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# Poincaré gauge theory from self-coupling

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**Abstract.** Poincaré gauge theory is derived from a linear theory by the method suggested by Gupta for deriving Einstein's general relativity from the linear theory of a spin-2 field. Non-linearity is introduced by requiring that a set of tensor fields be coupled to the Noether currents of the Poincaré group (energy-momentum and spin).

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## 1. Introduction

It was pointed out by Gupta (1952) that Einstein's gravitational theory is obtainable from the linear theory of a massless spin-2 field by an iterative process. If the symmetric (Belinfante) energy-momentum tensor of matter is taken as the source term in the linear field equations of a massless spin-2 field, the resulting equations are inconsistent with energy-momentum conservation, because the energy-momentum of the spin-2 field itself has not been included in the source. When it is included, the resulting nonlinear spin-2 equations are still not consistent with the conservation law, because the self-coupling term that has been introduced contributes to energy-momentum, and this portion of the energy-momentum tensor should be included in the source. Thus, for consistency one has to introduce an infinite sequence of self-coupling terms. This iterative procedure converges, and the non-linear spin-2 theory that it converges to is Einstein's gravitational theory. The reader is referred to the references given by Deser (1970) for details.

By starting from the first order form of the Lagrangian of the linear massless spin-2 theory, Deser (1970) was able to obtain Einstein's theory immediately by coupling the field to its own symmetric energy-momentum tensor. The infinite sequence described above did not arise. Deser applied the same principle to a set of massless spin-1 fields, the (first order) Lagrangian of which is invariant under the *global* action of a group G that transforms the set linearly according to the co-adjoint representation. If the Noether currents associated with this symmetry are taken as the sources of the spin-1 fields, the resulting nonlinear theory is the Yang-Mills theory of a *gauge* group *G*. Observe that this method of deriving Yang-Mills theory does not invoke the principle of gauge invariance. Instead, it relies on the idea of self-coupling. Gauge invariance arises spontaneously as a property of the resulting non-linear theory.

The aim of the present paper is to derive the Poincaré gauge theories (e.g. the ECKS theory or the Poincaré gauge theory of Hehl (1978)) by employing the method of Gupta and Deser.

In § 2 we introduce a class of first order Lagrangians for vector fields. We discuss the energy-momentum and spin tensors of the resulting linear theories and derive the gauge theory of a group $G$ by introducing self-coupling through coupling to Noether currents. In § 3 the linear theories are generalised for tensor fields rather than vector fields, and we show that Poincaré gauge theories result when the tensor fields are coupled to energy-momentum and spin—the Noether currents of the Poincaré group.

## 2. Vector theories

Let $\mathscr{L}$ be the Lagrangian for a special-relativistic theory of a vector field $A$, which contains derivatives only to first order, and only in the combination $\partial_i A_j - \partial_j A_i$. Define

$$\mathscr{H}^{ij} = \partial \mathscr{L} / \partial \partial_i A_j = -\mathscr{H}^{ji}. \tag{1}$$

In terms of a Cartesian coordinate system, the action of an infinitesimal Poincaré transformation on Minkowski space is

$$x^i \to x^i - \xi^i, \quad \xi^i = a^i + x^j \omega_j{}^i, \quad \omega^{ij} = -\omega^{ji}.$$

The corresponding change in the components of $A$ is

$$\delta A_j = \xi^i \partial_i A_j + \omega_j{}^i A_i. \tag{2}$$

$\mathscr{L}$ is required to be invariant. The Noether currents associated with the Poincaré invariance are contained in

$$\theta^i = \mathscr{H}^{ij} \delta A_j - \xi^i \mathscr{L} + \partial_j \mathscr{X}^{ji}. \tag{3}$$

The Noether conservation laws are

$$\partial_i \theta^i = 0. \tag{4}$$

The tensor $\mathscr{X}^{ji}$ satisfies $\mathscr{X}^{ji} = -\mathscr{X}^{ij}$ and is linear in $\xi^k$,

$$\mathscr{X}^{ji} = \mathscr{X}_k{}^{ji} \xi^k, \tag{5}$$

but is otherwise arbitrary. An energy-momentum tensor $\theta^i{}_k$ and a spin tensor $\tau^i{}_{jk}$ are obtained as the coefficients of the Poincaré group parameters, in $\theta^i$:

$$\theta^i = \theta^i{}_k \xi^k + \tfrac{1}{2} \tau^i{}_{kl} \omega^{kl} = \theta^i{}_k a^k + \tfrac{1}{2}(\tau^i{}_{kl} - 2\theta^i{}_k x_l) \omega^{kl}. \tag{6}$$

In terms of these tensors, the Noether identity (4) is

$$\partial_i \theta^i{}_k = 0, \tag{7}$$

$$\partial_i \tau^i{}_{jk} = \theta_{kj} - \theta_{jk}. \tag{8}$$

The explicit expressions for the energy-momentum and spin tensors are

$$\theta^i{}_k = \mathscr{H}^{ij}\partial_k A_j - \delta^i_k \mathscr{L} + \partial_j \mathscr{X}^{ji}{}_k, \tag{9}$$

$$\tau^i{}_{jk} = \mathscr{H}^i{}_j A_k - \mathscr{H}^i{}_k A_j + \mathscr{X}^i{}_{jk} - \mathscr{X}^i{}_{kj}. \tag{10}$$

There is an arbitrariness in the definitions, corresponding to the freedom to choose $\mathscr{X}^{ji}$ (see Hehl, 1976). The canonical choice is $\mathscr{X}^{ji}=0$. We shall adopt a different choice. Observe that the Lagrangian is invariant under $A_i \to A_i + \partial_i \lambda$ because we have stipulated that the derivative occurs only as a curl. If we insist that $\theta^i$ be invariant under this transformation, we are led to the choice

$$\mathscr{X}^{ji}{}_k = \mathscr{H}^{ji} A_k. \tag{11}$$

The tensors (9) and (10) are then

$$\theta^i{}_k = \mathscr{H}^{ij}\partial_k A_j - \partial_j(\mathscr{H}^{ij} A_k) - \delta^i_k \mathscr{L}, \tag{12}$$

$$\tau^i{}_{jk} = 0. \tag{13}$$

The Lagrangians that we shall be employing are generalisations of the Lagrangian

$$\mathscr{L} = \mathscr{H}^{ij}(\partial_i A_j - \tfrac{1}{2}F_{ij}) + \mathscr{G}(F), \tag{14}$$

in which $\mathscr{H}^{ij}(= -\mathscr{H}^{ji})$ and $F_{ij}(= -F_{ji})$ are auxiliary fields, and $\mathscr{G}$ is a Lorentz scalar constructed from the $F_{ij}$ (not containing derivatives). The Euler-Lagrange equations (obtained by varying $F$ and $A$, respectively) are

$$F_{ij} = \partial_i A_j - \partial_j A_i, \tag{15}$$

$$\mathscr{H}^{ij} = \partial \mathscr{G} / \partial F_{ij}, \tag{16}$$

$$\partial_i \mathscr{H}^{ij} = 0. \tag{17}$$

The energy-momentum tensor (12) is, in this case, simply

$$\theta^i{}_k = \mathscr{H}^{ij} F_{kj} - \delta^i_k \mathscr{G}. \tag{18}$$

Now let $A_i{}^a$ be the components of a set of vector fields which transform among themselves according to the co-adjoint representation of a global symmetry group $G$:

$$\delta A^a_i = \varepsilon^b A^c_i c_{bc}{}^a, \tag{19}$$

where $c_{bc}{}^a$ are the structure constants of $G$ and $\varepsilon^a$ are the infinitesimal parameters. The associated Noether currents are

$$\mathscr{J}^i_a = \mathscr{H}^{ij}{}_c A_j{}^b c_{ab}{}^c \tag{20}$$

(where $\mathscr{H}^{ij}{}_a = \partial \mathscr{L} / \partial \partial_i A_j{}^a = -\mathscr{H}^{ji}{}_a$). We shall take the Lagrangian for the vector fields $A$ to be of the form

$$\mathscr{L} = \mathscr{H}^{ij}{}_a(\partial_i A_j{}^a - \tfrac{1}{2}F_{ij}{}^a) + \mathscr{G}(F), \tag{21}$$

which leads to the Euler-Lagrange equations

$$F_{ij}{}^a = \partial_i A_j{}^a - \partial_j A_i{}^a, \tag{22}$$

$$\mathscr{H}^{ij}{}_a = \partial\mathscr{G}/\partial F_{ij}{}^a, \tag{23}$$

$$\partial_i \mathscr{H}^{ij}{}_a = 0. \tag{24}$$

We now wish to introduce the Noether current into (24) as a source term. The total Noether current is $\mathscr{J}^i_a + \mathscr{J}_M{}^i{}_a$ where $\mathscr{J}_M{}^i{}_a$ is the Noether current of matter fields. The desired source term appears in (24) if the Lagrangian is modified by introducing the coupling term

$$\tfrac{1}{2} A_i{}^a \mathscr{J}^i{}_a + A_i{}^a \mathscr{J}_M{}^i{}_a \tag{25}$$

(the factor $\frac{1}{2}$ is necessary because $\mathscr{J}^i{}_a$ already contains $A_i{}^a$ linearly, so the first term in (25) is quadratic in $A$). We have absorbed the coupling constant in the definition of the fields $A$. The modified Euler-Lagrange equations are then

$$F_{ij}{}^a = \partial_i A_j{}^a - \partial_j A_i{}^a - A_i^b A_j^c c_{bc}{}^a, \tag{26}$$

$$\mathscr{H}^{ij}{}_a = \partial\mathscr{G}/\partial F_{ij}{}^a, \tag{27}$$

$$\partial_i \mathscr{H}^{ij}{}_a = -c_{ab}{}^c A_i{}^b \mathscr{H}^{ij}{}_c + \mathscr{J}_M{}^i{}_a. \tag{28}$$

Observe that the introduction of the self-coupling term does not affect the form (20) of the Noether current of the fields $A$, so no further iteration is required (the contribution of the self-coupling term to the current is taken care of *implicitly*, through the extra term that has appeared in $F_{ij}{}^a$, (26)).

We now see that the self-coupling has converted the original linear theory to a *gauge* theory of the group $G$. Equation (28) can be recast in the manifestly gauge covariant form

$$D_i \mathscr{H}^{ij}{}_a = \partial_i \mathscr{H}^{ij}{}_a + c_{ab}{}^c A_i{}^b \mathscr{H}^{ij}{}_c = \mathscr{J}_M{}^j{}_a. \tag{29}$$

Using (12) to work out $\theta^i_k$ for the gauge fields, we find

$$\theta^i{}_k = \mathscr{H}^{ij}{}_a F_{kj}{}^a - \delta^i_k \mathscr{G} + \mathscr{J}_M{}^i{}_a A_k{}^a. \tag{30}$$

The final term is the contribution from the coupling to matter. The remaining tensor is formally a sum of terms like the tensor (18) of the linear theory, the extra term in $F_{ij}{}^a$ gives rise implicitly to a contribution to energy-momentum from the self-coupling.

## 3. The tensor theories

Let $\rho$ be a representation of the Lorentz group, with generators $G_{ij} = -G_{ji}$, and consider a set $A$ of fields with components that transform, under an infinitesimal Poincaré transformation $x^i \to x^i - \xi^i$, $\xi^i = a^i + x_j \omega^{ji}$, $\omega^{ij} = -\omega^{ji}$, according to

$$\delta A_j = \xi^i \partial_i A_j + \omega_j{}^i A_i + \omega A_j, \quad \omega = \tfrac{1}{2}\omega^{ij} G_{ij}. \tag{31}$$

That is, $A$ is a collection of tensors (or, more generally, spin-tensors). One covariant vector index has been indicated explicitly, while the remaining indices are implicit in the matrix notation.

Let $\mathscr{L}$ be a Lagrangian for $A$, containing derivatives only to first order and only in the combination $\partial_i A_j - \partial_j A_i$. Define $\mathscr{H}^{ij} = \partial \mathscr{L} / \partial_i A_j$. The Poincaré invariance of $\mathscr{L}$ implies the Noether identity

$$\partial_i \theta^i = 0 \tag{32}$$

where

$$\theta^i = \mathscr{H}^{ij} \cdot \delta A_j - \xi^i \mathscr{L} + \partial_j \mathscr{X}^{ji} \tag{33}$$

(the dot denotes contraction over the implicit indices). We choose

$$\mathscr{X}^{ij}{}_k = \mathscr{H}^{ij} \cdot A_k \tag{34}$$

in analogy with (11) and find the energy-momentum and spin tensors

$$\theta^i{}_k = \mathscr{H}^{ij} \cdot \partial_k A_j - \partial_j(\mathscr{H}^{ij} \cdot A_k) - \delta^i_k \mathscr{L}, \tag{35}$$

$$\tau^i{}_{kl} = \mathscr{H}^{ij} \cdot G_{kl} A_j, \tag{36}$$

satisfying (7) and (8).

There is an alternative method for obtaining the canonical energy-momentum and spin tensors, which is analogous to the method of obtaining the symmetrised (Belinfante) energy-momentum tensor by going over to a curvilinear coordinate system and working out the functional derivative of the Lagrangian density with respect to the metric. Let $L_M(\psi, \partial_i \psi)$ be a Lagrangian for a field $\psi$, in Minkowski space. Let $f_{\alpha\beta}$ denote the generators of the representation of the Lorentz group to which $\psi$ belongs. Introduce an orthonormal tetrad $e_i{}^\alpha$ ($i$ is a coordinate-based (holonomic) index and $\alpha$ is an anholonomic index labelling the four vectors). We shall write $e_\alpha{}^i$ for the elements of the matrix inverse to the matrix of components $e_i{}^\alpha$, and we shall write $e$ for the determinant $e = |e_i{}^\alpha|$. In addition, introduce a set of spin coefficients $\Gamma_i{}^{\alpha\beta}$. One can now write the theory in a form that is covariant under general coordinate transformations as well as under Lorentz rotations of the tetrad. We simply replace $L$ by the density

$$\mathscr{L}_M(\psi, D_i\psi, e_i{}^\alpha, \Gamma_i{}^{\alpha\beta}) = eL_M(\psi, D_\alpha \psi) \tag{37}$$

where

$$D_i \psi = \partial_i \psi - \tfrac{1}{2} \Gamma_i{}^{\alpha\beta} f_{\alpha\beta} \psi, \tag{38}$$

$$D_\alpha \psi = e_\alpha{}^i D_i \psi. \tag{39}$$

Note that, in this context, there is no geometrical or dynamical significance to the tetrad and spin coefficients. They are introduced purely as a device for computing the energy-momentum and spin tensors. It is now not difficult to deduce that

$$-\frac{\partial \mathscr{L}_M}{\partial e_i{}^\alpha} = \frac{\partial \mathscr{L}_M}{\partial \partial_i \psi} D_\alpha \psi - e_i{}^\alpha \tag{40}$$

and that

$$-\frac{\partial \mathscr{L}_M}{\partial \Gamma_i{}^{\alpha\beta}}=\frac{\partial \mathscr{L}_M}{\partial \partial_i \psi} f_{\alpha\beta}\psi. \tag{41}$$

These expressions are covariant generalisations of the canonical energy-momentum tensor and canonical spin tensor for the field and reduce to these quantities when we revert to the Cartesian coordinate system by setting $e_i{}^\alpha=\delta_i^\alpha$, $\Gamma_i{}^{\alpha\beta}=0$.

We shall apply the above trick to the set of tensor fields $A$, but with this difference: the explicit covariant vector index on $A_i$ will be kept holonomic.

The transformation laws for the tetrad components and spin coefficients are

$$\delta e_i{}^\alpha=\xi^j\partial_j e_i{}^\alpha+e_j{}^\alpha\partial_i\xi^j-e_i{}^\gamma\varepsilon_\gamma{}^\alpha, \tag{42}$$

$$\delta\Gamma_i{}^{\alpha\beta}=\xi^j\partial_j\Gamma_i{}^{\alpha\beta}+\Gamma_j{}^{\alpha\beta}\partial_i\xi^j-\Gamma_i{}^{\gamma\beta}\varepsilon_\gamma{}^\alpha-\Gamma_i{}^{\alpha\gamma}\varepsilon_\gamma{}^\beta+\partial_i\varepsilon^{\alpha\beta}. \tag{43}$$

The three transformation laws (31), (42) and (43) can be reformulated in terms of the parameters

$$\lambda^{\alpha\beta}=\varepsilon^{\alpha\beta}+\xi^i\Gamma_i{}^{\alpha\beta},$$

$$\xi^\alpha=\xi^i e_i{}^\alpha.$$

We find

$$\delta A_i=\xi^\alpha\bar{\nabla}_\alpha A_i+A_\alpha D_i\xi^\alpha+\lambda A_i,\quad \lambda=\tfrac{1}{2}\lambda^{\alpha\beta}G_{\alpha\beta}, \tag{44}$$

$$\delta e_i{}^\alpha=\xi^\gamma\Omega_{\gamma i}{}^\alpha+D_i\xi^\alpha-\lambda_i^\alpha, \tag{45}$$

$$\delta\Gamma_i{}^{\alpha\beta}=\xi^\gamma\Omega_{\gamma i}{}^{\alpha\beta}+D_i\lambda^{\alpha\beta}, \tag{46}$$

where

$$\Omega_{ij}{}^\beta=D_ie_j{}^\beta-D_je_i{}^\beta=\partial_ie_j{}^\beta-\partial_je_j{}^\beta-e_i{}^\gamma\Gamma_{j\gamma}{}^\beta+e_j^\gamma\Gamma_{i\gamma}{}^\beta \tag{47}$$

and

$$\Omega_{ij\alpha}{}^\beta=\partial_i\Gamma_{j\alpha}{}^\beta-\partial_j\Gamma_{i\alpha}{}^\beta-\Gamma_{i\alpha}{}^\gamma\Gamma_{j\gamma}{}^\beta+\Gamma_{j\alpha}{}^\gamma\Gamma_{i\gamma}{}^\beta \tag{48}$$

are the torsion and curvature associated with the tetrad and spin coefficients, and $\bar{\nabla}_i$ denotes the covariant derivative constructed from the anholonomic connection $\Gamma_{i\alpha}{}^\beta$ and the holonomic connection $\bar{\Gamma}_{ij}{}^k=\Gamma_{ji}{}^k=(\partial_je_i{}^\alpha+e_i\Gamma_{j\beta}{}^\alpha)e_\alpha{}^k$;

$$\bar{\nabla}_iA_j=\partial_iA_j-\Gamma_{ji}{}^kA_k-\tfrac{1}{2}\Gamma_i{}^{\alpha\beta}G_{\alpha\beta}A_j$$

(note the index order in the second term).

The fact that $\mathscr{L}$ is invariant under spacetime-dependent tetrad rotations and a scalar density of weight 1 under general coordinate transformations implies the identity

$$D_i(\mathscr{H}^{ij}\cdot\delta A_j-\xi^i\mathscr{L})=\Sigma^i{}_\alpha\delta e_i{}^\alpha+\tfrac{1}{2}\Sigma^i{}_{\alpha\beta}\delta\Gamma_i{}^{\alpha\beta} \tag{49}$$

where

$$\Sigma^i{}_\alpha=-\frac{\partial\mathscr{L}}{\partial e_i{}^\alpha},\quad \Sigma^i{}_{\alpha\beta}=-\frac{\partial\mathscr{L}}{\partial\Gamma_i{}^{\alpha\beta}}. \tag{50}$$

Substituting the variations (44), (45) and (46) into the identity (49) and equating coefficients of $D_i\xi^\alpha$ and $D_i\lambda^{\alpha\beta}$ gives

$$\Sigma^i{}_\alpha = \mathscr{H}^{ij}\cdot\bar{\nabla}_\alpha A_j - D_i(\mathscr{H}^{ij}\cdot A_\alpha) - e_\alpha{}^i\mathscr{L}, \tag{51}$$

$$\Sigma^i{}_{\alpha\beta} = \mathscr{H}^{ij}\cdot G_{\alpha\beta}A_j. \tag{52}$$

These are clearly the covariant generalisations of the expressions (35) and (36) for energy-momentum and spin, and reduce to them when we revert to the Cartesian system by setting $e_i{}^\alpha = \delta_i^\alpha$, $\Gamma_i{}^{\alpha\beta} = 0$. Equating coefficients of $\xi^\alpha$ and $\lambda^{\alpha\beta}$ gives

$$D_i\Sigma^i{}_\alpha = \Sigma^j{}_\beta\Omega_{\alpha j}{}^\beta + \tfrac{1}{2}\Sigma^j{}_{\beta\gamma}\Omega_{\alpha j}{}^{\beta\gamma}, \tag{53}$$

$$D_i\Sigma^i{}_{\alpha\beta} = \Sigma_{\beta\alpha} - \Sigma_{\alpha\beta}. \tag{54}$$

These reduce to the Noether identities (7) and (8) when we revert to the Cartesian system.

Now consider the special-relativistic theory given by the Lagrangian

$$\mathscr{L} = \mathscr{H}^{ij}\cdot(\partial_i A_j - \tfrac{1}{2}F_{ij}) + \mathscr{G}(F) + \mathscr{L}_M(\psi, \partial_i\psi) \tag{55}$$

in which the $A$-fields consists of a tensor $A_i^\alpha$ and a tensor $A_i{}^{\alpha\beta} = -A_i{}^{\beta\alpha}$. That is, written more explicitly,

$$\mathscr{L} = \mathscr{H}^{ij}{}_\alpha(\partial_i A_j{}^\alpha - \tfrac{1}{2}F_{ij}{}^\alpha) + \tfrac{1}{2}\mathscr{H}^{ij}{}_{\alpha\beta}(\partial_i A_j{}^{\alpha\beta} - F_{ij}{}^{\alpha\beta}) + \mathscr{G}(F) + \mathscr{L}_M(\psi, \partial_i\psi). \tag{56}$$

$\mathscr{L}_M$ is a matter Lagrangian. The energy momentum tensor and the spin tensor for the $A$-fields are

$$\begin{aligned}\theta^i{}_k &= \mathscr{H}^{ij}{}_\alpha\partial_k A_j{}^\alpha + \tfrac{1}{2}\mathscr{H}^{ij}{}_{\alpha\beta}\partial_k A_j{}^{\alpha\beta} - \partial_j(\mathscr{H}^{ij}{}_\alpha A_k{}^\alpha + \tfrac{1}{2}\mathscr{H}^{ij}{}_{\alpha\beta}A_k{}^{\alpha\beta}) - \delta^i_k\mathscr{G}\\ &= \mathscr{H}^{ij}{}_\alpha F_{kj}{}^\alpha + \tfrac{1}{2}\mathscr{H}^{ij}{}_{\alpha\beta}F_{kj}{}^{\alpha\beta} - \delta^i_k\mathscr{G},\end{aligned} \tag{57}$$

$$\tau^i{}_{\alpha\beta} = \mathscr{H}^{ij}{}_\beta A_{j\alpha} - \mathscr{H}^{ij}{}_\alpha A_{j\beta} + \mathscr{H}^{ij}{}_{\beta\gamma}A_{j\alpha}{}^\gamma - \mathscr{H}^{ij}{}_{\alpha\gamma}A_{j\beta}{}^\gamma. \tag{58}$$

At this stage, there is no distinction between Latin and Greek indices.

We wish to add coupling terms to (55) so that, in the resulting theory, total energy momentum and total spin will appear as sources on the right-hand sides of the Euler-Lagrange equations $\partial_i\mathscr{H}^{ij}{}_\alpha = 0$ and $\partial_i\mathscr{H}^{ij}{}_{\alpha\beta} = 0$.

First, consider the coupling to energy-momentum. Introduce a tetrad $e_i^\alpha$ and go to a curvilinear coordinate system. Then $\mathscr{L}$ becomes a density under general coordinate transformations. In this process, the Greek indices in (56) are to be treated as anholonomic indices, and the matter fields are also treated as anholonomic. Observe that $e_i^\alpha$ will appear only in $\mathscr{G} + \mathscr{L}_M$. The other terms become generally covariant densities simply by identifying the $\mathscr{H}^{ij}$ as tensor densities. The total energy-momentum tensor density is $-\partial\mathscr{L}/\partial e_i^\alpha$. So to begin with we introduce the coupling term $A_i^\alpha\partial\mathscr{L}/\partial e_i^\alpha$. That is, we replace $\mathscr{L}$ by $\mathscr{L} + A_i^\alpha\partial\mathscr{L}/\partial e_i^\alpha$. This changes the energy-momentum density to $-\partial(\mathscr{L} + A_j^\beta\partial\mathscr{L}/\partial e_j^\beta)/\partial e_i^\alpha$. To include the extra energy momentum in the source we need another coupling term $\tfrac{1}{2}A_i{}^\alpha A_j{}^\beta\partial^2\mathscr{L}/\partial e_i{}^\alpha\partial e_j{}^\beta$ (the factor $\tfrac{1}{2}$ is needed because this term is quadratic in $A_i{}^\alpha$). Again, the energy-momentum tensor density changes and we have to introduce yet another coupling term, and so on. Finally, we conclude that the necessary change is brought about by replacing $\mathscr{L}$ by

$$\mathscr{L}+A_i^{\alpha}\frac{\partial\mathscr{L}}{\partial e_i^{\alpha}}+\tfrac{1}{2}A_i^{\alpha}A_j^{\beta}\frac{\partial^2\mathscr{L}}{\partial e_i^{\alpha}\partial e_j^{\beta}}+\tfrac{1}{6}A_i^{\alpha}A_j^{\beta}A_k^{\gamma}\frac{\partial^3\mathscr{L}}{\partial e_i^{\alpha}\partial e_j^{\beta}\partial e_k^{\gamma}}+\ \ldots \tag{59}$$

In other words, we have to replace $e_i^{\alpha}$, wherever it occurs in $\mathscr{L}$, by

$$B_i^{\alpha}=e_i^{\alpha}+A_i^{\alpha}. \tag{60}$$

We now use the letter $\mathscr{L}$ to denote the final form of the Lagrangian density, so that the total energy-momentum density is now $-\partial\mathscr{L}/\partial e_i^{\alpha}=-\partial\mathscr{L}/\partial B_i^{\alpha}$, and the Euler-Lagrange equation obtained from varying $B_j^{\alpha}$ becomes $\partial_i\mathscr{H}^{ij}{}_{\alpha}=\partial\mathscr{L}/\partial B_i^{\alpha}$, as required.

Now turn to the problem of coupling to spin. Introduce spin coefficients $\Gamma_i{}^{\alpha\beta}$ and replace $\partial_i B_j^{\alpha}$ by $\partial_i B_j^{\alpha}+B_j{}^{\gamma}\Gamma_{i\gamma}{}^{\alpha}$, $\partial_i A_j{}^{\alpha\beta}$ by $\partial_i A_j{}^{\alpha\beta}+A_j{}^{\gamma\beta}\Gamma_{i\gamma}{}^{\alpha}+A_j{}^{\alpha\gamma}\Gamma_{i\gamma}{}^{\beta}$, and $\partial_i\psi$ by $\partial_i\psi-\tfrac{1}{2}\Gamma_i{}^{\alpha\beta}f_{\alpha\beta}\psi$ ($f_{\alpha\beta}$ being the generators for the representation of the Lorentz group to which $\psi$ belongs). The total spin-tensor density is now $-\partial\mathscr{L}/\partial\Gamma_i{}^{\alpha\beta}$. Let $\mathscr{L}_0$ be the terms in $\mathscr{L}$ that do not contain (undifferentiated) $A_i{}^{\alpha\beta}$ and let $\mathscr{L}_1$ denote the term linear in $A_i{}^{\alpha\beta}$ (namely $\mathscr{H}^{ij}{}_{\alpha\beta}A_j{}^{\gamma\beta}\Gamma_{i\gamma}{}^{\alpha}$). Then the appropriate initial coupling term is

$$\left(\frac{1}{2}A_i{}^{\alpha\beta}\frac{\partial\mathscr{L}_0}{\partial\Gamma_i{}^{\alpha\beta}}\right)+\frac{1}{2}\left(\frac{1}{2}A_i{}^{\alpha\beta}\frac{\partial\mathscr{L}_1}{\partial\Gamma_i{}^{\alpha\beta}}\right). \tag{61}$$

(The factors $\tfrac{1}{2}$ attached to $A_i{}^{\alpha\beta}$ are simply included to avoid double counting in the summation over the stewsymmetric index pair $\alpha\beta$. The factor $\tfrac{1}{2}$ outside the bracket in the second term is included because this term is quadratic in $A_i{}^{\alpha\beta}$). Now, if $\mathscr{L}_M$ contains $\Gamma_i{}^{\alpha\beta}$ only linearly, the above coupling scheme will not contain $\Gamma_i{}^{\alpha\beta}$ and therefore will not contribute *explicitly* to the total spin density. The process then terminates. If $\mathscr{L}_M$ contains boson fields with spin greater than zero, it may be quadratic in $\Gamma_i{}^{\alpha\beta}$. In which case (28) does contribute to spin density and we have to add another coupling term

$$\frac{1}{2}\left[\left(\frac{1}{4}A_i{}^{\alpha\beta}A_j{}^{\gamma\delta}\frac{\partial^2\mathscr{L}_0}{\partial\Gamma_i{}^{\alpha\beta}\partial\Gamma_j{}^{\gamma\delta}}\right)+\frac{1}{2}\left(\frac{1}{4}A_i{}^{\alpha\beta}A_j{}^{\gamma\delta}\frac{\partial^2\mathscr{L}_1}{\partial\Gamma_i{}^{\alpha\beta}\partial\Gamma_j{}^{\gamma\delta}}\right)\right], \tag{62}$$

and the iterative process terminates here. In any case, when the process of adding on these spin couplings has terminated the effect has been to replace $\Gamma_i{}^{\alpha\beta}$ by $\Gamma_i{}^{\alpha\beta}+A_i{}^{\alpha\beta}$ in $\mathscr{L}_0$ and by $\Gamma_i{}^{\alpha\beta}+\tfrac{1}{2}A_i{}^{\alpha\beta}$ in $\mathscr{L}_1$. Again, retaining the symbol $\mathscr{L}$ to denote the final form of the Lagrangian density, one can easily verify that the total spin density $-\partial\mathscr{L}/\partial\Gamma_i{}^{\alpha\beta}$ is the same as $-\partial\mathscr{L}/\partial A_i{}^{\alpha\beta}$, in the limit $\Gamma_i{}^{\alpha\beta}=0$. The spin-couplings terms that we have introduced contribute to energy-momentum but since $e_i^{\alpha}$ has already been replaced by $B_i^{\alpha}$ in the expressions $\mathscr{L}_0$ and $\mathscr{L}_1$ before we began to construct the spin coupling, this extra energy-momentum is taken care of.

We obtain the final form for the generalisation of (56) by setting $e_i^{\alpha}=\delta_i^{\alpha}$, $\Gamma_i{}^{\alpha\beta}=0$. We get

$$\begin{aligned}\mathscr{L}=\ &\mathscr{H}^{ij}{}_{\alpha}(\partial_i B_j^{\alpha}+B_j{}^{\gamma}A_{i\gamma}{}^{\alpha}-\tfrac{1}{2}F_{ij}{}^{\alpha})\\&+\tfrac{1}{2}\mathscr{H}^{ij}{}_{\alpha\beta}(\partial_i A_j{}^{\alpha\beta}+A_j{}^{\gamma\beta}A_{i\gamma}{}^{\alpha}-\tfrac{1}{2}F_{ij}{}^{\alpha\beta})+\mathscr{G}(F,B_i^{\alpha})\\&+\mathscr{L}_M(\psi,\partial_i\psi,B_i^{\alpha},A_i{}^{\alpha\beta}).\end{aligned} \tag{63}$$

The Euler-Lagrange equations from variation of $\mathscr{H}$ are

$$F_{ij}{}^{\beta}=\partial_i B_j^{\beta}-\partial_j B_i^{\beta}-B_i^{\gamma}A_{j\gamma}{}^{\beta}+B_j^{\gamma}A_{i\gamma}{}^{\beta}, \tag{64}$$

$$F_{ij\alpha}{}^{\beta}=\partial_i A_{j\alpha}{}^{\beta}-\partial_j A_{i\alpha}{}^{\beta}-A_{i\alpha}{}^{\gamma}A_{j\gamma}{}^{\beta}+A_{j\alpha}{}^{\gamma}A_{i\gamma}{}^{\beta}. \tag{65}$$

From the variation of $F$, we get

$$\mathscr{H}^{ij}{}_{\alpha}=\partial\mathscr{G}/\partial F_{ij}{}^{\alpha}, \tag{66}$$

$$\mathscr{H}^{ij}{}_{\alpha\beta}=\partial\mathscr{G}/\partial F_{ij}{}^{\alpha\beta}. \tag{67}$$

Finally, variation of $B$ and $A$ gives

$$\partial_i\mathscr{H}^{ij}{}_{\alpha}=\partial\mathscr{L}/\partial B_j^{\alpha}, \tag{68}$$

$$\partial_i\mathscr{H}^{ij}{}_{\alpha\beta}=\partial\mathscr{L}/\partial A_j^{\alpha\beta}. \tag{69}$$

It would be obtuse to continue to regard (61) as the Lagrangian of a theory in Minkowski space. $\mathscr{L}$ is now a scalar density under general coordinate transformations and invariant under spacetime-dependent tetrad rotations, provided we interpret the $\mathscr{H}^{ij}$ as tensor densities, $B_i^{\alpha}$ as components of a tetrad, and $A_i^{\alpha\beta}$ as a set of spin coefficients. It is then natural to take the metric to be the one for which the tetrad $B$ is orthonormal,

$$g_{ij}=B_i^{\alpha}B_j^{\beta}\eta_{\alpha\beta} \tag{70}$$

and to take the holonomic connection to be the one associated with the anholonomic connection $A$, namely

$$\Gamma_{ij}{}^{k}=(\partial_i B_j^{\alpha}+B_j^{\beta}A_{i\beta}{}^{\alpha})B_{\alpha}{}^{k}. \tag{71}$$

We are then in a $U_4$. The tensors $F_{ij}{}^{\alpha}$ and $F_{ij}{}^{\alpha\beta}$, (64) and (65), are its torsion and curvature. We shall use $B_j^{\alpha}$ and its inverse $B_{\alpha}{}^{j}$ to convert between Latin and Greek indices in the usual way, and shall denote the covariant derivative for anholonomic field components by $D_i$.

The total energy-momentum tensor density $-\partial\mathscr{L}/\partial B_i^{\alpha}$ appearing as the source term in (68) is given by

$$-\frac{\partial\mathscr{L}}{\partial B_i^{\alpha}}=A_{i\alpha}{}^{\beta}\mathscr{H}^{ij}{}_{\beta}+\frac{\partial\mathscr{G}}{\partial B_i^{\alpha}}+\frac{\partial\mathscr{L}_M}{\partial B_i^{\alpha}}. \tag{72}$$

The first term represents the energy-momentum contributed by the coupling to spin. The second term is given by

$$-\frac{\partial\mathscr{G}}{\partial B_i^{\alpha}}=\mathscr{H}^{ij}{}_{\beta}F_{\alpha j}{}^{\beta}+\tfrac{1}{2}\mathscr{H}^{ij}{}_{\beta\gamma}F_{\alpha j}{}^{\beta\gamma}-B_{\alpha}{}^{i}\mathscr{G} \tag{73}$$

(a straightforward generalisation of (57)), and

$$-\frac{\partial\mathscr{L}_M}{\partial B_i^{\alpha}}=\frac{\partial\mathscr{L}_M}{\partial\partial_i\psi}D_{\alpha}\psi-B_{\alpha}{}^{i}\mathscr{L}_M=\theta_M{}^{i}{}_{\alpha} \tag{74}$$

is the canonical energy-momentum density of matter. The total spin density occurring as the source in (69) is made up of the spin of the $B$ and $A$ fields and the spin of the matter:

$$-\frac{\partial \mathscr{L}}{\partial A_i^{\alpha\beta}} = \mathscr{H}^{ij}{}_\beta B_{j\alpha} - \mathscr{H}^{ij}{}_\alpha B_{j\beta} + \mathscr{H}^{ij}{}_{\beta\gamma} A_{j\alpha}{}^\gamma - \mathscr{H}^{ij}{}_{\alpha\gamma} A_{j\beta}{}^\gamma + \tau_M{}^i{}_{\alpha\beta}, \quad (75)$$

$$\tau_M{}^i{}_{\alpha\beta} = \frac{\partial L_M}{\partial \partial_i \psi} f_{\alpha\beta}\psi. \quad (76)$$

We are now in a position to write the field equations (68) and (69) in a manifestly covariant form:

$$D_i \mathscr{H}^{ij}{}_\alpha = \partial_i \mathscr{H}^{ij}{}_\alpha - A_{i\alpha}{}^\beta \mathscr{H}^{ij}{}_\beta = -\mathscr{E}^j{}_\alpha - \theta_M{}^j{}_\alpha, \quad (77)$$

$$D_i \mathscr{H}^{ij}{}_{\alpha\beta} = \partial_i \mathscr{H}_{\beta\alpha}{}^{ij} - A_{i\alpha}{}^\gamma \mathscr{H}^{ij}{}_{\gamma\beta} - A_{i\beta}{}^\gamma \mathscr{H}^{ij}{}_{\gamma\alpha} = -\mathscr{E}^j{}_{\alpha\beta} - \tau_M{}^j{}_{\alpha\beta}, \quad (78)$$

where

$$\mathscr{E}^i{}_\alpha = \mathscr{H}^{ij}{}_\beta F_{\alpha j}{}^\beta + \tfrac{1}{2}\mathscr{H}^{ij}{}_{\beta\gamma} F_{\alpha j}{}^{\beta\gamma} - B_\alpha{}^i \mathscr{G}, \quad (79)$$

$$\mathscr{E}^i{}_{\alpha\beta} = \mathscr{H}^i{}_{\beta\alpha} - \mathscr{H}^i{}_{\alpha\beta}, \quad (80)$$

The equations (77) and (78) are the Euler-Lagrange equations of a Poincaré gauge theory, in the Maxwellian form first given by Hehl (1978) Poincaré gauge theory is usually arrived at from a Lagrangian density of the form

$$\mathscr{G}(F_{ij}{}^\alpha, F_{ij}{}^{\alpha\beta}, B_i^\alpha) + \mathscr{L}_M(\psi, D_i\psi, B_i^\alpha) \quad (81)$$

with (64) and (65) regarded as definitions.

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