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Superconformal transformations of the $N = 4$ supersymmetric Yang-Mills theory

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Abstract. We obtain the superconformal transformation laws of the $N = 4$ supersymmetric Yang-Mills theory and explicitly demonstrate the closure of the algebra.

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The $N = 4$ supersymmetric Yang-Mills theory (SSYM) has many interesting properties, one of them being its finiteness to all orders in perturbation (Mandelstam 1983; Howe *et al* 1983; Brink *et al* 1983; Salam *et al* 1983). One of the ways to prove finiteness (Sohnius and West 1981), relies on the fact that the $N = 4$ superconformal (SC) algebra does not involve the chiral-R charge. In this paper we obtain the SC transformation laws for the $N = 4$ SSYM supermultiplet and explicitly verify that the representation of the SC algebra does not involve R charge as expected. We give a simple way of obtaining the SC transformation laws and explicitly verify the closure of the algebra.

The $N = 4$ SC algebra is well known (Sohnius 1985). In addition to the usual fifteen generators of the conformal group, the $N = 4$ SC algebra in 3 + 1 dimensions consists of eight four-component Majorana fermionic charges corresponding to the SUSY charges $Q^M{}_i$ and SC charges $S^M{}_i$, $i = 1 - 4$, where M implies the Majorana fermions.

These fermionic charges can be expressed in terms of eight two-component Weyl fermionic charges $Q_{i\alpha}$ and $S_{i\alpha}$ and their hermitian conjugates $\bar{Q}_{i\dot{\alpha}}$ and $\bar{S}_{i\dot{\alpha}}$ (α and $\dot{\alpha}$ are Weyl fermion indices) as,

$$Q^M{}_i = \begin{pmatrix} Q_{i\alpha} \\ \bar{Q}_{i\dot{\alpha}} \end{pmatrix} \quad S^M{}_i = \begin{pmatrix} S_{i\alpha} \\ \bar{S}_{i\dot{\alpha}} \end{pmatrix}. \tag{1}$$

Further, the algebra contains the generators of global SO(4) transformations R_{ij} which rotate $Q_{i\alpha}$ (and $S_{i\alpha}$) among themselves. In the two-component Weyl fermion notation, there will be a larger SU(4) symmetry. However, unlike the $N = 1$ and $N = 2$ cases, the $N = 4$ SC algebra does not involve the R charge which generates chiral transformations. Thus the $N = 4$ SC algebra is similar to the $N = 1$ or 2 SC algebra (Sohnius 1985) except for the following,

$$\begin{aligned} \{Q_{i\alpha}, S_j^{\beta}\}_+ &= \delta_{ij}[M_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta + 2iD\delta_\alpha{}^\beta] - 4\delta_\alpha{}^\beta R_{ij} \\ \{\bar{S}_i^{\dot{\alpha}}, Q_{j\beta}\}_+ &= \delta_{ij}[M_{\mu\nu}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_\beta + 2iD\delta^{\dot{\alpha}}{}_\beta] - 4\delta^{\dot{\alpha}}{}_\beta R_{ij}. \end{aligned} \tag{2}$$

With $R = \sum_i R_{ii} = 0, R_{ij}^\dagger = R_{ji}$

$$[Q_{i\alpha}, R_{jk}]_- = \delta_{ik} Q_{j\alpha} - (1/4)\delta_{jk} Q_{i\alpha}$$

$$[\bar{Q}_{i\dot{\alpha}}, R_{jk}]_- = -\delta_{ij}\bar{Q}_{k\dot{\alpha}} + (1/4)\delta_{jk}\bar{Q}_{i\dot{\alpha}}$$

$$[R_{ij}, R_{kl}]_- = \delta_{il} R_{kj} - \delta_{jk} R_{il}.$$

We will use

$$(\sigma_\mu)_{\alpha\beta} = (\sigma_0, \sigma_i)_{\alpha\beta}, \quad (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} = (\sigma_0, -\sigma_i)^{\dot{\alpha}\beta}, \quad \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix},$$

$$\Sigma_{\mu\nu} = \tfrac{1}{4}[\gamma_\mu, \gamma_\nu]_-, \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3.$$

A representation of the algebra

In four-dimensional flat spacetime the largest SUSY which can be represented on a particle multiplet with spins $\leqslant 1$ is $N = 4$. The $N = 4$ super Yang-Mills multiplet will contain gauge bosons A_μ^a(a is the gauge group index) and hence must be massless (Sohnius 1985). In addition, there are four Weyl fermions λ_i^a $i = 1-4$. A gauge boson has two Bose degrees of freedom and each Weyl fermion has two fermionic degrees of freedom. To match Bose-Fermi degrees of freedom, there must be six more bosonic degrees of freedom. For each a, the gauge group index, these are the real selfdual $A_{ij}^a = 1/2\varepsilon_{ijkl}A_{kl}^a$ and self anti-dual $B_{ij}^a = -1/2\varepsilon_{ijkl}B_{kl}^a$ fields, where ε_{ijkl} is the completely antisymmetric SO(4) tensor.

The SUSY transformation laws for these fields are,

$$\begin{aligned}
&\delta_Q(\varepsilon)\varphi_{ij}^a = \bar{\varepsilon}_{Li}\lambda_j^a - \bar{\varepsilon}_{Lj}\lambda_i^a + \varepsilon_{ijkl}\bar{\varepsilon}_{Rk}\lambda_l^a \\
&\delta_Q(\varepsilon)A_\mu^a = i\bar{\varepsilon}_k\gamma_\mu\lambda_k^a \\
&\delta_Q(\varepsilon)\lambda_i^a = \{\textstyle\sum\cdot F^a\delta_{ik} + i(\not{D}(A_{ik} - i\gamma_5 B_{ik}))^a \\
&\qquad + i/2[(A_{ij} - i\gamma_5 B_{ij}) \times (A_{jk} + i\gamma_5 B_{jk})]^a\}\varepsilon_k,
\end{aligned} \tag{4}$$

where $\varphi_{ij}^a = 1/2(A_{ij}^a + iB_{ij}^a)$ and, $\bar{\varepsilon}_{\substack{Li\\R}} = \bar{\varepsilon}_i[\frac{1}{2}(1 \pm \gamma_5)]$.

Here the SUSY transformation parameters $\bar{\varepsilon}_i$ are the four-component Majorana fermionic Grassmann variables, and $(A \times B)^a = f^{abc}A^bB^c$. The $N = 4$SSYM Lagrangian is (Sohnius 1985)

$$\begin{aligned}
\mathrm{L} = &-1/4F_{\mu\nu}^a F^{\mu\nu a} + i/2\bar{\lambda}_i^a(\not{D}\lambda_i)^a \\
&+ 1/8(D_\mu A_{ij})^a(D^\mu A_{ij})^a + 1/8(D_\mu B_{ij})^a(D^\mu B_{ij})^a \\
&+ i/2\bar{\lambda}_i \times \lambda_j\cdot A_{ij} + 1/2\bar{\lambda}_i \times \gamma_5\lambda_j\cdot B_{ij} \\
&+ 1/32[A_{ij} \times B_{kl}]\cdot[A_{ij} \times B_{kl}] \\
&+ 1/64[A_{ij} \times A_{kl}]\cdot[A_{ij} \times A_{kl}] \\
&+ 1/64[B_{ij} \times B_{kl}]\cdot[B_{ij} \times B_{kl}].
\end{aligned}$$

The SC transformation laws of a generic field X^a can be obtained, as was done in the case of $N = 2$ theory (Mehta 1987). One demands that the SC, SUSY and conformal transformations satisfy the SC algebra. In particular, requiring

$$[Q^M{}_i, K_\mu]_- = \gamma_\mu S^M{}_i$$

yields

$$\begin{aligned}4[X^a, S^M{}_i]_- &= [X^a, [\gamma_\mu Q^M{}_i, K^\mu]_-]_- \\ &= -[[X^a, \gamma_\mu Q^M{}_i]_-, K^\mu]_- + [[X^a, K^\mu]_-, \gamma_\mu Q^M{}_i]_-, \end{aligned} \tag{5}$$

where K_μ are the generators of conformal transformations.

Substituting the known SUSY and conformal transformation properties of X^a on the right hand side one can solve the SC transformation laws of X^a.

However the calculation can be simplified and made more transparent by using a slightly different method which involves exploiting another closure relation

$$[S^M{}_i, P_\mu]_- = \gamma_\mu Q^M{}_i$$

which yields

$$4[X^a, Q^M{}_i]_- = -[[X^a, \gamma_\mu S^M{}_i]_-, P^\mu]_- + [[X^a, P^\mu]_-, \gamma_\mu S^M{}_i]_-. \tag{6}$$

Equation (6) is easier to calculate than (5) since the action of boosts P_μ is simpler than that of conformal boosts K_μ. First we write down all the possible terms which can appear in the SC transformation of a generic field X^a. These terms must contain the SC transformation parameter ζ_i and must have the same mass dimension and the same Lorentz, gauge, SO(4) and the parity transformation property as X^a. Since SUSY transformation parameter ε_i has a mass dimension $-1/2$ and the superconformal transformation parameter ζ_i has mass dimension $+1/2$ one possible set of terms are obtained by taking over all the terms which appear in SUSY transformation of a generic field X^a and replacing the SUSY transformation parameter (mass dimension $-1/2$) with $ai\not{x}\zeta_i$ (mass dimension $-1/2$), where a is a real constant to be fixed. It can be easily seen from the algebra (6) that these terms containing x are the only possible explicitly x-dependent terms. However, there can be terms which do not contain x. For bosons there cannot be any such term since it will have mass dimension greater than one. The most general form of such a term in the SC transformation of a fermion is $b(A^a_{ij} + ci\gamma_5 B^a_{ij})$, where b and c are arbitrary real constants which, along with the constant a introduced earlier, are fixed by the closure relation (6) to be $a = -1$, $b = 2$, $c = 1$. The $N = 4$ SC transformation laws thus obtained are

$$\begin{aligned}\delta_S(\zeta)\varphi^a_{ij} &= (i\bar{\zeta}_i\not{x})_L\lambda^a_j - (i\bar{\zeta}_j\not{x})_L\lambda^a_i + \varepsilon_{ijkl}(i\bar{\zeta}_k\not{x})_R\lambda^a_l, \\ \delta_S(\zeta)A^a_\mu &= (i\bar{\zeta}_i\not{x})\gamma_\mu\lambda^a_i, \\ \delta_S(\zeta)\lambda^a_i &= \{\textstyle\sum\cdot F^a\delta_{ik} + i(\not{D}(A_{ik} - i\gamma_5 B_{ik}))^a \\ &\quad + i/2[(A_{ij} - i\gamma_5 B_{ij}) \times (A_{jk} + i\gamma_5 B_{jk})]^a\}(-i\not{x}\zeta_k) \\ &\quad + 2(A_{ik} + i\gamma_5 B_{ik})\zeta_k. \end{aligned} \tag{7}$$

The additional piece in the transformation of fermion, which will be absent in the case of $N = 1$ SSYM arises due to the fact that the $N = 4$ SSYM consists of an $N = 1$ gauge multiplet interacting with three $N = 1$ chiral multiplets. The SC transformation of the fermions belonging to the chiral multiplet will pick up exactly this additional piece. When we restrict $i = 1,2$ and replace A^a_{ij} by S^a and B^a_{ij} by P^a the $N = 4$ SC and SUSY transformation laws reduce to those of the $N = 2$ (Mehta 1987).

Let us see if the SC transformation laws satisfy the rest of the algebra also. Consider the action of commutator of SUSY and SC transformation on φ^a_{ij}. We will denote $i\bar{\zeta}\not{x}$ by $\bar{\eta}$.

$$
\begin{aligned}
-[\delta_Q(\varepsilon), \delta_S(\zeta)]_- \varphi^a_{ij}
&= \delta_Q(\varepsilon)[\bar{\eta}_{Li}\lambda^a_j - \bar{\eta}_{Lj}\lambda^a_i + \varepsilon_{ijkl}\bar{\eta}_{Rk}\lambda^a_l] \\
&\quad - \delta_S(\zeta)[\bar{\varepsilon}_{Li}\lambda^a_j - \bar{\varepsilon}_{Lj}\lambda^a_i + \varepsilon_{ijkl}\bar{\varepsilon}_{Rk}\lambda^a_l] \\
&= [\bar{\eta}_{Li}\{\textstyle\sum\cdot F^a\delta_{jk} + i(\not{D}(A_{jk} - i\gamma_5 B_{jk}))^a \\
&\quad + i/2[(A_{jl} - i\gamma_5 B_{jl}) \times (A_{lk} + i\gamma_5 B_{lk})]^a\}\varepsilon_k \\
&\quad - \{i \leftrightarrow j\} \\
&\quad + \varepsilon_{ijkl}\bar{\eta}_{Rk}\{\textstyle\sum\cdot F^a\delta_{lm} + i(\not{D}(A_{lm} - i\gamma_5 B_{lm}))^a \\
&\quad + i/2[(A_{ln} - i\gamma_5 B_{ln}) \times (A_{nm} + i\gamma_5 B_{nm})]^a\}\varepsilon_m] \\
&\quad - [\eta \leftrightarrow \varepsilon] \\
&\quad - 2\bar{\varepsilon}_{Li}(A_{jk} + i\gamma_5 B_{jk})^a\zeta_k + 2\bar{\varepsilon}_{Lj} \\
&\quad \times (A_{ik} + i\gamma_5 B_{ik})^a\zeta_k - 2\varepsilon_{ijkl}\bar{\varepsilon}_{Rk} \\
&\quad \times (A_{lm} + i\gamma_5 B_{lm})^a\zeta_m,
\end{aligned}
\tag{8}
$$

where the only $\not{x}$ independent terms are the last three terms. Using the duality property of A^a_{ij} and B^a_{ij} and the formulae for the product of ε_{ijkl} tensors we can extract φ^a_{ij} out of those terms. Thus the $\not{x}$ independent terms can be rewritten as

$$
\begin{aligned}
&-2\varphi^a_{ij}\bar{\varepsilon}_k\zeta_k - 4\varphi^a_{kj}\{\bar{\varepsilon}_{Li}\zeta_k - \bar{\varepsilon}_{Rk}\zeta_i - \delta_{ik}/4\bar{\varepsilon}_l\gamma_5\zeta_l\} \\
&\qquad - 4\varphi^a_{ik}\{\bar{\varepsilon}_{Lj}\zeta_k - \bar{\varepsilon}_{Rk}\zeta_i - \delta_{jk}/4\bar{\varepsilon}_l\gamma_5\zeta_l\}.
\end{aligned}
$$

The first term constitutes a part of the dilatation transformation of φ^a_{ij}. The last two terms can be interpreted as the SO(4) transformation of φ^a_{ij} with transformation parameters being the expressions in the curly brackets. Note that the trace of the SO(4) transformation parameter i.e. of each curly bracket is zero. Therefore, unlike the $N = 1,2$ cases, there is no chiral $U(1)$ transformation involved here.

The $\not{x}$-dependent terms can be rearranged into a Lorentz transformation, a field-dependent gauge transformation, and a part of dilatation of φ^a_{ij}. The same commutator is evaluated on other fields also. The result can be expressed for a generic field X^a as

$$
\begin{aligned}
[\delta_Q(\varepsilon), \delta_S(\zeta)]_- X^a = -[X^a, 2i[(\bar{\zeta}_i\Sigma_{\mu\nu}\varepsilon_i)M^{\mu\nu} \\
+ \bar{\zeta}_i\varepsilon_i D + \Lambda^b G^b] - 4Y_{ij}R_{ij}]_- + \text{EOM},
\end{aligned}
\tag{9}
$$

where the first and second terms in the bracket correspond to the generators of the Lorentz transformation and dilatation with parameters of the transformation being $\bar{\zeta}_i\sum_{\mu\nu}\varepsilon_i$ and $\bar{\zeta}_i\varepsilon_i$ respectively. The third term corresponds to the generator of local gauge transformation with parameter of transformation being

$$\Lambda^b = \bar{\varepsilon}_i(i\not{A}^a)(-i\not{x}\zeta_i) - \bar{\varepsilon}_i(A^a_{ij} - i\gamma_5 B^a_{ij})(-i\not{x}\zeta_j).$$

The fourth term is the generator of SO(4) transformation with transformation parameter

$$Y_{ij} = \bar{\varepsilon}_{Li}\zeta_j - \bar{\varepsilon}_{Rj}\zeta_i$$

and

$$[\lambda^a_i, Y_{kl}R_{kl}]_- = Y_{ki}\gamma_5\lambda^a_k - Y_{kk}/4\gamma_5\lambda^a_i$$

$$[\varphi^a_{ij}, Y_{kl}R_{kl}]_- = (-Y_{ki}\varphi^a_{kj} + Y_{kk}/4\varphi^a_{ij}) + (-Y_{kj}\varphi^a_{ik} + Y_{kk}/4\varphi^a_{ij}). \tag{10}$$

The algebra closes only up to a field-dependent local gauge transformation because we are working in the Wess-Zumino gauge. Further, in the case of fermions, the closure requires the use of equations of motion. This is because the transformation laws are on-shell.

The commutator of two SC transformation can be similarly evaluated. The result is

$$[\delta_S(\zeta^1), \delta_S(\zeta^2)]_- X^a = 2[X^a, (\bar{\zeta}^1_k\gamma_\mu\zeta^2_k)K^\mu + \tilde{\Lambda}^b G^b]_- + \mathrm{EOM} \tag{11}$$

where the first term on the right hand side corresponds to the generator of conformal transformations with parameter $\bar{\zeta}^1_k\gamma_\mu\zeta^2_k$. The second term is the generator of local gauge transformation with parameter

$$\tilde{\Lambda}^b = (i\bar{\zeta}^1_i\not{x})(i\not{A}^a)(-i\not{x}\zeta^2_i) + (i\bar{\zeta}^1_i\not{x})(A^a_{ij} - i\gamma_5 B^a_{ij})(-i\not{x}\zeta^2_j).$$

Again the algebra closes only upto the field-dependent local gauge transformation and the equations of motion for the same reasons as mentioned earlier.

Thus we have obtained the superconformal transformation laws for the $N = 4$ SSYM and explicitly verified the closure of the algebra and the absence of the R charge.

Further the action is invariant under SUSY and conformal transformations and, since the superconformal transformations satisfy the algebra, the action is also invariant under superconformal transformations.

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