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Effect of phase fluctuation on a system of rotating superfluid

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Abstract. It has been shown that when the root-mean-square of the gradient of phase fluctuation exceeds the inverse of the coherence length a system of superfluid rotating with angular velocity exceeding the critical angular velocity has an instability.

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It was Anderson (1966) who first pointed out the importance of phase in superfluid ^{4}He. He introduced phase as an operator conjugate to the number operator. It was however pointed out by Carruthers and Nioto (1968) that a phase operator conjugate to the number operator does not exist. We take a different approach in introducing the phase. In earlier publications (see Biswas and Rama Rao 1971) we have illustrated our approach of introducing phase in a consistent manner. As stated earlier we start with the boson annihilation operator $\psi(r)$. Let us introduce the coherent states $|\psi\rangle$ from the definition

$$\psi(r)|\alpha\rangle = \alpha(r)|\alpha(r)\rangle. \tag{1}$$

$\alpha(r, t)$ is thus the eigenvalue of the annihilation operator, we write

$$\alpha = \{[J(r, t)]/\hbar\}^{\frac{1}{2}} \exp[i\phi(r, t)]. \tag{2}$$

Let

$$H = -\int \frac{\hbar^2}{2m} \psi^\dagger \nabla \psi^2 \mathbf{d}r + \int V(|r - r'|)\psi^\dagger(r)\psi(r)\psi^\dagger(r')\psi(r')\,\mathrm{d}r\,\mathrm{d}r' \tag{3}$$

be the Hamiltonian. We have shown earlier (Biswas and Rama Rao 1975) that $\phi(r)$ and $J(r)$ satisfy conjugate relations

$$\langle \dot{\phi}(r)\rangle = \langle \delta H/\delta J(r)\rangle, \tag{4}$$

$$\langle \dot{J}(r)\rangle = -\langle \delta H/\delta\phi(r)\rangle. \tag{5}$$

The angular brackets in (4) and (5) denote averages with respect to the distribution function $\rho(\alpha^*, \alpha) = \langle \alpha|\rho|\alpha\rangle = \rho(J, \phi, t)$. Equation (4) gives

$$\frac{\partial\langle\phi\rangle}{\delta t} + \frac{\hbar}{2m}\langle(\nabla\phi)^2\rangle = \frac{\hbar}{2m}\left\langle\frac{\nabla^2\sqrt{J}}{\sqrt{J}}\right\rangle - \frac{1}{\hbar}\int V(|r - r'|)\langle J(r')\rangle \mathrm{d}r \tag{6}$$

and

$$\frac{\partial\langle J(r)\rangle}{\delta t}+\frac{\hbar}{m}\nabla\langle(J\nabla\phi)\rangle=0. \tag{7}$$

We have $\langle J(r)\rangle=[\langle N(r)\rangle+1]h$ where $N(r)$ is the number of particles at a volume $\partial \mathbf{r}$ around $\mathbf{r}$. Equations (6) and (7) are the phase equation and the equation of continuity respectively. We have shown earlier (Biswas and Rama Rao 1975) how the two-fluid equations follow from the conservation of energy equation and conservation of momentum equation and equations (6) and (7). In this note our main equation is equation (6). We are considering a system in which the superfluid is contained in a circular cylinder of very large radius. The cylinder rotates with an angular velocity Ω which exceeds the critical velocity of the superfluid. The whole superfluid is thus rotating as a rigid body with angular velocity Ω. We get from (6)

$$\frac{\partial\langle\phi\rangle}{\delta t}+\frac{\hbar}{2m}[\nabla\langle\phi\rangle]^2+\frac{\hbar}{2m}\langle(\nabla\tilde{\phi})^2\rangle=\frac{\hbar}{2m}\frac{\nabla^2 f}{f}, \tag{8}$$

where we have neglected the interaction. $\nabla\tilde{\phi}$ is the gradient of the fluctuation in ϕ. We have replaced $[\langle J\rangle/V]^{\frac{1}{2}}$ by f the square root of the density and neglected the fluctuation in J since the fluctuation in density is very small at low temperatures. For a stationary state we put

$$\partial\langle\phi\rangle/\delta t=-\mu/\hbar, \tag{9}$$

where μ is the chemical potential of the system.

Since Ω is the angular velocity of a fluid element we have

$$\frac{h}{m}|\nabla\langle\phi\rangle|=\Omega r, \quad \text{or} \quad [\nabla\langle\phi\rangle]^2=\frac{m^2}{h^2}\Omega^2 r^2.$$

We further put $\langle(\nabla\tilde{\phi})^2\rangle=g^2$. Then (2) becomes

$$\nabla^2 f+\left(\frac{2m\mu}{\hbar^2}-g^2-\frac{m^2\Omega^2 r^2}{\hbar^2}\right)f=0 \tag{10}$$

or

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r}+[\gamma^2-\beta^2 r^2]f=0, \tag{11}$$

where

$$\gamma^2=(2m\mu/\hbar^2)-g^2, \quad \beta^2=m^2\Omega^2/\hbar^2.$$

The solution of (11) for a cylinder of infinite radius is (with the boundary condition that $f\to 0$ as $r\to\infty$)

$$f=A\exp(-\beta r^2/2)\int_0^\infty t\gamma^2/2\beta\exp(-t^2/4\beta)^{\frac{1}{2}}J_0(tr)\,\mathrm{d}t, \tag{12}$$

where $J_0(x)$ is the Bessel's function of order zero. We see from (12) that if $\gamma^2<0$ there is an instability in the system. Hence if $g>(2m\mu)^{\frac{1}{2}}/\hbar$ i.e. the root-mean-square of the gradient of phase fluctuation exceeds the inverse of the coherence length there is an instability in the system. From the present analysis we cannot pin-point the experimental implication of the instability. It may be that a sort of superfluid turbulence develops in the system.

References

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