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The two-body multipole problem of electrodynamics

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Abstract. A 2-body system composed of two objects having arbitrary distributions of charge and current is discussed. An expression for the velocity dependent potential between these two objects has been obtained in the non-relativistic approximation. This potential consists of two parts viz. a velocity independent scalar potential Φ_{eff} and another part which is linearly dependent on the relative velocity between the objects. The second part naturally suggests a vector potential $\mathbf{A}_{\text{eff}}$. The potentials have been expanded into multipole terms. It has been found that Φ_{eff} is a sum of two components viz. Φ_{EE} and Φ_{MM} such that each multipole term in Φ_{EE} represents an interaction between the electric multipoles of the two systems, each term in Φ_{MM} represents an interaction between their magnetic multipoles whereas each term in $\mathbf{A}_{\text{eff}}$ represents an interaction between an electric multipole of one and a magnetic multipole of the other. The results have been applied to the interaction between an electric dipole and a magnetic dipole. The symmetry among the multipole terms in $\mathbf{A}_{\text{eff}}$ suggests vanishing vector potential between two identical objects. A corollary of this appears to be absence of spin orbit interaction between two identical particles in the same spin state.

Keywords. Multipole expansion; multipole moments; irreducible spherical tensors; two-body scalar and vector potentials; spin-orbit interaction; electric dipole-magnetic dipole interaction.

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1. Introduction

The simplest problem involving a charged object moving in an external magnetic field assumes a uniform magnetic field $\mathbf{B}$ and a point charge e. The associated interaction Hamiltonian is the following familiar expression (Messiah 1966).

$$H_{\text{int}} = -\frac{e}{2mc}\mathbf{B}\cdot\mathbf{L}, \tag{1}$$

where m is the mass and $\mathbf{L}$ is the orbital angular momentum of the test charge.

The next order of generalization treats the field as non-uniform. Decomposing the source into magnetic multipole components and letting $\mathbf{B}_l$ represent the magnetic field due to the lth magnetic multipole, the following two expressions for the interaction Hamiltonian can be obtained (Datta 1984, 1985).

$$\left.\begin{aligned} H_{\text{int}}^{<} &= -\frac{e}{mc}\left\{\sum_{l=1}^{\infty}\frac{\mathbf{B}_l}{l+1}\right\}\cdot\mathbf{L} \\ H_{\text{int}}^{>} &= \frac{e}{mc}\left\{\sum_{l=1}^{\infty}\frac{\mathbf{B}_l}{l}\right\}\cdot\mathbf{L}. \end{aligned}\right\} \tag{2}$$

The first expression envisages a point test charge moving in an interior domain of radius R, the sources all lying outside this sphere. The second expression does the same thing when all the sources are confined within a sphere of radius R and the trajectory of the test charge is assumed to lie exterior to this sphere.

The initial objective of the present article was to generalize the problem further by assuming that the test charge is endowed with all orders of electric and magnetic moments. It was soon found that a further enlargement of the scope of the problem was possible, without additional expenditure of time and effort, such that both the source object and the test object could be treated on the same footing, each endowed with all orders of electric and magnetic moments. Therefore, the problem under discussion is a two-body problem of electrodynamics in which two extended objects, characterized by their own charge and current densities, are moving with arbitrary velocities and, in the process, are interacting with each other through electromagnetic forces. Figure 2 gives a schematic representation of this problem.

The primary objectives of this article are, therefore, I. to obtain the velocity dependent potential U between the above two objects, A and B, and II. to expand this potential into a multipole series.

The simplest 2-body problem in electrodynamics—involving interaction between two point charges is itself considerably difficult (Jackson 1975) and can be handled by making some approximations, like smallness of the velocities of the participating particles (Synge 1972). An approximate 2-body potential, correct to order $(v/c)^2$ can be obtained and is known as Darwin potential (Darwin 1920). We have handled the present problem in the non-relativistic approximation—i.e. using Lorentz transformation in the non-relativistic limit—retaining terms only up to first order in v/c. The set of assumptions that have gone into our analysis are as follows:

(i) It is possible to identify two *non-rotating* rest frames S_A and S_B of the objects A and B, their origins always coinciding with the centres of mass of the respective objects.

(ii) The velocities $c\boldsymbol{\beta}_A$ and $c\boldsymbol{\beta}_B$ of the rest frames S_A and S_B with respect to the laboratory frame S_L are non-relativistic. Consequently, it is permissible to perform non-relativistic Lorentz transformations from the above rest frames to the laboratory frame so that terms containing β_A^2, β_B^2, $\boldsymbol{\beta}_A\cdot\boldsymbol{\beta}_B$ and higher powers drop out and $\gamma_A \equiv 1/(1-\beta_A^2)^{1/2} \approx 1$, $\gamma_B \equiv 1/(1-\beta_B^2)^{1/2} \approx 1$.

(iii) The internal charge and the current distributions of the objects, as they appear in the respective rest frames, are steady.

(iv) The accelerations of the objects and/or characteristic time of interaction between the objects are not too large so that the effect of the radiation damping forces may be ignored. This last assumption has been explained and justified in § 5.

Under the above assumptions we have been able to express the desired potential as the sum of three terms viz.

$$U = \Phi_{\mathrm{EE}}(\mathbf{r}) + \Phi_{\mathrm{MM}}(\mathbf{r}) - \boldsymbol{\beta}\cdot\mathbf{A}_{\mathrm{EM}}(\mathbf{r}) \tag{3}$$

where r is the radius vector extending from the centre of mass of one of the objects, A, to the centre of mass of the other object B and $c\boldsymbol{\beta}$ is the corresponding relative velocity. Denoting their charge densities by ρ and their current densities by $\mathbf{J}$ the integral expressions of the three potentials appearing in (3) have been shown as

$$\left.\begin{aligned}\Phi_{\mathrm{EE}}(\mathbf{r}) &= \int_B \mathrm{d}^3\eta \int_A \mathrm{d}^3\xi \frac{\rho_B(\boldsymbol{\eta})\rho_A(\boldsymbol{\xi})}{|\mathbf{r}+\boldsymbol{\eta}-\boldsymbol{\xi}|} \\ \Phi_{\mathrm{MM}}(\mathbf{r}) &= -\frac{1}{c^2}\int_B \mathrm{d}^3\eta \int_A \mathrm{d}^3\xi \frac{\mathbf{J}_B(\boldsymbol{\eta})\cdot\mathbf{J}_A(\boldsymbol{\xi})}{|\mathbf{r}+\boldsymbol{\eta}-\boldsymbol{\xi}|} \\ \mathbf{A}_{\mathrm{EM}}(\mathbf{r}) &= \frac{1}{c}\int_B \mathrm{d}^3\eta \int_A \mathrm{d}^3\xi \frac{\rho_B(\boldsymbol{\eta})\mathbf{J}_A(\boldsymbol{\xi}) - \mathbf{J}_B(\boldsymbol{\eta})\rho_A(\boldsymbol{\xi})}{|\mathbf{r}+\boldsymbol{\eta}-\boldsymbol{\xi}|}\end{aligned}\right\} \tag{4}$$

The integrations have been performed over the rest volumes of the objects A and B.

It is seen that $\Phi_{\mathrm{EE}}(\mathbf{r})$ and $\Phi_{\mathrm{MM}}(\mathbf{r})$ are respectively the potentials of interaction between the rest frame charge densities and between the rest frame current densities of the individual systems. The vector potential $\mathbf{A}_{\mathrm{EM}}(\mathbf{r})$ on the other hand, determines the velocity dependent force between the two systems and depends exclusively on the interaction between the rest frame charge density of one system and the rest frame current density of the other.

The two scalar potentials Φ_{EE} and Φ_{MM} and the vector potential $\mathbf{A}_{\mathrm{EM}}$ have subsequently been expanded into multipole terms. The expansions are shown first in their irreducible spherical tensor forms and then in their non-tensorial forms. It has been found that each term in $\Phi_{\mathrm{EE}}(\mathbf{r})$ consists of interaction between the electric multipoles of the two systems, each term in $\Phi_{\mathrm{MM}}(\mathbf{r})$ represents an interaction between their magnetic multipoles, whereas each term in $\mathbf{A}_{\mathrm{EM}}(\mathbf{r})$ represents an interaction between an electric multipole of one and a magnetic multipole of the other.

We have utilized the above results to give a multipole character to the Hamiltonian of a two-body system. The potentials shown in (3) suggest the following Hamiltonian.

$$\left.\begin{aligned}H &= \frac{P^2}{2M} + H_{\mathrm{rel}}(\mathbf{r},\mathbf{p}) \\ H_{\mathrm{rel}}(\mathbf{r},\mathbf{p}) &= \frac{1}{2\mu}\left(\mathbf{p} - \frac{1}{c}\mathbf{A}_{\mathrm{EM}}(\mathbf{r})\right)^2 + \Phi_{\mathrm{EE}}(\mathbf{r}) + \Phi_{\mathrm{MM}}(\mathbf{r})\end{aligned}\right\}. \tag{5}$$

Here $\mathbf{P}$ and M represent the total momentum and the total mass of the system. $H_{\mathrm{rel}}(\mathbf{r},\mathbf{p})$ is the Hamiltonian of relative motion and $\mathbf{p}$ is the momentum of relative motion. μ is the reduced mass of the 2-body system. The above equation tells that the relative motion between two extended charge-current distributions can be visualized to be the motion of a single representative particle of unit positive charge and mass μ, moving in a static field determined by a scalar potential $\Phi_{\mathrm{EE}}(\mathbf{r}) + \Phi_{\mathrm{MM}}(\mathbf{r})$, and a vector potential $\mathbf{A}_{\mathrm{EM}}(\mathbf{r})$.

The derived multipole formulas have been used to obtain the interaction between an electric dipole and a magnetic dipole. The interaction Hamiltonian shows a coupling among the electric dipole moment of one object, the spin angular momentum of the other and the orbital angular momentum of their relative motion.

Our multipole expansion for the vector potential $\mathbf{A}_{\mathrm{EM}}$ shows a kind of symmetry between electric and magnetic multipoles due to which two identical objects having the same orientation fail to produce any $\mathbf{A}_{\mathrm{EM}}$ at all. The interaction force between two identical objects, therefore, appears to be velocity-independent. A corollary of this observation is that there is no interaction Hamiltonian coupling spin and orbital angular momentum between two identical particles in the same spin state.

Multipole moments and the associated fields form an old topic discussed in some details by Morse and Feshbach (1953). Blatt and Weisskopf (1952) and Jackson (1975) adopted complex multipole moments for discussion of the radiation field. Jackson also gives a complete coverage of the multipole electrostatic field. The magnetic counterpart of this treatment can be found in Bronzan (1971) and Gray (1978). The multipole expansion of the electrostatic potential between two distributions of charges is found in Stone (1979).

The two-body problem of this article has been defined in § 2 wherein we have also derived the basic Hamiltonian of the two-body system of charge and current written in terms of a scalar potential and a vector potential.

Multipole expansion of these potentials will follow from an expansion of $1/(|\mathbf{r}+\boldsymbol{\eta}-\boldsymbol{\xi}|)$ in an infinite series of products of spherical harmonics. We have worked out this expansion using the properties of irreducible spherical tensors, and have used this expansion to express the interaction potentials as three infinite series of multipole–multipole interaction terms. Details of these mathematical steps are quite involved and covered in a separate paper. The final results of this analysis are quoted in § 3.

The multipole formulas presented in this article can be applied to a variety of familiar situations like the interaction between two electric charges, between an electric charge and an electric dipole, between an electric charge and a magnetic dipole, between two electric dipoles, between two magnetic dipoles. One of the above interactions, viz. the one between an electric charge and a magnetic dipole will yield the familiar hydrogen atom spin–orbit coupling potential. We have however, shown in § 4 only one application of the multipole formulas to obtain an unfamiliar interaction Hamiltonian—corresponding to the interaction between an electric dipole and a magnetic dipole. In § 5 we have provided a brief analysis of the limitations of our formalism.

2. Non-relativistic potential between two distributions of charge and current

Our discussion is based on the following principle. Let A denote a system consisting of a large number N of particles, the ith particle being at the radius vector $\mathbf{r}_i$ with velocity $\mathbf{v}_i$ at the time t. The force on this particle is the vector sum of the external force $\mathbf{f}_i^{(\mathrm{ext})}$ and the internal force $\mathbf{f}_i^{(\mathrm{int})}$, the latter being the vector sum of the interaction forces on this particle due to all the other particles in the system. We assume a velocity dependent potential $u_i(\mathbf{r}_i, \mathbf{v}_i)$ such that

$$\mathbf{f}_i^{(\mathrm{ext})} = -\nabla_{\mathbf{r}_i} u_i + \frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\mathbf{v}_i} u_i, \tag{6}$$

where the symbol $\nabla_{\mathbf{a}}$ stands for $(\partial/\partial a_x, \partial/\partial a_y, \partial/\partial a_z)$.

The velocity dependent potential U of the system is *defined* as

$$U = \sum_{i=1}^{N} u_i(\mathbf{r}_i, \mathbf{v}_i). \tag{7}$$

Let the system A be endowed with f degrees of freedom so that there exist f generalized co-ordinates $\{q_\alpha;\ \alpha = 1, 2, \ldots, f\}$. Assuming homogeneity of space it can be shown that the generalized force Q_α associated with the co-ordinate q_α whose virtual displacement does not change the relative distances between the system particles, is

$$Q_\alpha = -\frac{\partial U}{\partial q_\alpha} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{q}_\alpha}\right). \tag{8}$$

In particular, the total force on the systems comes out to be

$$\mathbf{F} = -\nabla_{\mathbf{R}} U + \frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\mathbf{V}} U, \tag{9}$$

where $\mathbf{R}$ and $\mathbf{V}$ represent the radius vector and the velocity of the centre of mass of the system, with respect to the lab frame S_L as shown in figure 1. The motion of the system is assumed to be non-relativistic.

The system A of our interest consists of charged particles placed in an external electromagnetic field whose scalar and vector potentials $(\Phi, \mathbf{A})$ are specified. Together, they form a 4-vector potential A^μ. Each particle has now a velocity dependent potential given as (Goldstein 1980)

$$u_i(\mathbf{r}_i, \mathbf{v}_i) = e_i\{\Phi(ct, \mathbf{r}_i) - \frac{1}{c}\mathbf{v}_i \cdot \mathbf{A}(ct, \mathbf{r}_i)\}, \tag{10}$$

where e_i is the charge of the ith particle. The total potential of the system is now the sum in (7).

The density of particles in our system is assumed large so that their distribution is specified in terms of a charge density $\rho(ct, \mathbf{r})$ and a current density $\mathbf{J}(ct, \mathbf{r})$. Together, they form a 4-current density vector $J^\mu(x) = (c\rho(ct, \mathbf{r}), \mathbf{J}(ct, \mathbf{r}))$. The total potential of the system can now be written as the following integral.

$$U = \int \left(\rho\Phi - \frac{1}{c}\mathbf{J}\cdot\mathbf{A}\right)\mathrm{d}^3 r. \tag{11}$$

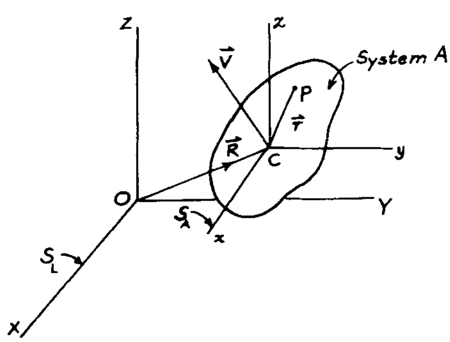

Figure 1. The radius vector **R**, the velocity vector **V** and the frame of reference S_A of the system A in relation to the laboratory frame S_L.

It should be noted here that the components of the 4-vectors A^μ and J^μ have been evaluated in the laboratory frame S_L.

Let us imagine a non-rotating rest frame of references S_A associated with the system with its origin C tied to the centre of mass of A. A frame of reference is considered non-rotating, if three freely suspended gyroscopes, with their axes oriented once along the X, Y, Z directions, remain so oriented for ever.

Let $(ct, \mathbf{R}+\mathbf{r})$ be the co-ordinates of an event P in S_L. Choosing the time origin of S_A appropriately and applying homogeneous Lorentz transformation between S_L and S_A we get the co-ordinates $(ct', \mathbf{r}')$ of this event in S_A (Jackson 1975, equation (11.19))

$$\left.\begin{aligned} ct' &= -\gamma \boldsymbol{\beta}\cdot\mathbf{r} \\ \mathbf{r}' &= \left(\boldsymbol{I} + \frac{\gamma-1}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}\right)\cdot\mathbf{r} \end{aligned}\right\} \tag{12}$$

where $\boldsymbol{I}$ is the identity dyadic.

If $V^\mu(x') = (V^\circ(x'), \mathbf{V}(x'))$ be the components of a vector field in S_A, its components in S_L are

$$\left.\begin{aligned} V_L^\circ(ct, \mathbf{R}+\mathbf{r}) &= \gamma\{V^\circ(x') + \boldsymbol{\beta}\cdot\mathbf{V}(x')\} \\ \mathbf{V}_L(ct, \mathbf{R}+\mathbf{r}) &= \left(\boldsymbol{I} + \frac{\gamma-1}{\beta^2}\boldsymbol{\beta}\boldsymbol{\beta}\right)\cdot\mathbf{V}(x') + \gamma\boldsymbol{\beta}V^\circ(x'). \end{aligned}\right\} \tag{13}$$

The transformation equations (12) and (13) take the following form in the non-relativistic limit of Lorentz transformation, as suggested in assumption (ii) in § 1.

$$\left.\begin{aligned} V_L^\circ(ct, \mathbf{R}+\mathbf{r}) &= V^\circ(\mathbf{r}) + \boldsymbol{\beta}\cdot\mathbf{V}(\mathbf{r}) \\ \mathbf{V}_L(ct, \mathbf{R}+\mathbf{r}) &= \mathbf{V}(\mathbf{r}) + \boldsymbol{\beta}V^\circ(\mathbf{r}). \end{aligned}\right\} \tag{14}$$

We shall now obtain the velocity dependent potential between two steady localized distributions of charge and current. Figure 2 shows these two systems A and B with their centres at O and C and their non-rotating rest frames of reference S_A and S_B having velocities $c\boldsymbol{\beta}_A$ and $c\boldsymbol{\beta}_B$ respectively. These velocities are considered non-relativistic.

In their respective rest frames, the systems A and B are characterized by their steady current density 4-vectors.

$$\left.\begin{aligned} &J_A^\mu = (c\rho_A(\xi), \mathbf{J}_A(\xi)); \quad J_B^\mu = (c\rho_B(\boldsymbol{\eta}), \mathbf{J}_B(\boldsymbol{\eta})) \\ &\nabla\cdot\mathbf{J}_A = 0; \quad \nabla\cdot\mathbf{J}_B = 0. \end{aligned}\right\} \tag{15}$$

In its own rest frame the system A produces a steady 4-potential $A_A^\mu = (\Phi_A, \mathbf{A}_A)$ given as:

$$\Phi_A(\xi) = \int d^3r' \frac{\rho_A(\mathbf{r}')}{|\xi - \mathbf{r}'|}; \quad \mathbf{A}_A(\xi) = \frac{1}{c}\int d^3r' \frac{\mathbf{J}_A(\mathbf{r}')}{|\xi - \mathbf{r}'|}. \tag{16}$$

The corresponding potentials in the laboratory frame S_L as given by the non-

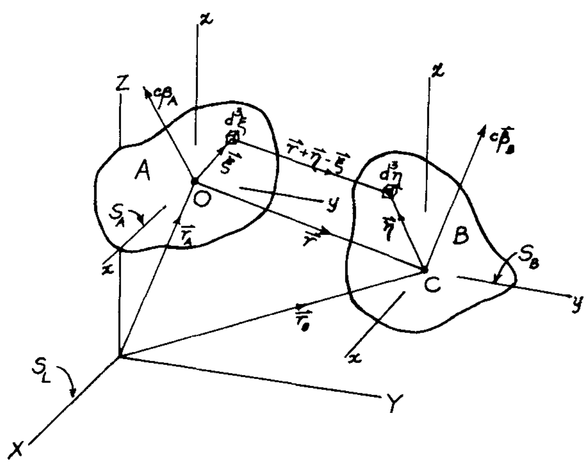

Figure 2. The radius vectors $\mathbf{r}_A, \mathbf{r}_B$, the velocity vectors $c\boldsymbol{\beta}_A, c\boldsymbol{\beta}_B$ and the frames of reference S_A, S_B of the systems A, B in relation to the laboratory frame S_L.

relativistic approximation equation (14), are as follows:

$$\left.\begin{aligned}\Phi_{A,\mathrm{Lab}}(ct, \mathbf{r}_A + \xi) &= \Phi_A(\xi) + \boldsymbol{\beta}_A(t)\cdot\mathbf{A}_A(\xi)\\ \mathbf{A}_{A,\mathrm{Lab}}(ct, \mathbf{r}_A + \xi) &= \mathbf{A}_A(\xi) + \boldsymbol{\beta}_A(t)\Phi_A(\xi).\end{aligned}\right\} \tag{17}$$

The interaction potential between A and B can be viewed as the velocity dependent potential U of the charge-current densities of B immersed in the 4-potentials shown in (17). The relevant densities are to be evaluated in S_L. In the non-relativistic approximation they are

$$\left.\begin{aligned}c\rho_{B,\mathrm{Lab}}(ct, \mathbf{r}_B + \boldsymbol{\eta}) &= c\rho_B(\boldsymbol{\eta}) + \boldsymbol{\beta}_B(t)\cdot\mathbf{J}_B(\boldsymbol{\eta})\\ \mathbf{J}_{B,\mathrm{Lab}}(ct, \mathbf{r}_B + \boldsymbol{\eta}) &= \mathbf{J}_B(\boldsymbol{\eta}) + \boldsymbol{\beta}_B(t)c\rho_B(\boldsymbol{\eta}).\end{aligned}\right\} \tag{18}$$

We now get the interaction potential using the formula (11)

$$\begin{aligned}U(\mathbf{r}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2) &= \int_B \left\{\rho_B(\boldsymbol{\eta}) + \frac{1}{c}\boldsymbol{\beta}_B(t)\cdot\mathbf{J}_B(\boldsymbol{\eta})\right\}\{\Phi_A(\mathbf{r}+\boldsymbol{\eta}) + \boldsymbol{\beta}_A(t)\cdot\mathbf{A}_A(\mathbf{r}+\boldsymbol{\eta})\}\,\mathrm{d}^3\eta\\ &\quad - \frac{1}{c}\int_B \{\mathbf{J}_B(\boldsymbol{\eta}) + c\boldsymbol{\beta}_B(t)\rho_B(\boldsymbol{\eta})\}\cdot\{\mathbf{A}_A(\mathbf{r}+\boldsymbol{\eta}) + \boldsymbol{\beta}_A(t)\Phi_A(\mathbf{r}+\boldsymbol{\eta})\}\,\mathrm{d}^3\eta\\ &\approx \Phi_{\mathrm{EE}}(\mathbf{r}) + \Phi_{\mathrm{MM}}(\mathbf{r}) + U_{\mathrm{EM}}(\mathbf{r}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2)\end{aligned} \tag{19}$$

where

$$\begin{aligned}\Phi_{\mathrm{EE}}(\mathbf{r}) &= \int_B \mathrm{d}^3\eta\,\rho_B(\boldsymbol{\eta})\Phi_A(\mathbf{r}+\boldsymbol{\eta})\\ &= \int_B \mathrm{d}^3\eta \int_A \mathrm{d}^3\xi\,\frac{\rho_B(\boldsymbol{\eta})\rho_A(\xi)}{|\mathbf{r}+\boldsymbol{\eta}-\xi|}.\end{aligned} \tag{20a}$$

$$\Phi_{\mathrm{MM}}(\mathbf{r}) = -\frac{1}{c}\int_B \mathrm{d}^3\eta\, \mathbf{J}_B(\boldsymbol{\eta})\cdot A_A(\mathbf{r}+\boldsymbol{\eta})$$

$$= -\frac{1}{c^2}\int_B \mathrm{d}^3\eta \int_A \mathrm{d}^3\xi \frac{\mathbf{J}_B(\boldsymbol{\eta})\cdot \mathbf{J}_A(\boldsymbol{\xi})}{|\mathbf{r}+\boldsymbol{\eta}-\boldsymbol{\xi}|}. \tag{20b}$$

$$U_{\mathrm{EM}}(\mathbf{r},\boldsymbol{\beta}_1,\boldsymbol{\beta}_2) = \frac{1}{c}\boldsymbol{\beta}_B\cdot\int_B \mathrm{d}^3\eta\{\mathbf{J}_B(\boldsymbol{\eta})\Phi_A(\mathbf{r}+\boldsymbol{\eta}) - c\rho_B(\boldsymbol{\eta})\mathbf{A}_A(\mathbf{r}+\boldsymbol{\eta})\}$$

$$-\frac{1}{c}\boldsymbol{\beta}_A\cdot\int_B \mathrm{d}^3\eta\{\mathbf{J}_B(\boldsymbol{\eta})\Phi_A(\mathbf{r}+\boldsymbol{\eta}) - c\rho_B(\boldsymbol{\eta})\mathbf{A}_A(\mathbf{r}+\boldsymbol{\eta})\}$$

$$= -\boldsymbol{\beta}\cdot\mathbf{A}_{\mathrm{EM}}(\mathbf{r}) \tag{20c}$$

where

$$\mathbf{A}_{\mathrm{EM}}(\mathbf{r}) = \frac{1}{c}\int_B \mathrm{d}^3\eta \int_A \mathrm{d}^3\xi \frac{\rho_B(\boldsymbol{\eta})\mathbf{J}_A(\boldsymbol{\xi}) - \mathbf{J}_B(\boldsymbol{\eta})\rho_A(\boldsymbol{\xi})}{|\mathbf{r}+\boldsymbol{\eta}-\boldsymbol{\xi}|} \tag{20d}$$

and

$$\boldsymbol{\beta} \equiv \boldsymbol{\beta}_B - \boldsymbol{\beta}_B. \tag{20e}$$

Terms containing β_A^2, β_B^2 and $\boldsymbol{\beta}_B\cdot\boldsymbol{\beta}_A$ have been dropped in line with our assumption (ii). As a consequence, there is no appreciable difference between the laboratory measures of the volume elements $\mathrm{d}^3\eta$ and $\mathrm{d}^3\xi$ (as appearing in the above equations) and their values measured in the respective comoving frames.

Equations (19) and (20) are our basic equations. They give us the velocity dependent interaction potential $U(\mathbf{r},\boldsymbol{\beta})$ of a slowly moving 2-body system consisting of two globules of steady charge and current with their centres of mass separated by a radius vector $\mathbf{r}$ and moving apart with a relative velocity $c\boldsymbol{\beta}$. Comparison with (10) shows that this interaction potential has the same form as that of a single particle of unit positive charge moving with a velocity $c\boldsymbol{\beta}$ in an external effective 4-potential

$$\left.\begin{aligned} &A^{\mu}_{\mathrm{eff}}(\mathbf{r}) = (\Phi_{\mathrm{eff}}(\mathbf{r}), \mathbf{A}_{\mathrm{eff}}(\mathbf{r})) \\ \text{where}\quad &\\ &\Phi_{\mathrm{eff}}(\mathbf{r}) = \Phi_{\mathrm{EE}}(\mathbf{r}) + \Phi_{\mathrm{MM}}(\mathbf{r}) \end{aligned}\right\} \tag{21}$$

is the effective scalar potential and $\mathbf{A}_{\mathrm{eff}}(\mathbf{r}) = \mathbf{A}_{\mathrm{EM}}(\mathbf{r})$ is the effective vector potential.

As a consequence, the Hamiltonian of a 2-body system consisting of two distributions of steady charge and current takes a simple form. Let m_A and m_B represent the masses of the systems A and B respectively. Their total mass and the reduced mass are respectively

$$M = m_A + m_B; \quad \mu = \frac{m_A m_B}{m_A + m_B}.$$

The Hamiltonian of the system is

$$H = H_{\mathrm{CM}} + H_{\mathrm{rel}}$$

where

$$\left.\begin{aligned} H_{\mathrm{CM}} &= \frac{P^2}{2M} \\ H_{\mathrm{rel}} &= \frac{1}{2\mu}\left(\mathbf{p} - \frac{1}{c}A_{\mathrm{EM}}(\mathbf{r})\right)^2 + \Phi_{\mathrm{EE}}(\mathbf{r}) + \Phi_{\mathrm{MM}}(\mathbf{r}), \end{aligned}\right\} \tag{22}$$

P and **p** being, respectively, the total momentum and the momentum of relative motion.

An approximate form of equation (22) under the weak field approximation is

$$H_{\mathrm{rel}} = \frac{p^2}{2\mu} + \Phi_{\mathrm{EE}}(\mathbf{r}) + \Phi_{\mathrm{MM}}(\mathbf{r}) - \frac{1}{\mu c}\mathbf{A}_{\mathrm{EM}}(\mathbf{r})\cdot\mathbf{p}. \tag{23}$$

This form may be useful in quantum mechanics when the Coulomb gauge is used.

3. Multipole expansion of the interaction potentials

Conversion of the scalar and the vector potentials of (20) into a series of multipole terms progresses through a number of mathematical steps involving irreducible spherical tensors. We shall omit these steps in this article for the sake of brevity and quote only the final forms of the significant results. The details of the mathematical steps omitted in this paper may be covered in a separate paper.

The key to the multipole expansion lies in an expansion of the factor $(1/|\mathbf{r} + \boldsymbol{\eta} - \boldsymbol{\xi}|)$ appearing in (20), in an infinite series involving spherical harmonics. We have adopted the same spherical harmonics $Y_{lm}(\theta, \varphi)$ as used by Jackson (1975) and Blatt and Weisskopf (1952) and write them as $Y_m^{(l)}(\hat{r})$. The superscript (l) written within parantheses is to remind us of the tensorial character of $Y_{lm}(\theta, \varphi)$. The desired expansion is as follows.

$$\frac{1}{|\mathbf{r} + \boldsymbol{\eta} - \boldsymbol{\xi}|} = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(-1)^l G(l,k)[[\mathscr{Y}^{>(l+k)}(\mathbf{r})\otimes\mathscr{Y}^{<(k)}(\boldsymbol{\eta})]^{(l)}\otimes\mathscr{Y}^{<(l)}(\boldsymbol{\xi})]^{(0)}, \tag{24a}$$

$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(-1)^l G(l,k)[\mathscr{Y}^{>(l+k)}(\mathbf{r})\otimes[\mathscr{Y}^{<(k)}(\boldsymbol{\eta})\otimes\mathscr{Y}^{<(l)}(\boldsymbol{\xi})]^{(l+k)}]^{(0)}. \tag{24b}$$

where

$$\mathscr{Y}_m^{>(l)}(\mathbf{r}) \equiv \frac{Y_m^{(l)}(\hat{r})}{r^{l+1}}; \quad \mathscr{Y}_m^{<(l)}(\mathbf{r}) \equiv r^l Y_m^{(l)}(\hat{r}) \tag{24c}$$

$$G(l,k) \equiv (4\pi)^{3/2}\left[\frac{(2l+2k-1)!!(l+k)!}{(2l+1)!!(2k+1)!!l!k!}\right]^{1/2} \tag{24d}$$

and

$$(-1)!! \equiv 1.$$

Equivalence of the two forms can be easily established.

Sometimes the non-tensorial form of the above expansion will be useful.

$$\frac{1}{|\mathbf{r}+\boldsymbol{\eta}-\boldsymbol{\xi}|}=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=-l}^{l}\sum_{q=-k}^{k}(-1)^k F(l,k)$$
$$\times\{\xi^l Y_{lm}(\hat{\xi})\}^*\{\eta^k Y_{kq}(\hat{\eta})\}^*\frac{Y_{l+k,m+q}(\hat{r})}{r^{l+k+1}}, \tag{25}$$

where

$$F(l,k)=\frac{4\pi^{3/2}}{[(2l+2k+1)(2l+1)(2k+1)]^{1/2}}$$
$$\times\left\{\frac{(l+k+m+q)!(l+k-m-q)!}{(l+m)!(l-m)!(k+q)!(k-q)!}\right\}^{1/2}.$$

We shall define the electric and magnetic multipole moments, $Q_m^{(l)}$ and $M_m^{(l)}$ respectively, as the following spherical tensors.

$$\left.\begin{aligned} Q_m^{(l)} &\equiv \int \rho(\mathbf{r}) r^l Y_m^{(l)}(\hat{r})\,\mathrm{d}^3 r \\ M_m^{(l)} &\equiv -\frac{i}{c}\left[\frac{l}{l+1}\right]^{1/2}\int \mathbf{J}(\mathbf{r})\cdot r^l \mathbf{X}_m^{(l)}(\hat{r})\,\mathrm{d}^3 r \\ &= -\frac{1}{c(l+1)}\int \nabla\cdot(\mathbf{r}\times\mathbf{J}) r^l Y_m^{(l)}(\hat{r})\,\mathrm{d}^3 r. \end{aligned}\right\} \tag{26}$$

Here $\mathbf{X}_m^{(l)}(\hat{r})$ is the vector spherical harmonic defined (Blatt and Weisskopf 1952; Jackson 1975) as

$$\mathbf{X}_m^{(l)}(\hat{r})\equiv\mathbf{X}_{lm}(\theta,\varphi)\equiv\frac{1}{i}\frac{\mathbf{r}\times\nabla}{[l(l+1)]^{1/2}}Y_{lm}(\theta,\varphi). \tag{27}$$

The multipole moments Q_{lm}, M_{lm} defined by Jackson (1975) and Blatt and Weisskopf (1952) are related to our moments as follows.

$$\left.\begin{aligned} Q_{lm} &= [Q_m^{(l)}]^* = (-1)^m Q_{-m}^{(l)} \\ M_{lm} &= [M_m^{(l)}]^* = (-1)^m M_{-m}^{(l)} \end{aligned}\right\}. \tag{28}$$

Using the series expansion (24) and the definitions (26), the scalar moment $\Phi_{\mathrm{EE}}(\mathbf{r})$ gets expanded quite easily.

$$\Phi_{\mathrm{EE}}(\mathbf{r})=\int_B \mathrm{d}^3\eta\int_A \mathrm{d}^3\xi\,\frac{\rho_B(\boldsymbol{\eta})\rho_A(\boldsymbol{\xi})}{|\mathbf{r}+\boldsymbol{\eta}-\boldsymbol{\xi}|}$$
$$=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(-1)^l G(l,k)[\mathscr{Y}^{>(l+k)}(\mathbf{r})\otimes[Q_B^{(k)}\otimes Q_A^{(l)}]^{(k+l)}]^{(0)}. \tag{29}$$

Here, $Q_A^{(l)}=\{Q_{A,m}^{(l)}; m=-l,\ldots,l\}$ and $Q_B^{(k)}=\{Q_{B,q}^{(k)}; q=-k,\ldots,k\}$ are the electric multipole moments of the systems A and B respectively.

The multipole expansion of $\Phi_{\mathrm{MM}}(\mathbf{r})$ is not as easily obtained. The assumption $\nabla\cdot\mathbf{J}=0$

however, will reduce $\Phi_{\mathrm{MM}}(\mathbf{r})$ ultimately to a similar form.

$$\Phi_{\mathrm{MM}}(\mathbf{r}) = -\frac{1}{c^2}\int_B \mathrm{d}^3\eta \int_A \mathrm{d}^3\xi \frac{\mathbf{J}_B(\boldsymbol{\eta})\cdot\mathbf{J}_A(\boldsymbol{\xi})}{|\mathbf{r}+\boldsymbol{\eta}-\boldsymbol{\xi}|}$$

$$= \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}(-1)^l G(l,k)[\mathscr{Y}^{>(l+k)}(\mathbf{r})\otimes[M_B^{(k)}\otimes M_A^{(l)}]^{(k+l)}]^{(0)}. \tag{30}$$

The vector potential $\mathbf{A}_{\mathrm{EM}}(\mathbf{r})$ gets similarly expanded with the help of (24). This time, however, we obtained three infinite series. Two of these series can be shown to possess zero curl and, therefore, can be dropped using the gauge freedom. The non-zero curl part of the expansion has the following form.

$$A_{\mathrm{EM}}(\mathbf{r}) = i\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^l\left(\frac{l+k+1}{l+k}\right)^{1/2} G(l,k)$$

$$\times[\mathbf{X}^{>(l+k)}(\mathbf{r})\otimes\{[Q_A^{(l)}\otimes M_B^{(k)}]^{(l+k)}-[Q_B^{(k)}\otimes M_A^{(l)}]^{(k+l)}\}]^{(0)} \tag{31}$$

where

$$\mathbf{X}_m^{>(l)}(\mathbf{r}) \equiv \frac{\mathbf{X}_m^{(l)}(\hat{r})}{r^{l+1}} = \frac{\mathbf{r}\times\nabla}{i[l(l+1)]^{1/2}}\left\{\frac{Y_m^{(l)}(\hat{r})}{r^{l+1}}\right\}.$$

Non-tensorial forms of the above three expansions, using the multipole moments Q_{lm} and M_{lm} as defined in (28), are presented below.

$$\Phi_{\mathrm{EE}}(\mathbf{r}) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=-l}^{l}\sum_{q=-k}^{k}(-1)^k F(l,k) Q_{lm}(A) Q_{kq}(B) \times\frac{Y_{l+k,m+q}(\theta,\varphi)}{r^{l+k+1}}. \tag{32a}$$

$$\Phi_{\mathrm{MM}}(\mathbf{r}) = \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=-l}^{l}\sum_{q=-k}^{k}(-1)^k F(l,k) M_{lm}(A) M_{kq}(B) \times\frac{Y_{l+k,m+q}(\theta,\varphi)}{r^{l+k+1}}. \tag{32b}$$

$$\mathbf{A}_{\mathrm{EM}}(\mathbf{r}) = \sum_l\sum_k\sum_{m=-l}^{l}\sum_{q=-k}^{k}(-1)^k\left(\frac{l+k+1}{l+k}\right)^{1/2} F(l,k) \times\{Q_{lm}(A)M_{kq}(B)-Q_{kq}(B)M_{lm}(A)\}\frac{\mathbf{X}_{l+k,m+q}(\theta,\varphi)}{r^{l+k+1}} \tag{32c}$$

where

$$F(l,k) = \frac{(4\pi)^{3/2}}{[(2l+1)(2k+1)(2l+2k+1)]^{1/2}} \times\left[\frac{(l+k+m+q)!(l+k-m-q)!}{(l+m)!(l-m)!(k+q)!(k-q)!}\right]^{1/2}. \tag{32d}$$

A cursory look at the above results shows that, within the limits of the assumptions made in §1, there is no vector potential $\mathbf{A}_{\mathrm{EM}}(\mathbf{r})$ between two identical particles. The interaction force between two identical particles is velocity independent and is derivable from a purely scalar potential consisting of electric-electric and magnetic-magnetic interactions and permits no electric-magnetic coupling. A corollary of this conclusion is that there is no spin-orbit or L-S coupling force between two identical particles e.g. between a pair of electrons or protons or hydrogen atoms in the same quantum state.

4. Electric dipole-magnetic dipole interaction

The multipole formulas (29), (30), (31) can be used to reproduce such familiar interaction potentials as the ones between two magnetic dipoles, two electric dipoles as well as the familiar hydrogen atom spin-orbit interaction. They can also be used to obtain the interaction Hamiltonian between an electric dipole and a magnetic dipole, a result which is not familiar.

Consider the object A having the electric dipole moment $\mathbf{p}_A = (p_x, p_y, p_z)$ and another object B having the magnetic dipole moment $\mathbf{m}_B = (m_x, m_y, m_z)$. The corresponding tensor moments are

$$\left.\begin{aligned} Q^{(1)}_{\pm 1} &= \left(\frac{3}{8\pi}\right)^{1/2} (\mp p_x - ip_y); \quad Q^{(1)}_0 = \left(\frac{3}{4\pi}\right)^{1/2} p_z \\ M^{(1)}_{\pm 1} &= \left(\frac{3}{8\pi}\right)^{1/2} (\mp m_x - im_y); \quad M^{(1)}_0 = \left(\frac{3}{4\pi}\right)^{1/2} m_z. \end{aligned}\right\} \tag{33}$$

Setting $l = k = 1$ in (31) and using the multipole quantities shown in (33) it can be shown in a straightforward way that

$$A_{\mathrm{EM}}(\mathbf{r}) = \tfrac{1}{2}\mathbf{r} \times \nabla\left[\frac{r^2 \mathbf{p}_A \cdot \mathbf{m}_B - 3(\mathbf{r}\cdot\mathbf{p}_A)(\mathbf{r}\cdot\mathbf{m}_B)}{r^5}\right]. \tag{34}$$

When this vector potential is used in (23) we get the required interaction Hamiltonian.

$$\begin{aligned} H_{\mathbf{p}_A - \mathbf{m}_B - \mathbf{L}} &\simeq -(1/\mu c)\mathbf{A}_{\mathrm{EM}}(\mathbf{r})\cdot\mathbf{p} \\ &= \frac{1}{2\mu c}\nabla\left[\frac{r^2 \mathbf{p}_A \cdot \mathbf{m}_B - 3(\mathbf{r}\cdot\mathbf{p}_A)(\mathbf{r}\cdot\mathbf{m}_B)}{r^5}\right]\cdot\mathbf{L}. \end{aligned} \tag{35}$$

This can be re-written as a dipole-spin-orbit coupling.

$$H_{\mathbf{p}_A - \mathbf{s}_B - \mathbf{L}} = \left(\frac{1}{2\mu c}\right)\left(\frac{g e_B}{2\mu_B c}\right)\nabla\left[\frac{r^2 \mathbf{p}_A \cdot \mathbf{s}_B - 3(\mathbf{r}\cdot\mathbf{p}_A)(\mathbf{r}\cdot\mathbf{s}_B)}{r^5}\right]\cdot\mathbf{L}, \tag{36}$$

where $\mu_B, e_B, \mathbf{s}_B$ and g represent, respectively, the mass, charge, spin and Lande factor of B.

5. Summary and an analysis of the limitations of the formalism

We have considered a two-body system interacting through electromagnetic interaction and have obtained the Hamiltonian (equation 22). The Hamiltonian of the relative motion has been found to be the same as that of a representative particle of unit positive charge and mass μ equal to the reduced mass of the 2-body system, moving in a static field determined by a scalar potential and a vector potential viz. $\Phi_{EE} + \Phi_{MM}$ and $\mathbf{A}_{EM}$ (equations 19 and 20). The velocity of this representative particle equals the relative velocity between the centres of mass of the individual systems. The assumptions needed for the derivation are listed in §1.

Subsequently, we have obtained a multipole expansion of the potentials $\Phi_{EE}(\mathbf{r})$, $\Phi_{MM}(\mathbf{r})$ and $\mathbf{A}_{EM}(\mathbf{r})$, first in the irreducible spherical tensor forms (equations 29, 30, 31) and then in the non-tensorial forms (equations 32). From the symmetry among the multipole moments appearing in the expression for $\mathbf{A}_{EM}$ it appears that two objects having the same set of electric and magnetic moments do not produce any vector potential $\mathbf{A}_{EM}$ between them. Their interaction is purely velocity independent.

The formalism can be illustrated through applications to a few typical cases. One of them is an interaction between an electric dipole and a magnetic dipole. It shows a coupling among the electric dipole, the spin and the orbital angular momentum.

We shall now examine briefly the limitations imposed on our formalism by assumptions (i) and (iv). Non-rotating frames are impossible because rotations arise from two unavoidable sources viz. (a) accelerations of the frames of reference resulting in the so-called Thomas precession and (b) mutual torques exerted by one system on the other.

It appears that the effect of Thomas precession can be calculated either by (a) making a Lorentz transformation from the accelerating frames of reference to the laboratory frame, or by (b) evaluating the extra currents produced by the Thomas precessional angular velocities and their contributions to the interaction potential. When applied to the hydrogen atom, either of these routes has contributed the same extra term, viz.

$$\Delta H = \frac{2r_0}{r}\left[\frac{e^2}{2m_e c^2}\frac{\mathbf{s}\cdot\mathbf{L}}{r^3}\right]$$

where $r_0 \equiv e^2/m_e^2 c^2 \approx 3 \times 10^{-13}$ cm, is the classical electron radius (Jackson 1975). Thus, the extra term is $\sim 10^{-3}$ of the spin-orbit term and, therefore, can be dropped.

As for the radiation effect, Jackson's criterion using the characteristic time

$$\tau = \frac{2}{3}\frac{e^2}{mc^3}$$

will be our guide. The largest value of characteristic time comes from an electron which equals $6{\cdot}26 \times 10^{-24}$ s. The period of classical electron orbit in the hydrogen atom is $\sim 10^{-16}$ s, which is $\sim 10^7$ times τ. Therefore, radiation does not alter the two body potential of the hydrogen atom appreciably. This assertion is even much stronger for particles heavier than electron.

It appears, therefore, that the one defect our formulas suffer from is the assumption that the interacting systems do not respond to their mutual torques and remain non-rotating. We would like to address ourselves to a formalism which is free from this defect in a future communication.

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