_1)I^A(\Delta_2) = I^A(\Delta_1 \cap \Delta_2) \tag{42}$$

for all $\Delta_1, \Delta_2 \in B(\bar{R})$, and also the compatibility property

$$I^A(\Delta_1)I^B(\Delta_2) = I^B(\Delta_2)I^A(\Delta_1) \tag{43}$$

for all $\Delta_1, \Delta_2 \in B(\bar{R})$, whenever A, B are compatible. Finally, when A is an observable with a purely discrete spectrum and when we consider a normal or density operator state, then the action of $I^A(\Delta)$ given by the general collapse postulate (41) *reduces to the conventional form* (4) *given by the von Neumann-Lüders collapse postulate.*

Proof: For each $f \in C(\bar{R})$ and self-adjoint operator A, $f(A)$ is a bounded operator and since ε_η^A is a linear projection map from $B(\mathscr{H})$ onto $\mathscr{U}'_A$ we can directly see that $I^A(f)$ is a well-defined linear transformation on V for each $f \in C(\bar{R})$. We shall now show that $I^A(f)$ is positive whenever $f \geqslant 0$. For this, we need only to note that if $f \geqslant 0$ then $f(A) \geqslant 0$ and if $B \geqslant 0$ then $\varepsilon_\eta^A(B)$ is a positive element of $\mathscr{U}'_A$ so that their product $f(A)\varepsilon_\eta^A(B)$ is positive. Hence we get

$$\langle I^A(f)\varphi, B\rangle = \langle \varphi, f(A)\varepsilon_\eta^A(B)\rangle \geqslant 0 \tag{44}$$

whenever $f \geqslant 0$ and $B \geqslant 0$. Thus (36) is satisfied. Finally if $e(x) = 1$ for all $x \in \bar{R}$ we can straightaway see that

$$\langle I^A(e)\varphi, I\rangle = \langle \varphi, \varepsilon_\eta^A(I)\rangle = \langle \varphi, I\rangle \tag{45}$$

so that (37) is also satisfied. Thus the general collapse postulate (41) associates with each self-adjoint operator A, a family $\{I^A(f)\}$ of elements of $B(V)$ which satisfy (36) and (37). Hence they characterize a unique $B^+(V)$-valued measure, $\Delta \to I^A(\Delta)$ given by

$$\int_{\bar{R}} f(\lambda)\langle I^A(\mathrm{d}\lambda)\varphi, B\rangle = \langle I^A(f)\varphi, B\rangle$$
$$= \langle \varphi, f(A)\varepsilon_\eta^A(B)\rangle \tag{46}$$

for all $f \in C(\bar{R})$, $B \in B(\mathscr{H})$. Thus the general collapse postulate (41) serves to associate a $B^+(V)$-valued measure $\Delta \to I^A(\Delta)$ with each self adjoint operator A.

Now the joint probability measure for successive observations can be calculated via the generalised Wigner formula (39). If $f_1, f_2, \ldots, f_n$ are arbitrary elements of $C(\bar{R})$ then we use (39) and (38) to get

$$\int_{\bar{R}^n} \cdots \int f_1(\lambda_1)\ldots f_n(\lambda_n)\, \mathrm{Pr}^{\varphi}_{A_1(t_1),\ldots,A_n(t_n)}(\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_n)$$
$$= \int_{\bar{R}^n} \cdots \int f_1(\lambda_1)\ldots f_n(\lambda_n)\langle I^{A_n(t_n)}(\mathrm{d}\lambda_n)\ldots$$
$$\times\, I^{A_1(t_1)}(\mathrm{d}\lambda_1)\varphi, I\rangle$$
$$= \langle I^{A_n(t_n)}(f_n)\ldots I^{A_1(t_1)}(f_1)\varphi, I\rangle$$
$$= \langle \varphi, f_1(A_1(t_1))\varepsilon_\eta^{A_1(t_1)}(\ldots \varepsilon_\eta^{A_{n-1}(t_{n-1})}(f_n(A_n(t_n)))\ldots)\rangle. \tag{47}$$

Equation (47) gives us a characterization of the joint probability measure $\mathrm{Pr}^{\varphi}_{A_1(t_1),\ldots,A_n(t_n)}(\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_n)$ for any arbitrary state φ. From (47) it is clear that the joint probabilities are affine functions of the state φ.

Further when the starting state of the ensemble is taken to be a normal or density operator state, (47) clearly reduces to (21) which we had proposed as the general law of quantum mechanical joint probabilities for normal states. Thus our general collapse postulate leads to the appropriate generalisation, namely (47), of the general law of quantum mechanical joint probabilities valid for non-normal states also.

The repeatability property (42) and the compatibility property (43) are equivalent to the following relations

$$I^A(f_1)I^A(f_2) = I^A(f_1 f_2) \tag{48}$$

for all $f_1, f_2 \in C(\bar{R})$ where $(f_1 f_2)(x) = f_1(x) f_2(x)$ for all $x \in \bar{R}$; and

$$I^A(f_1)I^B(f_2) = I^B(f_2)I^A(f_1) \tag{49}$$

for all $f_1, f_2 \in C(\bar{R})$ whenever A, B are compatible. Relations (48) (49) can be easily shown to follow from the following properties of the conditional expectations (derived in Srinivas 1980)

$$\varepsilon_\eta^A(f(A)B) = f(A)\varepsilon_\eta^A(B) \tag{50}$$

for all $f \in C(\bar{R})$ and $B \in B(\mathscr{H})$; and

$$\varepsilon_\eta^A \varepsilon_\eta^B = \varepsilon_\eta^B \varepsilon_\eta^A \tag{51}$$

whenever A and B are compatible.

Finally, let A be an observable with a purely discrete spectrum, with the spectral resolution:

$$A=\sum_i \lambda_i P^A(\lambda_i).$$

For such an observable, (41) reduces to

$$\begin{aligned}\int_{\bar{R}} f(\lambda)\langle I^A(\mathrm{d}\lambda)\varphi, B\rangle &= \langle I^A(f)\varphi, B\rangle \\ &= \langle \varphi, f(A)\varepsilon_\eta^A(B)\rangle \\ &= \left\langle \varphi, \sum_i f(\lambda_i)P^A(\lambda_i)BP^A(\lambda_i)\right\rangle,\end{aligned} \tag{52}$$

where we have employed the relation

$$f(A)=\sum_i f(\lambda_i)P^A(\lambda_i) \tag{53}$$

and the formula for the conditional expectation associated with an observable with a purely discrete spectrum (derived in Srinivas 1980)

$$\varepsilon_\eta^A(B)=\sum_i P^A(\lambda_i)BP^A(\lambda_i). \tag{54}$$

Now (52) can be further simplified for the case of a normal or density operator state ρ by exploiting its ultraweak continuity property, to yield

$$\begin{aligned}\int_{\bar{R}} f(\lambda)\langle I^A(\mathrm{d}\lambda)\rho, B\rangle &= \sum_i f(\lambda_i)\langle \rho, P^A(\lambda_i)BP^A(\lambda_i)\rangle \\ &= \sum_i f(\lambda_i)\langle P^A(\lambda_i)\rho P^A(\lambda_i), B\rangle\end{aligned} \tag{55}$$

for all $f\in C(\bar{R})$ and $B\in B(\mathscr{H})$. From (55) we straightaway recover the von Neumann-Lüders collapse expression

$$I^A(\Delta)\rho=\sum_{\lambda_i\in\Delta} P^A(\lambda_i)\rho P^A(\lambda_i) \tag{56}$$

valid for all density operators ρ when A has a purely discrete spectrum. This completes the proof of Theorem 2.

Theorem 2 shows that the general collapse postulate (41) can be employed as the fundamental principle of quantum theory, from which the joint probabilities and the entire statistics of successive observations can be derived in a natural manner, for both normal and non-normal states. Of course, whether we start from the general collapse postulate or not, the general law of quantum mechanical joint probabilities given in (21) has a natural extension to the case of non-normal states also, and can be stated as follows.

General law of quantum mechanical joint probabilities: If an ensemble of systems prepared in state φ is subjected to a sequence of measurements of observables $A_1(t_1), A_2(t_2), \ldots, A_n(t_n)(t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n)$ then the expectation value of an arbitrary function $f_1(\lambda_1)f_2(\lambda_2)\ldots f_n(\lambda_n)$ of the outcomes (where each $f_i \in C(\bar{R})$) and the σ-additive joint probability measure $\mathrm{Pr}^{\varphi}_{A_1(t_1),\ldots,A_n(t_n)}(\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_n)$ on $(\bar{R}^n, B(\bar{R}^n))$ are given by the equation

$$\begin{aligned}
&\exp^{\varphi}_{A_1(t_1),\ldots,A_n(t_n)}[f_1(A_1(t_1))\ldots f_n(A_n(t_n))] \\
&= \int_{\bar{R}^n}\cdots\int f_1(\lambda_1)\ldots f_n(\lambda_n)\,\mathrm{Pr}^{\varphi}_{A_1(t_1),\ldots,A_n(t_n)}(\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_n) \\
&= \langle \varphi, f_1(A_1(t_1))\varepsilon^{A_1(t_1)}_{\eta}(\ldots\varepsilon^{A_{n-1}(t_{n-1})}_{\eta}(f_n(A_n(t_n)))\ldots)\rangle. \qquad (57)
\end{aligned}$$

The above formula (57) gives a very general prescription for quantum mechanical joint probabilities for successive measurements of arbitrary sequence of observables, carried out on any ensemble in a normal or non-normal state. There is of course a certain arbitraryness in these joint probabilities (57) as well as in the general collapse postulate (41) which arises due to the arbitraryness in the choice of the invariant mean η on the additive group R which plays a crucial role in the determination of the conditional expectation ε^A_η: $B(\mathscr{H}) \to \mathscr{U}'_A$, which occurs prominantly both in our collapse postulate (41) and in the general law for the joint probabilities (57). It is of course true that for the particular case of compatible observables, or for the case of successive observations of discrete observables performed on an ensemble of systems in a normal or density operator state to start with, we recover the conventional prescriptions for joint probabilities and the conventional von Neumann-Lüders collapse postulate, which are totally independent of the invariant mean chosen in (41) or (57). However the dependence on the choice of the invariant mean does show up in the general case, as it will be demonstrated in the next section for the case of joint probabilities of successive observation of position immediately followed by that of momentum.

For the case of the collapse postulate (16) which was proposed earlier (Srinivas 1980) it was suggested that the physical significance of employing different invariant means could perhaps be related to different ways of measuring an observable. Recently Ozawa (1987a) has indeed demonstrated that, for each invariant mean η, the change of state as given by the collapse pastulate (16) for the case $\Delta = R$, can be derived from a Hamiltonian model of the system interacting with a measuring apparatus by choosing a suitable (η-independent) interaction Hamiltonian and by assuming that the initial state of the apparatus is a particular (η-dependent) non-normal state referred to as an η-Dirac state. Of course the same demonstration goes through for the collapse postulate (41) also, as the change of state characterized by the collapse postulate (41) coincides with that characterized by (16) for the particular choice $\Delta = R$, as is clear from (45). Still the question of the physical significance of choosing different invariant means η in (41) or (57) does demand further detailed investigation.

6. An example: Successive measurement of position followed immediately by measurement of momentum

There is indeed a very curious feature of the σ-additive joint probabilities $\mathrm{Pr}^{\rho}_{A_1(t_1),\ldots,A_n(t_n)}(\mathrm{d}\lambda_1,\ldots,\mathrm{d}\lambda_n)$ defined by (21) or the more general

$\Pr^{\varphi}_{A_1(t_1),\ldots,A_n(t_n)}(d\lambda_1,\ldots,d\lambda_n)$ defined by (57) which is that they are σ-additive probability measures on $(\bar{R}^n, B(\bar{R}^n))$ where $\bar{R} = R \cup \{-\infty, \infty\}$. This feature, of course, played a crucial role in our proof of the existence of these joint probability measures in Theorems 1 and 2, based on the Reisz-Markov-Kakutani theorem which has a natural formulation on the space of continuous functions on a compact space such as $\bar{R}^n$.

We shall now see the physical significance of the fact that the joint probabilities of successive measurements are in general σ-additive measures on $\bar{R}^n$ and not R^n.

We shall consider a nonrelativistic single particle system in one dimension and consider the situation where an ensemble in state ρ is first subjected to a position measurement and immediately thereafter to a momentum measurement. If we employ the expectation value formula (57) then we get

$$\exp^{\varphi}_{Q,P}[f_1(Q)f_2(P)] = \langle \varphi, f_1(Q)\varepsilon^Q_\eta(f_2(P))\rangle \tag{58}$$

for all $f_1, f_2 \in C(\bar{R})$. Recently Ozawa (1987a) has proved the important result

$$\varepsilon^Q_\eta(f_2(P)) = \eta(f_2)I, \tag{59}$$

where I is the identity operator in $B(\mathscr{H})$. Using (59) in (58) we get

$$\exp^{\varphi}_{Q,P}[f_1(Q)f_2(P)] = \eta(f_2)\langle \varphi, f_1(Q)\rangle \tag{60}$$

for all $f_1, f_2 \in C(\bar{R})$. Since functions of the form $f_1(\lambda_1)f_2(\lambda_2)$ are dense in $C(\bar{R}^2)$, we can easily conclude from (60) that

$$\Pr^{\varphi}_{Q,P}(d\lambda_1, d\lambda_2) = \Pr^{\varphi}_Q(d\lambda_1)\mu_\eta(d\lambda_2), \tag{61}$$

where the σ-additive measure μ_η on $(\bar{R}, B(\bar{R}))$ is defined (via Riesz-Markov-Kakutani theorem) by

$$\int_{\bar{R}} f(\lambda)\mu_\eta(d\lambda) = \eta(f) \tag{62}$$

for all $f \in C(\bar{R})$.

Equation (61) gives the general formula for the joint probability measure for a position measurement immediately followed by a momentum measurement. It shows the curious feature that in any state the outcomes in a successive measurement of Q immediately followed by that of P are statistically independent (in the sense defined in Srinivas 1978), for we have

$$\Pr^{\varphi}_{Q,P}(d\lambda_1, d\lambda_2) = \Pr^{\varphi}_{Q,P}(\bar{R}, d\lambda_2)\Pr^{\varphi}_{Q,P}(d\lambda_1, \bar{R}) \tag{63}$$

where

$$\Pr^{\varphi}_{Q,P}(\bar{R}, d\lambda_2) = \mu_\eta(d\lambda_2) \tag{64}$$

and

$$\Pr^{\varphi}_{Q,P}(d\lambda_1, \bar{R}) = \Pr^{\varphi}_Q(d\lambda_1). \tag{65}$$

Of course if the starting state of the ensemble is a normal state characterized by the density operator ρ then we have

$$\Pr^{\rho}_{Q,P}(d\lambda_1, \bar{R}) = \Pr^{\rho}_Q(d\lambda_1) = \mathrm{Tr}\,(\rho P^Q(d\lambda_1)). \tag{66}$$

Equation (64) reveals the following interesting features of the probability measure of momentum when momentum is measured immediately after a position measurement on an ensemble in state φ. Firstly this probability $\Pr^{\varphi}_{Q,P}(\bar{R}, d\lambda_2)$ is totally independent of the initial state φ and is very different from the probability measure of momentum in the initial state given by the formula

$$\int_{\bar{R}} f_2(\lambda_2) \Pr^{\varphi}_{P}(d\lambda_2) = \langle \varphi, f_2(P) \rangle \tag{67}$$

for all $f_2 \in C(\bar{R})$. Of course the fact that the probabilities $\Pr^{\varphi}_{Q,P}(\bar{R}, d\lambda_2)$ and $\Pr^{\varphi}_{P}(d\lambda_2)$ are very different is nothing but the well-known '*quantum interference of probabilities*' (de Broglie 1948; Srinivas 1975).

To understand the nature of the probability distribution (64) better, we can use the invariant mean η_α defined for $0 \leqslant \alpha \leqslant 1$ by

$$\eta_\alpha(f) = \alpha f(-\infty) + (1-\alpha) f(\infty) \tag{68}$$

for all $f \in C(\bar{R})$. If we employ the invariant mean (68) in (64) we get

$$\Pr^{\varphi}_{Q,P}(\bar{R}, d\lambda_2) = \alpha \delta_{-\infty}(d\lambda_2) + (1-\alpha)\delta_{\infty}(d\lambda_2), \tag{69}$$

where the Dirac measure $\delta_x(d\lambda)$ on $\bar{R}$ is defined for any $x \in \bar{R}$ by

$$\int_{\bar{R}} f(\lambda)\delta_x(d\lambda) = f(x) \tag{70}$$

for all $f(\lambda) \in C(\bar{R})$. Note that the Dirac measure is indeed a σ-additive probability measure on $(\bar{R}, B(\bar{R}))$. Thus we see from (69) that the *probability distribution of momentum immediately following a position measurement is entirely concentrated at* $\pm\infty$. It is also clear from (69) that this probability distribution depends on the choice of the invariant mean η_α which is employed to characterize the collapse or change of state associated with the first position measurement.

These features were derived earlier also (Srinivas (1980)) from the collapse postulate (16). But there we had only a *finitely additive* probability measure on $(R, B(R))$ of the form

$$\Pr^{\varphi}_{Q,P}(R, (-\infty, a]) = 1 - \Pr^{\varphi}_{Q,P}(R, [a, \infty)) = \alpha \tag{71a}$$

$$\Pr^{\varphi}_{Q,P}(R, [a, b]) = 0 \tag{71b}$$

for all finite a, b. As we can easily see the σ-additive measure (69) on $(\bar{R}, B(\bar{R}))$ when reduced to $(R, B(R))$ leads precisely to the finitely additive measure (71).

Clearly we can see that $(\bar{R}^n, B(\bar{R}^n))$ is a much more natural setting for defining the quantum mechanical joint probabilities, a fact recently noted by Ozawa (1987b). Our result that the probability distribution of momentum immediately following a position measurement, is entirely concentrated at $\pm\infty$ provides in some sense a justification, for the first time from the fundamental postulates of quantum theory, for the original interpretation of uncertainty principle as propounded in the celebrated paper of Heisenberg (1927). From an analysis of the so called 'gamma-ray microscope experiment', Heisenberg concluded that: "At the moment of the position determination that is, when the quantum of light is being diffracted by the electron, the latter changes its momentum discontinuously. This change is greater the smaller the wave length of light, that is the more precise the position determination. Hence, at the moment when the position of the electron is being ascertained its momentum can be known only upto a magnitude that corresponds to the discontinuous change; thus the more accurate the position determination, the less accurate the momentum determination and vice versa" (English translation cited from Jammer 1974, p. 63).

In the absence of an appropriate collapse postulate for the continuous spectrum observables position and momentum, there was no way of deducing (from fundamental postulates of quantum theory) the sort of interpretation of uncertainty principle implied in the above statement of Heisenberg for the limiting or ideal case of a precise measurement of position or momentum. As is well known (Jammer 1974; Gnanapragasam and Srinivas 1979) the uncertainty relations of Heisenberg do not really capture the above feature of uncertainty principle, as (according to the basic principles of quantum mechanics) they refer only to independent measurements of position and momentum performed on *identical but different* ensembles of systems and *not* to successive measurement of position and momentum performed on the *same* ensemble of systems.

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