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On the solutions of the Wick-Cutkosky model in the instantaneous approximation

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Abstract. By reexamining the analysis of Basu and Biswas, based on the stereographic projection method of Fock and Levy, it is shown that the general solution of the Wick-Cutkosky model in the instantaneous approximation, hitherto unreported, involves only one quantum number; this is contrasted with the well-known solution which involves two quantum numbers, but for which the spectrum is degenerate with respect to one of them. The latter situation is shown to hold under a rather special circumstance.

Keywords. Bethe-Salpeter equation; Wick-Cutkosky model; stereographic projection; Funk-Hecke theorem; second order difference equation; infinite continued fraction.

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1. Introduction

To explore the ramifications of finite temperature field theory, it is important to extend its scope beyond the calculation of effective potentials at nonzero temperature for simple field-theoretic models. Efforts in this direction have been reviewed by Donoghue *et al* (1985). In this spirit, it seems to be of general interest to ask as to how the energy spectrum of bound states of a field theory changes with temperature. Such an investigation was recently carried out for the Wick-Cutkosky model within the framework of instantaneous approximation (Malik and Pande 1987). In the high temperature limit, we were able to find an exact solution (eigenfunction) characterized by a single quantum number l. This is to be contrasted with the case at zero temperature where the *known* solution depends on two quantum numbers n and l, though the energy-spectrum in the limit of small binding energy is $O(4)$-degenerate, i.e., it involves only the quantum number n (Scarf 1954; Basu and Biswas 1969).

The above finding is in accord with intuition: when the temperature is large enough, only the gross features of the system survive (the eigenfunction depends on one rather than two quantum numbers). Recall that the double-well of the spontaneously broken ϕ^4 field theory goes over to a single well at a temperature high enough (Linde 1979). This paper reexamines the usual Wick-Cutkosky model ($T=0$) and investigates whether or not it has a solution depending on one quantum number. Our results are: (i) the well-known two-quantum-number solution with the usual $O(4)$-degenerate energy spectrum exists, but under a very special circumstance, and (ii) the general solution is a one-quantum-number solution, which has so far been overlooked in the literature.

In § 2 we briefly reproduce the known treatment of the model as given by Basu and

Biswas (1969); its exclusion from this note would make no sense of our analysis in § 3, where it is shown how a careful treatment of the relevant difference equation leads to the solutions discussed above.

2. The model and its earlier analysis

The equation describing the bound states of two equal mass scalar particles of mass m, interacting via the exchange of a zero mass scalar particle, is given in the Wick-Cutkosky model by

$$\left[\left(\frac{P}{2}+p\right)^2+m^2\right]\left[\left(\frac{P}{2}-p\right)^2+m^2\right]\psi(p)=-\frac{i\lambda}{\pi^2}\int\frac{d^4p'\psi(p')}{(p-p')^2-i\varepsilon} \tag{1}$$

where P is the sum of the four-momenta of the external particles forming the bound state, p their relative four-momentum, λ the square of the coupling constant and $\psi(p)$ the Bethe-Salpeter amplitude. We reproduce below Basu and Biswas's (1969) analysis of the above equation based on the method of Fock (1935) and Levy (1950).

Using the specialized frame

$$P=(0,0,0,iE), \tag{2}$$

and working in the instantaneous approximation, one can put (1) in the form

$$S(\mathbf{p})=-\frac{i\lambda}{\pi^2}\int\frac{d^4p'}{(\mathbf{p}-\mathbf{p}')^2}\frac{S(\mathbf{p}')}{[\mathbf{p}'^2-(p_0'+(E/2))^2+m^2][\mathbf{p}'^2-(p_0'+(E/2)^2+m^2]}, \tag{3}$$

where

$$S(\mathbf{p})=[\mathbf{p}^2-(p_0+(E/2))^2+m^2][\mathbf{p}^2-(p_0-(E/2))^2+m^2]\psi(p). \tag{4}$$

After performing the p_0'-integration, and introducing the function $\phi(p)$ given by

$$\phi(\mathbf{p})=\frac{S(\mathbf{p})}{(\mathbf{p}^2+m^2)^{1/2}[\mathbf{p}^2+m^2-(E^2/4)]}, \tag{5}$$

one gets from (3):

$$(\mathbf{p}^2+m^2)^{1/2}(\mathbf{p}^2+m^2-(E^2/4))\phi(\mathbf{p})=\frac{\lambda}{2\pi}\int\frac{d^3p'\,\phi(\mathbf{p}')}{(\mathbf{p}-\mathbf{p}')^2}. \tag{6}$$

Since this equation possesses $O(3)$ symmetry, one can write,

$$\phi(\mathbf{p})=g(p)\,Y_l^m(\theta,\phi),\quad p\equiv|\mathbf{p}|. \tag{7}$$

The angular integrations can now be easily performed; this leaves one with a one-dimensional integral equation for the function $g(p)$, which, following Fock and Levy, can be projected onto the surface of a unit sphere in four dimensions. With the definitions

$$p = mc\rho, \quad c = (1-\varepsilon^2)^{1/2}, \quad \varepsilon = E/2m \tag{8}$$

and

$$\alpha = \left(1 + \frac{c^2}{2}\right), \quad \beta = \left(1 - \frac{c^2}{2}\right), \quad \frac{\nu}{2\pi^2} = \frac{\lambda}{4\pi c m^2}, \tag{9}$$

one can introduce the Fock variables

$$\rho = \tan\frac{\psi}{2}, \quad \rho' = \tan\frac{\psi'}{2} \tag{10}$$

and the function $f(\psi)$ defined by

$$f(\psi) = \sec^4\frac{\psi}{2}\, g\left(\tan\frac{\psi}{2}\right). \tag{11}$$

The desired integral equation is then given by

$$(\alpha + \beta\cos\psi) f(\psi) = \frac{\nu}{2\pi^2}(1+\cos\psi)\int\frac{d\Omega' f(\psi')\, P_l(\cos\theta')}{2(1-\cos\Theta)}, \tag{12}$$

where $d\Omega'$ is the element of the solid angle in four dimensions and Θ is the angle between two unit vectors of polar angles $(\psi, 0, 0)$ and (ψ', θ', ϕ'). For a detailed derivation of this equation refer Basu and Biswas (1969) and Basu (1969, unpublished).

A solution of (12) is now attempted in the form

$$f(\psi) = \sum_{k=0}^{\infty} a_k P^{(2)}_{n+k,\,l}(\cos\psi), \quad n \geqslant l, \tag{13}$$

where the $P^{(2)}_{n,\,l}(\cos\psi)$ are defined in terms of the Gegenbauer polynomials $C^l_n(\cos\psi)$ by the relation

$$P^{(2)}_{n,\,l}(\cos\psi) = \frac{1}{n+1}\sin^l\psi\; C^{l+1}_{n-l}(\cos\psi). \tag{14}$$

On using the Funk-Hecke theorem (Erdelyi 1953) for integral equations in the form

$$\int\frac{P^{(2)}_{n,\,l}(\cos\psi')\, P_l(\cos\theta')}{2(1-\cos\Theta)} = \lambda_n P^{(2)}_{n,\,l}(\cos\psi), \tag{15}$$

where

$$\lambda_n = \frac{2\pi^2}{n+1}, \tag{16}$$

and the recurrence relation

$$\cos\psi\, P^{(2)}_{n,\,l}(\cos\psi) = \frac{(n+2)(n-l+1)}{2(n+1)^2} P^{(2)}_{n+1,\,l}(\cos\psi) + \frac{n(n+l+1)}{2(n+1)^2} P^{(2)}_{n-1,\,l}(\cos\psi), \tag{17}$$

one can easily reduce (12) to:

$$\sum_{k=0}^{\infty}\left(\alpha-\frac{\nu}{n+k+1}\right)a_k P^{(2)}_{n+k,l}(\cos\psi)$$

$$+\sum_{k=1}^{\infty}\left(\beta-\frac{\nu}{n+k}\right)\frac{(n+k+1)(n+k-l)}{2(n+k)^2}a_{k-1}P^{(2)}_{n+k,l}(\cos\psi)$$

$$+\sum_{k=-1}^{\infty}\left(\beta-\frac{\nu}{n+k+2}\right)\frac{(n+k+1)(n+k+l+2)}{2(n+k+2)^2}a_{k+1}P^{(2)}_{n+k,l}(\cos\psi)=0. \quad (18)$$

Since $P^{(2)}_{n,l}(\cos\psi)$ is nonvanishing for all $n\geqslant l$, (18) is equivalent to the following set of equations

$$\left(\beta-\frac{\nu}{n+1}\right)\frac{n(n+l+1)}{2(n+1)^2}a_0 P^{(2)}_{n-1,l}(\cos\psi)=0 \quad (19)$$

$$\left(\beta-\frac{\nu}{n+2}\right)\frac{(n+1)(n+l+2)}{2(n+2)^2}a_1+\left(\alpha-\frac{\nu}{n+1}\right)a_0=0 \quad (20)$$

and

$$M_k a_{k+1}+L_k a_k+K_k a_{k-1}=0;\quad k\geqslant 1, \quad (21)$$

where

$$M_k=\left(\beta-\frac{\nu}{n+k+2}\right)\frac{(n+k+1)(n+k+l+2)}{2(n+k+2)^2},$$

$$L_k=\left(\alpha-\frac{\nu}{n+k+1}\right) \quad (22)$$

and

$$K_k=\left(\beta-\frac{\nu}{n+k}\right)\frac{(n+k+1)(n+k-l)}{2(n+k)^2}.$$

Further, (20) and (21) can be put together into the form

$$M_k a_{k+1}+L_k a_k+K_k a_{k-1}=0;\quad k\geqslant 0, \quad (23)$$

with the boundary condition

$$a_{-1}=0. \quad (24)$$

3. Investigation of the condition for the existence of the nontrivial solutions of the difference equation, etc.

The criteria for the existence of nontrivial solutions of (19) and (23) determine the energy eigenvalue problem. In the earlier analysis, in essence, the former of these equations seems to have been totally overlooked. In this manner, using a theorem of

Poincaré and an identity due to Thiele, Basu and Biswas (1969) obtained the condition for the existence of a nontrivial solution of the difference equations (23) in the form of an infinite continued fraction:

$$\frac{L_0}{M_0}=\cfrac{(K_1/M_1)}{(L_1/M_1)-\cfrac{(K_2/M_2)}{(L_2/M_2)-\dots}}$$

$$\equiv I(\lambda, \alpha, \beta, n, l). \tag{25}$$

An elementary and intuitive derivation of this expression has been given elsewhere (Malik and Pande 1987) and will not be repeated here.

For (25) to have a meaning, one must first ensure through (19) that $a_0 \neq 0$. This necessitates distinguishing between two cases i.e. $n > l$ and $n = l$.

Case (i): $n > l$

(a) In this case $P^{(2)}_{n-1,l} \neq 0$, and if $[\beta - \nu/(n+1)] \neq 0$, $a_0 = 0$. Equation (20) now implies that $a_1 = 0$, if $[\beta - \nu/(n+2)] \neq 0$. The first of (21) then leads to $a_2 = 0$, if $[\beta - \nu/(n+3)] \neq 0$. Similarly, $a_k = 0$ for all $k \geqslant 0$ if $[\beta - \nu/(n+k+1)] \neq 0$. Equation (25) is thus rendered meaningless. On the other hand, for example, let $[\beta - \nu/(n+2)] = 0$; now $a_1 \neq 0$. This implies that the assumed solution is of the form [see (13)]

$$f(\psi) = a_1 P^{(2)}_{n+1,l} + a_2 P^{(2)}_{n+2,l} + \dots$$

Now, however, the substitution of this solution into (12) will yield—via the Funk-Hecke theorem and the use of the recurrence relation (17)—a term on the right hand side of the form $P^{(2)}_{n,l}$ which is absent on the left hand side. Thus, in either of the cases considered, there is no solution.

(b) Even when $P^{(2)}_{n-1,l} \neq 0$, one can ensure that $a_0 \neq 0$ through the relation

$$\left(\beta - \frac{\nu}{n+1}\right) = 0. \tag{26}$$

This leads to the result $K_1 = 0$, so that (25) reduces to

$$\frac{L_0}{M_0} = 0, \tag{27}$$

implying

$$\left(\alpha - \frac{\nu}{n+1}\right) = 0. \tag{28}$$

Equation (23) then leads to

$$a_1 = a_2 = a_3 = \dots = 0. \tag{29}$$

The series solution given in (13) then shrinks to

$$f(\psi) = a_0 P^{(2)}_{n,l}(\cos\psi). \tag{30}$$

Further, (9), (26) and (28) now imply

$$\alpha=\beta=\frac{\nu}{n+1}=1, \tag{31}$$

so that (12) itself gets simplified to

$$f(\psi)=\frac{\nu}{2\pi^2}\int\frac{f(\psi')\,P_l(\cos\theta')\,\mathrm{d}\Omega'}{2(1-\cos\Theta)}, \tag{32}$$

and it is simple to check that (30) indeed is its solution. The corresponding spectrum is given by

$$\nu=\frac{\lambda\pi}{2cm^2}=n+1, \tag{33}$$

which is the standard $O(4)$-degenerate result found by Scarf (1954), and Basu and Biswas (1969). Thus, this two-quantum-number solution (albeit degenerate in the energy spectrum with respect to one quantum number) exists under the rather stringent condition given by (31), which corresponds to the limit $c^2 \ll 1$. In this limit, of course, the results (30) and (33) could have been obtained directly from (12), without going through the derivation of the infinite continued fraction in (25).

Case (ii): $n=l$.

In this case, in (19),

$$P^{(2)}_{n-1,l}(\cos\psi)=P^{(2)}_{l-1,l}(\cos\psi)=0, \tag{34}$$

and we do not have to impose (26) for a_0 to be nonzero. With the substitution $n=l$, (23) then enables one to calculate all the other a_k's $(k>0)$ in terms of a_0. Indeed, the same substitution is to be made in (25), so that we have

$$\frac{L_0(n=l)}{M_0(n=l)}=I(\lambda,\alpha,\beta,n=l,l). \tag{35}$$

This equation expresses the binding energy (per unit mass) c, which appears in it through α and β, as a function of the coupling constant λ and the quantum number l, and thus represents the discrete bound state spectrum of the model under consideration. Recalling the definition of $I(\lambda,\alpha,\beta,n,l)$ given in (25), one may remark that the expression for c in terms of λ and l may be calculated to any desired accuracy by using standard numerical methods. We wish to emphasize, however, that the general solution of the problem under investigation involves only one quantum number, i.e. not only are the energy levels characterized by one quantum number, but so also is the eigenfunction, since with $n=l$, (13) reads

$$f(\psi)=\sum_{k=0}^{\infty}a_k P^{(2)}_{l+k,l}(\cos\psi), \tag{36}$$

which is equivalent to

$$f(\psi) = \sum_{k=l}^{\infty} \tilde{a}_k P_{k,l}^{(2)}(\cos\psi). \tag{37}$$

This situation is clearly distinct from the rather special case when $n > l$ discussed above, and has somehow not been reported in the literature so far.

We wish to conclude by pointing out that it is the general solution of the type given by (35)–(37) which survives when one considers the present model in the high temperature limit (Malik and Pande 1987).

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