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The $(q\bar{q})$-pion and its decay constant in a chiral potential model

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Abstract. Pion mass and its decay constant have been studied in a chiral symmetric potential model of independent quarks. The non-perturbative multi-gluon interaction which is responsible for quark confinement in a hadron is phenomenologically represented here by an effective potential $U(r)=\frac{1}{2}(1+\gamma^0)(ar^2+V_0)$. The residual interactions due to quark-pion coupling arising out of the chiral symmetry preservation and that due to quark-gluon coupling arising out of single-gluon exchange are treated as low order perturbations. The centre of mass correction is also taken into account appropriately. This leads to the $(q\bar{q})$-pion mass in consistency with that of the PCAC-pion and the pion decay constant in reasonable agreement with experiment.

Keywords. Pion mass; chiral symmetry; centre-of-mass correction; one-gluon exchange.

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1. Introduction

Quark model description of hadrons made up of low-mass quarks has been quite successful phenomenologically in reproducing the hadron masses and other properties (Chodos *et al* 1974a, b; De Grand *et al* 1975) satisfactorily. However the most illusive π-meson has proved to be a notable exception to this success. It is because of the fact that the pion of the quark model appears to be quite different from the pion of PCAC. Quantum chromodynamics (QCD), the fundamental theory of hadrons has a chiral SU(2) × SU(2) symmetry in the limit of vanishing up- and down-quark masses. Spontaneous breaking of this symmetry gives rise to pions as the associated massless Goldstone boson. But for small and finite quark masses, there is a slight departure from this description in providing the pion a small mass $\tilde{m}_\pi = 140$ MeV and giving rise to the so-called PCAC. On the other hand the quark model description of the pion is that of a $(q\bar{q})$-bound state. Non-relativistic two-body potential model studies of heavy meson spectra, when extended to ordinary light meson sector, realizes in certain cases (Barik and Jena 1981, 1982) a pion mass between 160 MeV to 180 MeV. However a relativistic treatment being more appropriate to this ordinary light meson sector, such quantitative results are never taken too seriously. The otherwise successful static bag model provides a relativistic description of the $(\rho\text{-}\pi)$ system through a

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bag-confinement of an independent quark and an antiquark in their ground states. When spin-dependent forces due to one-gluon exchange are taken into account perturbatively. Although, as a consequence, pion emerges naturally as the lightest state with m_π between 175 MeV to 280 MeV (De Grand *et al* 1975), it is found impossible to get a pion of $m_\pi = 140$ MeV.

However by treating the static cavity eigenstates as localized wave packets of true momentum eigenstates (Donoghue and Johnson 1980) in analogy with states in a non-relativistic shell model, it can be possible to realize a zero mass pion in the chiral limit. But in the static-cavity approximation for the quark confinement, the chiral symmetry is also lost at the bag surface which can only be restored by introducing an external pion field. This leads to the formulation of different versions of chirally symmetric bag models. Such a model (De Tar 1981) describing pion regards the external pion field as an approximation to the amplitude for finding the centre of mass of a composite $(q\bar{q})$-pion at a given space time point. In other words, while for distances comparable to the bag size, the pion is preferred to be represented by $(q\bar{q})$-bound states, at larger distances an elementary field description of it is believed to be adequate. Such a picture taking into account the mass-shift due to the lowest order pionic self energy, generates a pion mass between 268 MeV to 396 MeV (De Tar 1981) with a large quark gluon coupling constant $\alpha_s \sim 1{\cdot}5$.

We present here a modest venture for resolving this apparent dichotomy between the $(q\bar{q})$-pion and the PCAC-pion in the framework of a relativistic independent quark potential model with chiral symmetry. Such a scheme has been found to be quite successful for the ground state octet baryons in reproducing satisfactorily (Barik *et al* 1985a, b; Barik and Dash 1986a, b; c) the mass spectrum as well as the electromagnetic properties. Here hadrons are considered as an assembly of independent quarks confined, in a first approximation, by an effective potential,

$$U(r) = \tfrac{1}{2}(1+\gamma^0)(ar^2 + V_0), \tag{1}$$

which presumably represents the non-perturbative gluon interactions. This provides the zeroth order quark dynamics inside the hadronic core through a Lagrangian formulation. The residual interactions, due to quark-pion coupling arising out of the restoration of chiral symmetry in PCAC-limit in SU(2) sector and that due to one-gluon exchange at short distances, are treated perturbatively. The effect of the centre of mass motion is also taken into account following the prescription of Wong (1981). We extend the same model to the mesonic ground states with the particular motive to realize the controversial pion. The additional elementary pion field to be introduced over and above the confined independent quarks of non zero mass for restoration of chiral symmetry is considered to have a small mass $\tilde{m}_\pi = 140$ MeV in PCAC limit. Then our purpose here would be to realize a $(q\bar{q})$-pion with the same mass as that of the PCAC-pion in the event of small but non-zero quark masses. In the process we would also obtain for a consistency check an estimate of the $(\rho\text{-}\pi)$ mass difference as well as the pion decay constant f_π.

In § 2, we provide a brief outline of the potential model giving the zeroth order confined quark dynamics. This also deals with the corrections to the unperturbed energy of the $(q\bar{q})$-bound state due to the lowest order colour-electrostatic and magnetostatic energies arising out of one gluon exchange at short distances. Here again we take into account the energy shift due to quark-pion coupling arising out of the requirement of chiral symmetry restoration in PCAC-limit. This calculation in low order perturbation theory uses the experimental value of the pion-decay constant i.e. $f_\pi = 93$ MeV and the field pion mass $\tilde{m}_\pi = 140$ MeV. The prescription for taking into account the spurious centre of mass correction is also mentioned briefly. Now incorporating these corrections of § 2 to the unperturbed zeroth order energy of the $(q\bar{q})$-bound state system and considering the effect of the centre of mass motion, one can realize the mass of the $(q\bar{q})$-bound state in its static limit. Section 3 provides a simple expression for the pion-decay matrix element $F_\pi(p^2)$ as derived in the present model showing explicit momentum dependence. However $F_\pi(0)$ estimated in the static limit with m_π as obtained from the present model may be compared with f_π for a consistency check. Finally § 4 deals with the results and discussion.

2. Basic framework of the model

A meson in general is pictured in the present model as an assembly of a quark and an antiquark with appropriate interactions according to quantum chromodynamics (QCD). The quark-gluon interaction originating from one-gluon exchange at short distances and the quark-pion interaction in the non-strange flavour sector required to preserve chiral symmetry are presumed to be residual interactions compared to the dominant confining interaction. Therefore, to a first approximation, the confining part of the interaction is believed to provide the zeroth order quark dynamics inside the mesonic-core leading to the zeroth order energy of the $(q\bar{q})$-assembly of a light meson.

The confining interaction which is expected to be generated by the non-perturbative multi-gluon mechanism is impossible to calculate theoretically from first principle. Therefore, from a phenomenological point of view, the present model assumes that the quarks in a meson-core are independently confined by an average flavour independent potential of the form (Ferriera 1977; Barik *et al* 1985a, b)

$$U(r) = \tfrac{1}{2}(1+\gamma^0)\, V(r),$$

with,
$$V(r) = (ar^2 + V_0), \quad a > 0. \tag{1a}$$

The quark-Lagrangian density in zeroth order in such a picture is written as

$$\mathscr{L}_q^0(x) = \bar{q}(x)\left[\frac{i}{2}\gamma^\mu \overleftrightarrow{\partial}_\mu - U(r) - m_q\right] q(x). \tag{2}$$

Considering all the quarks in a meson-core in their ground $1S_{1/2}$-state, the normalized quark wave function $\Psi_q(\mathbf{r})$ satisfying the Dirac equation, obtainable from $\mathscr{L}_q^0(x)$, can be written in the two component form as,

$$\Psi_{q+}(\mathbf{r}) = \frac{1}{\sqrt{4\pi}}\begin{pmatrix} i g_q(r)/r \\ \boldsymbol{\sigma}\cdot\hat{r} f_q(r)/r \end{pmatrix}\chi_\uparrow, \tag{3}$$

for positive energy solution and

$$\Psi_{q-}(\mathbf{r})=\frac{1}{\sqrt{4\pi}}\begin{pmatrix} i\boldsymbol{\sigma}.\hat{r} f_q(r)/r \\ g_q(r)/r \end{pmatrix}\chi_\uparrow \tag{4}$$

for negative energy solutions. Taking

$$E'_q=(E_q-V_0/2),\quad m'_q=(m_q+V_0/2)$$
$$\lambda_q=(E'_q+m'_q)\quad \text{and}\quad r_{0q}=(a\lambda_q)^{-1/4}, \tag{5}$$

it can be shown that the reduced radial parts of the upper and lower component of $\Psi_{q+}(\mathbf{r})$ come out as,

$$g_q(r)=N_q(r/r_{0q})\exp(-r^2/2r_{0q}^2),$$
$$f_q(r)=-\frac{N_q}{\lambda_q r_{0q}}(r/r_{0q})^2\exp(-r^2/2r_{0q}^2). \tag{6}$$

The overall normalization factor N_q is given by the relation,

$$N_q^2=\frac{8\lambda_q}{\sqrt{\pi}r_{0q}}\frac{1}{(3E'_q+m'_q)}. \tag{7}$$

The ground-state individual quark binding energy $E_q=(E'_q+V_0/2)$ is obtainable from the energy eigenvalue condition

$$(E'^2_q-m'^2_q)r_{0q}^2=3. \tag{8}$$

The solutions through equations (4) to (8), provide the quark binding energy E_q which immediately leads to the energy of the meson-core in zeroth order as

$$E_M^0=\sum_q E_q \tag{9}$$

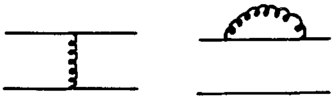
2.1 *One gluon-exchange correction*

The individual quarks in a meson-core are considered so far to be experiencing the only force coming from the average effective potential $U(r)$ in (1). All that remains inside the meson-core is the hopefully weak one-gluon exchange interaction provided by the interaction Lagrangian density

$$\mathscr{L}_I^q=\sum_a J_i^{\mu a}(x)A_\mu^a(x), \tag{10}$$

where $A_\mu^a(x)$ are the vector-gluon fields and $J_i^{\mu a}(x)$ is the ith-quark colour-current. Since at small distances the quarks should be almost free, it is reasonable to calculate the shift in the energy of the meson-core (arising out of the quark interaction energy due to its coupling to the coloured gluons) using a first order perturbation theory. Such an approach leads to the colour-electric and colour-magnetic energy shifts (Barik and Dash 1986a) as shown in figures 1a and b

$$(\Delta E_M)_g=(\Delta E_M)_g^{\mathscr{E}}+(\Delta E_M)_g^{\mathscr{M}} \tag{11}$$



(a) (b)

Figure 1. One-gluon exchange contribution to the energy of a $(q\bar{q})$-configuration.

when,

$$(\Delta E_M)_g^{\mathcal{E}} = \alpha_s \sum_{i,j} \left\langle \sum_a \lambda_i^a \lambda_j^a \right\rangle \frac{1}{\sqrt{\pi} R_{ij}} \left(1 - \frac{\alpha_i + \alpha_j}{R_{ij}^2} + \frac{3\alpha_i \alpha_j}{R_{ij}^4}\right)$$

$$(\Delta E_M)_g^{\mathcal{M}} = \alpha_s \sum_{i<j} \left\langle \sum_a \lambda_i^a \lambda_j^a \sigma_i \sigma_j \right\rangle \frac{256}{9\sqrt{\pi}} \frac{1}{(3E_i' + m_i')(3E_j' + m_j')} \frac{1}{R_{ij}^3}. \quad (12)$$

Here,

$$R_{ij}^2 = 3\left(\frac{1}{(E_i'^2 - m_i'^2)} + \frac{1}{(E_j'^2 - m_j'^2)}\right)$$

$$\alpha_i = 1/\lambda_i (3E_i' + m_i'). \quad (13)$$

λ_i^a are the usual Gellmann SU(3) matrices and α_s is the quark-gluon coupling constant.

Finally, taking into account the specific quark flavour and spin configurations in various ground-state mesons and using the relations

$$\left\langle \sum_a (\lambda_i^a)^2 \right\rangle = \frac{16}{3},$$

$$\left\langle \sum_a \lambda_i^a \lambda_j^a \right\rangle_{i \neq j} = -\frac{16}{3}. \quad (14)$$

One can write in general, the energy corrections due to one-gluon exchange as

$$(\Delta E_M)_g^{\mathcal{E}} = \alpha_s (b_{uu} I_{uu} + b_{us} I_{us} + b_{ss} I_{ss})$$

$$(\Delta E_M)_g^{\mathcal{M}} = \alpha_s (a_{uu} I_{uu} + a_{us} I_{us} + a_{ss} I_{ss}). \quad (15)$$

Here a_{ij} and b_{ij} are the numerical coefficients provided in table 1 for mesons (ω, ρ, π) considered here in the context of π-meson. The quantities I_{ij} are,

$$I_{ij} = \frac{16}{3\sqrt{\pi}} \frac{1}{R_{ij}} \left[1 - \frac{\alpha_i + \alpha_j}{R_{ij}^2} + \frac{3\alpha_i \alpha_j}{R_{ij}^4}\right]$$

$$I_{ij} = \frac{256}{9\sqrt{\pi}} \frac{1}{R_{ij}^3} \frac{1}{(3E_i' + m_i')(3E_j' + m_j')}. \quad (16)$$

One can note from table 1 that the colour-electric contributions for the mesons vanish when the constituent quark and antiquark masses in a meson-core are equal. Therefore the degeneracy among the mesons like ω, ρ and π is essentially removed at this level through the strong spin-spin interaction in the colour-magnetic part only.

Table 1. Coefficients appearing in the calculation of the colour-magnetic and -electric energy-corrections due to one-gluon exchange.

Mesons	a_{uu}	a_{us}	a_{ss}	b_{uu}	b_{us}	b_{ss}
w	2	0	0	0	0	0
ρ	2	0	0	0	0	0
π	−6	0	0	0	0	0

2.2 *Chiral symmetry and pionic correction*

Coming back again to the zeroth order Lagrangian density $\mathscr{L}_q^0$ which takes into account the non-perturbative gluon-interactions including gluon-self coupling through the phenomenological potential $U(r)$, one can note that under a global infinitesimal chiral transformation at least in the non-strange flavour sector,

$$q(x)\rightarrow q(x)-i\gamma^5\frac{(\boldsymbol{\tau}\cdot\boldsymbol{\epsilon})}{2}q(x), \tag{17}$$

the axial vector current of quarks is not conserved. This is because the scalar term in $\mathscr{L}_q^0$ proportional to $G(r)=(m_q+V(r)/2)$ is chirally odd. The vector part of the potential poses no problem in this respect. But in view of the experimental success of the partial conservation of axial-vector-current (PCAC) and hence the fact that chiral $SU(2)\times SU(2)$ is one of the best symmetries of strong interaction, it is desirable to conserve the total axial-vector current at least in the (u-d)-flavour sector. This is usually done at a phenomenological level (Chodos and Thorn 1975; Brown *et al* 1979a; Brown and Rho 1979b; Vento *et al* 1980; Theberge *et al* 1980, 1981; Thomas *et al* 1981; Thomas 1983) by introducing elementary pion field that also carries an axial current such that the four divergence of the total axial-vector current satisfies the PCAC-condition.

We therefore introduce in the usual manner, an elementary pion field $\boldsymbol{\Phi}(x)$ of small but finite mass $\tilde{m}_\pi=140$ MeV with the quark-pion interaction Lagrangian density,

$$\mathscr{L}_I^\pi=-\frac{i}{f_\pi}G(r)\bar{q}(x)\gamma^5(\boldsymbol{\tau}\cdot\boldsymbol{\Phi})q(x), \tag{18}$$

which is linear in isovector pion field $\boldsymbol{\Phi}(x)$. Here $f_\pi=93$ MeV is the phenomenological pion decay constant. Then the four-divergence of the total axial-vector current becomes,

$$\partial_\mu \mathbf{A}^\mu(x)=-f_\pi\tilde{m}_\pi^2\boldsymbol{\Phi}(x), \tag{19}$$

yielding the PCAC-relation. Consequently, the elementary pion field coupling to the non-strange quarks of the hadron, would give rise to pionic self-energy of the hadrons. This aspect can be studied in the usual perturbative approach (Thomas 1983; Barik and Dash 1986a).

The coupling of the pion-field to the non-strange quarks, shown in a minimal way through the single-loop self-energy diagram (figure 2), causes a shift in the energy of

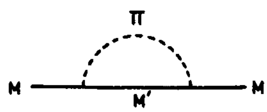


Figure 2. Pionic self-energy of a meson due to quark coupling with pions.

the meson-core. The self-energy of the meson-core due to pionic interaction is usually obtained from second order perturbation calculation as,

$$\Sigma_M(E_M^0)=\sum_k\sum_{M'}\frac{V_j^{MM'}(k)V_j^{MM'}(k)}{(E_M^0-w_k-m_{M'}^0)}, \tag{20}$$

when

$$\sum_k\equiv\sum_j\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \quad \text{and } M'$$

is the intermediate meson-core state. $V_j^{MM'}(k)$ is the meson-pion vertex function which in the present model is obtained as (Barik and Dash 1986a).

$$V_j^{MM'}(k)=i\frac{3g_A}{10f_\pi}\frac{ku(k)}{\sqrt{2w_k}}\left\langle M'\left|\sum_q(\boldsymbol{\sigma}_q\cdot\hat{k})\cdot(\tau_q)_j\right|M\right\rangle \tag{21}$$

where, $\quad g_A=\frac{5}{9}(5E_u'+7m_u')/(3E_u'+m_u')$ (22)

and the form factor $u(k)$ with

$$A=(E_u'-m_u')/2(5E_u'+7m_u') \tag{23}$$

comes out as,

$$u(k)=(1-Ar_{0u}^2k^2)\exp(-r_{0u}^2k^2/4). \tag{24}$$

Using the familiar Goldberger-Treimann relation $\sqrt{4\pi}f_{NN\pi}/\tilde{m}_\pi=g_A/2f_\pi$, one can rewrite (21), as,

$$V_j^{MM'}(k)=i\sqrt{4\pi}\frac{3}{5}\frac{f_{NN\pi}}{\tilde{m}_\pi}\frac{ku(k)}{\sqrt{2w_k}}\left\langle M'\left|\sum_q(\boldsymbol{\sigma}_q\cdot\hat{k})(\tau_q)_j\right|M\right\rangle. \tag{25}$$

Here $w_k=(k^2+m_\pi^2)^{1/2}$ is the pion-energy. For degenerate intermediate meson-states on mass shell with $m_M^0=m_M$, the self-energy becomes,

$$\delta m_M=\Sigma_H(E_M^0=m_M^0=m_{M'}^0)=-\sum_{k,M'}\frac{V_j^{MM'}(k)V_j^{MM'}(k)}{w_k}. \tag{26}$$

Now using the explicit expression for $V_j^{MM'}(k)$ as in (25), one gets

$$\delta m_M=-\frac{3}{25}f_{NN\pi}^2I_\pi\sum_{M'}C_{MM'} \tag{27}$$

when,

$$C_{MM'}=\left\langle M'\left|\sum_{q,q'}(\boldsymbol{\sigma}_q\cdot\boldsymbol{\sigma}_{q'})(\boldsymbol{\tau}_q\cdot\boldsymbol{\tau}_{q'})\right|M\right\rangle \tag{28}$$

and $$I_\pi = \frac{1}{\pi \tilde{m}_\pi^2} \int_0^\infty dk\, k^4 u^2(k)/w_k^2.$$

Now using (24) for $u(k)$ and substituting $z = \frac{1}{2}\tilde{m}_\pi^2 r_{0u}^2$, the integral I_π can be written in the form,

$$I_\pi = \frac{1}{\pi \tilde{m}_\pi^2}[\tilde{I}_4 - 2Ar_{0u}^2 \tilde{I}_6 + A^2 r_{0u}^4 \tilde{I}_8] \tag{29}$$

when the reduced integrals

$$\tilde{I}_{2n} = \int_0^\infty dk \frac{k^{2n}}{(k^2 + \tilde{m}_\pi^2)} \exp(-zk^2/\tilde{m}_\pi^2)$$

$$= (2z/r_{0u}^2)^{n-1/2}\left[(-1)^n \frac{\pi}{2}\exp(z) + \frac{\Gamma(n+\frac{1}{2})}{(2n-1)} F(1, \tfrac{3}{2} - n; z)\right]. \tag{30}$$

Now substituting (30) in (29) and using the identity

$$F(1, b; z) = 1 + \frac{z}{b} F(1, b+1; z),$$

one can finally get,

$$I_\pi = \left(\frac{\tilde{m}_\pi}{\pi}\right)\left[(1 + 4Az + 4A^2z^2)\left\{\frac{\pi}{2}\exp(z) + \frac{\sqrt{\pi}}{4} z^{-3/2} - \frac{\sqrt{\pi}}{2} z^{-1/2} F(1, \tfrac{1}{2}; z)\right\}\right.$$
$$\left. + A^2\left(\frac{15\sqrt{\pi}}{4} z^{-3/2} - \frac{3}{2}\sqrt{\pi} z^{-1/2}\right) - A\frac{3\sqrt{\pi}}{2} z^{-3/2}\right]. \tag{31}$$

Now using the values of $C_{MM'}$ summarized in table 2 (De Tar 1981) with appropriate intermediate meson-states MM' shown in figure 3, the self-energy δm_M for mesons like ω, ρ and π can be computed as,

$$(\delta m_w, \delta m_\rho, \delta m_\pi) \equiv (-72, -48, -72) f_{NN\pi}^2 I_\pi / 25. \tag{32}$$

Table 2. Contribution of $\langle \sum_{q,q'} (\sigma_q \cdot \sigma_{q'})(\tau_q \cdot \tau_{q'}) \rangle = C_{MM'}$ from the various intermediate mesons states.

Mesons	Intermediate meson state	$C_{MM'}$
w	ρ	24
ρ	w	8
	π	8
π	ρ	24

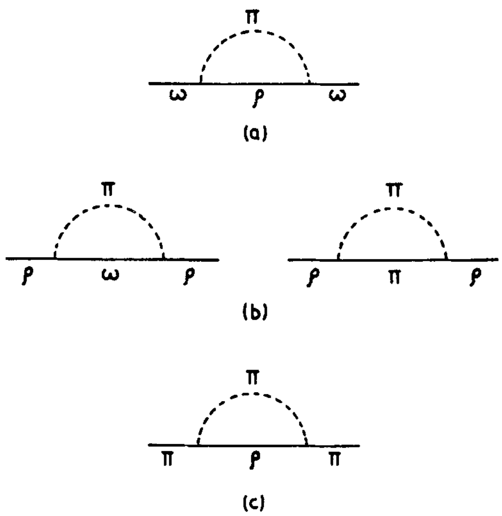


Figure 3. Relevant diagrams with appropriate intermediate states contributing to the pionic self-energies of mesons like (*a*) w, (*b*) ρ, (*c*) π.

The self-energy δm_M calculated here contains both the quark self-energy and the one-pion exchange contributions.

2.3 *Centre of mass momentum and the meson mass m_M*

In this shell-type relativistic independent quark model, the independent motion of quarks inside the hadron-core does not lead to a state of definite total momentum as it should, to represent the physical state of a hadron. This problem appears in the same way in nuclear physics in the case of ^{3}He and also in bag models, and therefore has to be resolved accordingly (Peierls and Yoccoz 1957; Donoghue and Johnson 1980; Carlson and Chachkhunashvili 1981; Hill and Wheeler 1953; Wong 1975). The energy associated with the spurious centre of mass motion must provide a further correction to the hadron energy obtained from the individual quark binding energy over and above the perturbative corrections discussed in §§2.2 and 2.3. This is accounted for following the prescriptions of Wong and other workers (Wong 1981; Duck 1978; Bertelski *et al* 1984; Eich *et al* 1985), which has been described in detail in our earlier work (Barik and Dash 1986a).

In such an approach the static meson-core state with core-centre at X is decomposed into components $\Phi(\mathbf{P})$ of plane-wave momentum eigen states as

$$|M(\mathbf{x})\rangle_c = \int \frac{d^3\mathbf{P}}{W_M(\mathbf{P})} \exp(i\mathbf{P}\cdot\mathbf{X})\Phi_M(\mathbf{P})|M(P)\rangle. \tag{33}$$

The inverse relation is

$$|M(\mathbf{P})\rangle = \frac{1}{(2\pi)^3}\frac{W_M(\mathbf{P})}{\Phi_M(\mathbf{P})}\int d^3\mathbf{X}\exp(-i\mathbf{P}\cdot\mathbf{X})|M(\mathbf{X})\rangle_c \tag{34}$$

with the normalization as follows,

$$\langle M(\mathbf{P}')|M(\mathbf{P})\rangle = (2\pi)^3 W_M(\mathbf{P})\delta(\mathbf{P}-\mathbf{P}') \tag{35}$$

and $W_M(\mathbf{P})=2w_p$. The momentum profile function $\Phi_M(\mathbf{P})$ can be obtained as,

$$\Phi_M^2(\mathbf{P})=\frac{W_H(\mathbf{P})}{(2\pi)^3}\tilde{I}_M(\mathbf{P}) \tag{36}$$

where $\tilde{I}_M(\mathbf{P})$ is the Fourier-transform of the Hill-Wheeler overlap function (Wong 1981). For mesons in the present model,

$$\tilde{I}_M(\mathbf{P})=\left(\frac{r_{0q}^2}{2\pi}\right)^{3/2}\exp(-P^2r_{0q}^2/2)(1-6C_q+15C_q^2)$$

$$\left[1+\frac{P^2r_{0q}^2(2C_q-10C_q^2+C_q^2P^2r_{0q}^2)}{(1-6C_q+15C_q^2)}\right] \tag{37}$$

when, $$C_q=(E_q'-m_q')/6(3E_q'+m_q'). \tag{38}$$

This permits ready estimates of the centre of mass momentum $\mathbf{P}$ as

$$\begin{aligned}\langle\mathbf{P}^2\rangle&=\int d^3\mathbf{P}\tilde{I}_M(\mathbf{P})\mathbf{P}^2\\&=\sum_q\langle\mathbf{p}^2\rangle_q.\end{aligned} \tag{39}$$

Here $\langle\mathbf{p}^2\rangle_q$ is the average value of the square of the individual quark-momentum taken over $1S_{1/2}$ single-quark state and is given by,

$$\langle\mathbf{p}^2\rangle_q=\frac{(11E_q'+m_q')(E_q'^2-m_q'^2)}{6(3E_q'+m_q')}. \tag{40}$$

Thus we find that the zeroth order energy E_M^0 in (9) for a ground-state meson M arising out of the binding energies of the constituent quark and antiquark confined independently by a phenomenological average potential $U(r)$ must be corrected for the energy shifts due to the residual quark-gloun [equations (15) and (16)] and quark-pion interaction [equation (32)], discussed in §§ 2.1 and 2.2. This would give the total energy of the $(q\bar{q})$-system in its ground state as,

$$E_M=E_M^0+[(\Delta E_M)_g^e+(\Delta E_M)_g^m]+\delta m_M. \tag{41}$$

Finally taking into account the centre of mass motion of the $(q\bar{q})$-system with the c.m. momentum $\mathbf{P}$ given as in (39) and (40), one can obtain the physical mass of $(q\bar{q})$-meson in its ground state as,

$$m_M=[E_M^2-\langle\mathbf{P}^2\rangle_M]^{1/2}. \tag{42}$$

Since our main objective here is to obtain the mass of the $(q\bar{q})$-pion, we would confine ourselves only to the non-strange meson-sector. Then (42) can yield the mass of the $(q\bar{q})$-pion together with that of ρ and w-mesons.

3. Pion-decay constant

The decay constant of the pion-core can also be studied in the present model by calculating the pion-decay matrix element F_π from vacuum to pion momentum eigen state defined by (Donoghue and Johnson 1980; Wong 1981)

$$\langle 0|\bar{\Psi}_{q^-}(\mathbf{r})\gamma^\mu\gamma^5\Psi_{q^+}(\mathbf{r})|\pi(\mathbf{P})\rangle = i\sqrt{2}F_\pi(\mathbf{P}^2)P^\mu \exp(i\mathbf{P}\cdot\mathbf{r}). \tag{43}$$

Taking the time component and transforming to the core state according to (33), we can have

$$\langle 0|\bar{\Psi}_{q^-}(\mathbf{r})\gamma^0\gamma^5\Psi_{q^+}(\mathbf{r})|\pi(0)\rangle_c$$
$$= i\sqrt{2}\int d^3\mathbf{P}\,P^0\frac{\Phi_\pi(\mathbf{P})}{W_\pi(\mathbf{P})}F_\pi(\mathbf{P}^2)\exp(i\mathbf{P}\cdot\mathbf{r}). \tag{44}$$

Then the pion-decay mitrix element is obtained in the form,

$$F_\pi(P^2) = \left[\frac{3(2\pi)^3 W_\pi(\mathbf{P})}{(P^0)^2\tilde{I}_\pi(\mathbf{P})}\right]^{1/2}\tilde{J}_A(\mathbf{P}) \tag{45}$$

when,

$$\tilde{J}_A(\mathbf{P}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{r}\exp(-i\mathbf{P}\cdot\mathbf{r})\langle 0|\bar{\Psi}_{q^-}(\mathbf{r})\gamma^0\gamma^5\Psi_{q^+}(\mathbf{r})|\pi(0)\rangle_c/i\sqrt{6}$$
$$= \frac{1}{(2\pi)^3}\int d^3\mathbf{r}\exp(-i\mathbf{P}\cdot\mathbf{r})[g_u^2(r) - f_u^2(r)]/4\pi r^2, \tag{46}$$

using the explicit form of $g_u(r)$ and $f_u(r)$ as given in (6), one can further simplify the expression (46) to,

$$\tilde{J}_A(\mathbf{P}) = \frac{1}{(2\pi)^3}\frac{(2-3\beta^2)}{(2+3\beta^2)}\left[1 + \frac{2+3\beta^2}{2-3\beta^2}C_u P^2 r_{0u}^2\right]\exp(-P^2 r_{0u}^2/4), \tag{47}$$

when $\beta = 1/\lambda_u r_{ou}$ and

$$C_u = \frac{E_u' - m_u'}{6(3E_u' + m_u')} = \frac{\beta^2}{(4+6\beta^2)}. \tag{48}$$

Now with $\tilde{I}_\pi(\mathbf{P})$ for π-meson from (37) and $\tilde{J}_A(\mathbf{P})$ from (47), it is straightforward to calculate the pion decay matrix element $F_\pi(P^2)$ from (45) in a more useful form as,

$$F_\pi(P^2) = F_\pi(0)\left(\frac{m_\pi W_\pi(\mathbf{P})}{2(P^0)^2}\right)^{1/2}\Lambda(P^2) \tag{49}$$

when

$$F_\pi(0) = \frac{\sqrt{6}}{B_u}((2\pi)^{3/2} m_\pi r_{0u}^3 D_u)^{-1/2} \tag{50}$$

and

$$\Lambda(P^2)=\frac{(1+B_u C_u P^2 r_{0u}^2)}{[1-(2-10\,C_u+P^2 r_{0u}^2 C_u)C_u P^2 r_{0u}^2/D_u]^{1/2}}. \tag{51}$$

Here $B_u=\left(\dfrac{2+3\beta^2}{2-3\beta^2}\right)$

and $$D_u=(1-6C_u+15C_u^2). \tag{52}$$

Such an expression has also been obtained by Wong (1981) in the bag model description of pion with the bag wave function for the independent quarks chosen ad hoc in Gaussian form. Now in the static limit if one identifies $F_\pi(0)$ as the pion-decay constant f_π, then it can be evaluated according to (50).

4. Results and discussion

For a quantitative evaluation of the $(q\bar{q})$-pion mass and the pion decay constant in the present model, one needs the potential parameters (a, V_0) and the non-strange quark mass $m_q(m_u=m_d)$ as model inputs. The potential parameters (a, V_0) describe the phenomenological average potential which is used in this model as a substitute for the long range part of the two body interaction. In our earlier work on baryons (Barik and Dash 1986a, b, c) a suitable choice of (a, V_0) has been found which together with a quark-gluon coupling constant $\alpha_s=0{\cdot}58$, has led us to a reasonable estimate of the physical masses (Barik and Dash 1986a) and magnetic moments (Barik and Dash 1986b, c) of the ground state octet baryons. But these parameters may be different in the present case since the quarks belonging to mesons and baryons may be acted upon by different long range potentials. For the short-range one gluon exchange interaction, it is well known that the two body quark-quark potential V_{qq} in a baryon is half of the quark antiquark potential $V_{q\bar{q}}$ in a meson. However any definite relationship of such type corresponding to long-range part of the interaction arising out of nonperturbative multi-gluon mechanism is not yet known in clear terms. Dosch and Miller (1976) have shown that in lattice approximation without including vacuum polarization, the three body potential for baryons may be written approximately as 0·54 times the two body $(q\bar{q})$-potential. However one does not really know what happens in the continuum limit. Therefore, it may not be totally unreasonable if we take the average central potential for quarks in a meson to be about the same as that in a baryon. The results so obtained may provide justifications a posteriori for such an assumption.

Therefore we retain the same set of parameters (a, V_0, m_q and α_s) as obtained in our earlier work on baryons (Barik and Dash 1986a, b, c) for the present study in mesonic sector.

$$(a, V_0)\equiv(0{\cdot}017166\ \text{GeV}^3,\ -137{\cdot}5\ \text{MeV}),$$

$$(m_u'=m_d',\ \alpha_s)=(10\ \text{MeV},\ 0{\cdot}58). \tag{53}$$

The energy eigen value condition (8) then yields $E'_u = E'_d = 540$ MeV, and hence the energy E^0_M of the meson-core according to (9). Then coming to calculate the energy shift due to residual one-gluon exchange interaction according to (15) and (16), one finds from table 1 that the energy-shift $(\Delta E_M)^{\mathscr{E}}_g$ due to colour-electrostatic interaction energy turns out to be zero here, while the one due to colour-magnetostatic interaction energy only contributes. For evaluating $(\Delta E_M)^{\mathscr{M}}_g$ one needs I_{uu} as given in (16) which when computed gives a value $I_{uu} = 64{\cdot}6812$ MeV. Then $(\Delta E_M)^{\mathscr{M}}_g$ for various different mesonic systems like (ρ, w, π), which are otherwise mass degenerate at the zeroth order, are calculated with $\alpha_s = 0{\cdot}58$. However if one believes that the quark-gluon coupling constant α_s may have some dependence on the scale size, one can always make a different choice for α_s in the present case of mesons which may enable one to realize the ($q\bar{q}$)-pion mass m_π consistent with the PCAC-pion mass $\tilde{m}_\pi = 140$ MeV. In view of this table 3 provides the calculated values of $(\Delta E_M)^{\mathscr{M}}_g$ for two choices of α_s like $\alpha_s = 0{\cdot}58$ and $0{\cdot}68$. The integral expression I_π in (31) is calculated for $z = \tilde{m}^2_\pi r^2_{0u}/2 \simeq 0{\cdot}1$ to give its value as $I_\pi = 291{\cdot}493$ MeV, which enables one to obtain the pionic self-energies of mesons (w, ρ, π) through (32). The values of δm_M so obtained with $f^2_{NN\pi} = 0{\cdot}08$ have also been presented in table 3. Finally the square momentum spread $\langle \mathbf{P}^2 \rangle_M$ of this meson system, calculated from (39) and (40) is also listed here together with E^0_M and the resulting physical masses m_π, m_ρ and m_w.

From the various energy corrections it is found that the degeneracy in mass due to SU(2)-symmetry between w and ρ-mesons is removed through the spin-isospin pionic corrections, while between w and π-mesons, it is lifted due to the gluonic correction. However, the mass degeneracy between ρ and π-mesons is effectively removed through both gluonic and pionic corrections. If one retains $\alpha_s = 0{\cdot}58$ found from the baryonic sector, then the physical masses of w and ρ are found in good agreement with the corresponding experimental ones. However, the mass of the pion remains almost twice that of the PCAC-pion with $m_\pi = 261$ MeV. But on the other hand with a slightly different choice $\alpha_s = 0{\cdot}68$, it is possible to obtain $m_\pi = \tilde{m}_\pi = 140$ MeV, when m_w and m_ρ are not too much different from the experimental values.

In estimating the ($q\bar{q}$)-meson mass in the present chiral model, the PCAC-pion in terms of an elementary field with $\tilde{m}_\pi = 140$ MeV and $f_\pi = 93$ MeV has been taken as an input. In the process we have recovered the ($q\bar{q}$)-pion mass to be of the same order as that of PCAC-pion with a suitable choice of $\alpha_s = 0{\cdot}68$. Now for a consistency check we estimate the pion decay constant f_π from the model through the expression (50) using

Table 3. Energy corrections and physical masses of ground state mesons (in MeV).

Mesons	E^0_M MeV	$\langle P^2 \rangle_M$ MeV^2	$(\Delta E_g)^{\mathscr{M}}$ $\alpha_s = 0{\cdot}58$ (MeV)	$(\Delta E_g)^{\mathscr{M}}$ $\alpha_s = 0{\cdot}68$ (MeV)	δm_M (MeV)	m_M (MeV) $\alpha_s = 0{\cdot}58$	m_M (MeV) $\alpha_s = 0{\cdot}68$	Expt.
w			75	87·85	−67·156	740·584	757·2	783
ρ	942·5	354688	75	87·85	−44·77	769·1	785·3	770
π			−225	−263·55	−67·156	261·264	140	140

this $(q\bar{q})$-pion mass $m_\pi = 140$ MeV. Identifying $f_\pi(0)$ as the pion decay constant in the static limit, we find that it comes out to be 118·4 MeV, which is in good agreement with the experimental value 93 MeV. In this model $F_\pi(0)$ is infinite in the limit of vanishing pion mass, which is unlike the observation of Donoghue and Johnson (1980).

Equation (49) shows that the pion decay matrix element $F_\pi(P^2)$ depends on squared three momentum P^2. This momentum dependence comes, primarily from a kinematical factor which for on-shell pion $(P^0 - w_0)$ is

$$\left(\frac{m_\pi W_\pi(\mathbf{P})}{2(P^0)^2}\right)^{1/2} = (m_\pi/w_p)^{1/2}$$

and also from a factor like $\Lambda(\mathbf{P}^2)$. The energy of the pion-core in the present model comes to be 438·3438 MeV which is evident from table 3. At finite $\langle \mathbf{P}^2 \rangle_\pi$ it is possible to calculate $\Lambda(\langle P^2 \rangle_\pi) \simeq 1{\cdot}283$ from (51). Consequently, $F_\pi(\langle P^2 \rangle_\pi)$ obtained is 86 MeV which is in rough agreement with the experimental value. The significance of this agreement is, however, not clear since $F_\pi(P^2)$ is valid only for the projected state of good momentum $\mathbf{P}$, while $\langle \mathbf{P} \rangle = 0$ holds for the unprojected core-state. It is likely that the usual pion decay matrix element containing a Lorentz-invariant decay constant F_π cannot be recovered completely unless the theoretical projected pion states also have a Lorentz-invariant internal structure. Here, the problem is concerned with centre-of-mass corrections of pion-core in the quark model.

In view of the reasonable agreement of the pion decay constant and the masses of the mesons like w, ρ and π in the non-strange flavour sector, with the corresponding experimental results it is reasonable to expect that the phenomenological effective central potential $U(r)$ confining the quarks in a baryon is about the same as the potential confining the quark and anti-quark in a meson. Furthermore, the present model indirectly indicates that by incorporating the appropriate gluonic, pionic and centre of mass corrections, it is possible for the traditional quark-model state of the pion to coincide with the PCAC pion to provide an effective way to reconcile these two seemingly very different physical pictures of the pion.

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