

## Effective medium in dispersed systems

R N PANDE,\* D R CHAUDHARY\*\* and F GORI

Departimento di Energetica, Università degli studi di Firenze Via Santa Marta 3, Firenze, Italy

\*Government College, Dholpur 328 001, India

\*\*Department of Physics, University of Rajasthan, Jaipur 302 004, India

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**Abstract.** A structural analysis of effective medium formed by dispersed systems from the viewpoint of flux modification at large dispersions is presented. The effective medium coefficient is investigated for its parametric dependence and the effective properties are estimated through this dependence. This estimation covers all highly dispersed two-phase systems including the effect of container.

**Keywords.** Effective medium; flux modification factor; dispersed systems.

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### 1. Introduction

The failure of the small dispersion technique for effective transport through highly dispersed systems led to the consideration of random phase distribution (Reynolds and Hough 1957; Hashin and Shtrikman 1962; Kumar and Chaudhary 1980; Jefferey 1973) and effective medium theories (Bruggeman 1935; Miller 1969; Hori and Yonezawa 1977). Bruggeman (1935) suggested that a large dispersion may be considered as the result of a number of successive small dispersions made in a parent medium and hence should obey the small dispersion laws at each stage of dispersion. In fact using this technique one may start from the first phase and may arrive at the second. However Bruggeman did not exploit this technique of successive dispersion. He suggested that by making simultaneous dispersions of the continuous and dispersed phases in the proportion of their volume fractions ( $\psi_c$  and  $\psi_d$ ) respectively in a two-phase system the parent medium practically does not alter. He defined this as an effective medium. Miller (1969) and Hori and Yonezawa (1977) introduced another effective medium which does not at all change by taking out or putting in a particle of the dispersed phase in the two-phase system. The effective transport coefficients also do not change.

A similar effective medium was introduced earlier (Pande and Chaudhary 1984b) using the successive dispersion technique. But as this medium is based on the concept of continuous character of the medium under consideration it comes out quite different from the previous ones. Therefore it was called an effective continuous medium (ECM). In this model, the zero level of porosity is replaced by another reference porosity level

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\* To whom correspondence should be addressed.

where the system is completely ordered and is composed of equal volume fractions of both the phases. The dispersions are made at this particular reference level ( $\psi = 0.5$ ) and it now becomes easier to generate any porosity in the vicinity of this reference level. Clearly the advantage of this kind of assumption is to allow the validity of small dispersion laws at this reference level (ECM). As the ECM is composed of both the phases in equal proportion of volume, making small dispersion of either phase in ECM yields highly dispersed systems.

The ECM as proposed now has the same structure for all kinds of phase compositions. However the value of thermal conductivity of ECM is different for different phase compositions. The aim of the present investigation is to study how the value of thermal conductivity of ECM changes with a change in phase composition of ECM. Once this dependence is known, the effective properties of highly dispersed systems can be easily estimated. Moreover the effective properties vary differently with the physical state of the dispersing medium i.e. whether the dispersing material is solid, liquid or gas. In this paper we aim to find a general representation for the ECM which may be independent of the nature of the material. Particular studies already exist (Powers 1961; Parrott and Stuckes 1975; Burrige *et al* 1982) but these seem unsuitable for a general structural analysis of ECM since they are valid only at small dispersions.

Recently it has been noticed that the effective properties are altered by the shape of the sample container (Chang 1986). In the case of packed beds when beds are kept within parallel plates, concentric cylinders and concentric spheres the values of stagnant effective properties are different in different cases and depend upon the characteristic length  $d_p/H$  where  $d_p$  is the particle size and  $H$  is the distance between packing plates. This type of behaviour of macroscopic medium where a phase influences from a distance without being actually present in the system, is difficult to account for on the basis of phase composition. ECM investigations may prove easier for explaining these kinds of influences.

## 2. Theoretical analysis

As proposed earlier (Pande and Chaudhary 1984b) ECM is the result of making  $n$  successive dispersions, each very small in amount, of dispersed or continuous phase in either phase of value  $\delta\psi$  such that  $n\delta\psi = 0.5$ . Using the earlier theory of small dispersion Pande and Chaudhary 1984a) and the self-consistency of ECM together, the effective transport coefficient ( $\lambda_{\text{ECM}}$ ) is defined as

$$\begin{aligned} \lambda_{\text{ECM}}^2 &= \lambda_c \lambda_d \left[ \left\{ 1 + 3.844(\lambda_d - \lambda_c) \delta\psi^{2/3} / (\lambda_d + 2\lambda_c) \right\}^n \right. \\ &\quad \left. \times \left\{ 1 + 3.844(\lambda_c - \lambda_d) \delta\psi^{2/3} / (\lambda_c + 2\lambda_d) \right\}^n \right] \\ &= \lambda_c \lambda_d \left[ 1 + 3.844 \left\{ (\lambda_d - \lambda_c)^2 \delta\psi^{2/3} / (\lambda_d + 2\lambda_c)(\lambda_c + 2\lambda_d) \right\} \right. \\ &\quad \left. - 14.776 \left\{ (\lambda_d - \lambda_c)^2 (\delta\psi^{2/3})^2 / (\lambda_d + 2\lambda_c)(\lambda_c + 2\lambda_d) \right\} \right]^n. \end{aligned} \quad (1)$$

Let the fraction  $\left\{ (\lambda_d - \lambda_c)^2 / (\lambda_d + 2\lambda_c)(\lambda_c + 2\lambda_d) \right\} = F$ ; equation (1) changes into

$$\lambda_{\text{ECM}}^2 = \lambda_c \lambda_d \left[ 1 + 3.844 F \delta\psi^{2/3} - 14.776 F (\delta\psi^{2/3})^2 \right]^n. \quad (2)$$

Case I.  $\lambda_d > \lambda_c$

Let  $\lambda_d/\lambda_c = a$ , then  $F$  changes as

$$\begin{aligned} F &= \{(\lambda_d/\lambda_c)^2 + 1 - 2(\lambda_d/\lambda_c)\} / \{2(\lambda_d/\lambda_c)^2 + 2 + 5(\lambda_d/\lambda_c)\} \\ &= (a-1)^2 / (2a^2 + 5a + 2). \end{aligned} \quad (3)$$

When  $a$  is much larger than unity  $F$  transforms as

$$F = (a-2)/(2a+5)$$

and (2) becomes

$$\begin{aligned} \lambda_{\text{ECM}}^2 &= \lambda_c \lambda_d [1 + \{3 \cdot 844(a-2)\delta\psi^{2/3}/(2a+5)\} \\ &\quad - \{14 \cdot 776(a-2)/(2a+5)\}(\delta\psi^{2/3})^2]^n. \end{aligned} \quad (4)$$

Expansion of (4) under identical conditions (Pande and Chaudhary 1984b) yields  $\lambda_{\text{ECM}}$ . However here we are interested in the bracketed term of (4). Let it be defined by  $b^2$  where  $b$  stands for the effective medium coefficient in a two-phase system. Thus

$$\lambda_{\text{ECM}} = b(\lambda_c \lambda_d)^{1/2} \quad (5)$$

$$\begin{aligned} \text{and } b^2 &= [1 + 3 \cdot 844n\delta\psi^{2/3}(a-2)/(2a+5) + \{2(1 \cdot 922n\delta\psi^{2/3})^2\} \\ &\quad \times \{(a-2)^2/(2a+5)^2\} - \{(3 \cdot 844n\delta\psi^{2/3})^2/n\}(a-2)/(2a+5) \\ &\quad - 2\{(1 \cdot 922n\delta\psi^{2/3})^2/n\}\{(a-2)/(2a+5)\}^2]. \end{aligned} \quad (6)$$

Here the value of  $b^2$  is written after the expansion of (4). Substituting the volume-averaged values for  $n\delta\psi^{2/3}$  from a previous derivation (Pande and Chaudhary 1984b) one finds

$$b^2 = [1 - 2 \cdot 82378\{(a-2)/(2a+5)\} + 3 \cdot 3930\{(a-2)/(2a+5)\}^2], \quad (7)$$

where  $a$  is much greater than unity.

Case II.  $\lambda_c > \lambda_d$

Let  $\lambda_c/\lambda_d = a$ , then by the same procedure

$$F = (a-2)/(2a+5).$$

Proceeding exactly in the same way one finds that

$$\lambda_{\text{ECM}} = b'(\lambda_c \lambda_d)^{1/2},$$

$$\text{and } b'^2 = [1 - 2 \cdot 82378 \{(a'-2)/(2a'+5)\} + 3 \cdot 393\{(a'-2)/(2a'+5)\}^2]. \quad (8)$$

Using (7) and (8) the thermal conductivity of ECM may be represented as

$$\begin{aligned} \{\lambda_{\text{ECM}}/\lambda_c\} &= b(\lambda_d/\lambda_c)^{1/2}, \\ &= b'(\lambda_d/\lambda_c)^{1/2}. \end{aligned} \quad (9)$$

Certainly  $\lambda_{\text{ECM}}$  depends upon  $b$  or  $b'$  for a given value of  $a$  or  $a'$ . The coefficient  $b$  changes

with the nature of the material. When  $\lambda_c = \lambda_d$ , one finds through (1) that

$$b = 1, \quad \text{and} \quad \lambda_{\text{ECM}} = (\lambda_c \lambda_d)^{1/2} = \lambda_c. \quad (10)$$

However when the dispersed phase ( $\lambda_d$ ) is air and  $\lambda_c$  is the solid, the ratio  $(\lambda_d/\lambda_c) = 128.6$ . Through (8)  $b$  is found as

$$b = 0.6574 \quad \text{and} \quad \lambda_{\text{ECM}} = 0.6574 (\lambda_s \lambda_d)^{1/2}. \quad (11)$$

Similarly when the continuous phase is soil and the dispersed phase is water, (8) yields

$$b = 0.7362 \quad \text{and} \quad \lambda_{\text{ECM}} = 0.7362 (\lambda_s \lambda_f)^{1/2}. \quad (12)$$

Thus using (8) and (12) one can generalize that for all practical purposes

$$0.6574 < b < 1. \quad (13)$$

### 3. Wall effects

The distant effect of the phases may be noticed through (7) and (8). Because of the wall there is a sharp change in the value of  $(\lambda_d/\lambda_c)$  near the wall. Differentiating (7) with respect to  $a$  one notices that

$$\frac{db}{da} = (1/2b) \left[ \frac{-(2.82378)}{(2a+5)} + \left\{ \frac{(6.21678)}{(2a+5)} \frac{2(a-2)}{(2a+5)} \right\} - \frac{3.393}{(2a+5)} \left\{ \frac{2(a-2)}{(2a+5)} \right\}^2 \right]. \quad (14)$$

When  $a > 2$ ,  $\{2(a-2)/(2a+5)\} < 1$ . For all values of  $a \rightarrow 0$  and  $a > 0$ ,  $(db/da)$  is negative. The fraction  $(db/da) \rightarrow 0$  when  $a \rightarrow \infty$ . Thus as the value of  $a$  increases from zero to infinity,  $db/da$  changes from negative to positive. This indicates that a local change in value of  $a$  is bound to invite a corresponding change in  $b$  and  $\lambda_E$ .

### 4. Discussion and results

One notices through (9) that  $b$  has a key role in changing the effective properties of ECM and a two-phase system. Hence to characterize the structure of ECM the function  $b$  is plotted with respect to the flux modification ratio  $a$  in figure 1. It shows a sharp decrease in the value of  $b$  for  $1 < a < 10$ . Also,  $b$  continues to decrease very slowly in the region  $10 < a < 20$ . In the region  $20 < a < 100$  there is slight increase in the value of  $b$  and for  $a > 100$ ,  $b$  is nearly constant. Using these values of  $b$  in (9)  $\lambda_E$  is calculated. The calculated values of  $\lambda_E/\lambda_f$  are plotted in figure 2 with respect to  $a$  for two values of  $\psi_f$ . For  $\psi_f = 0.5$ , the medium is ECM and hence (9) has been used to calculate  $\lambda_E/\lambda_f$ . While for  $\psi_f = 0.4$ , the medium is less porous and the equation for small dispersion has been taken into account along with the ECM relation (9). Here a small dispersion of solid phase equivalent to  $\xi_s = 0.5 - \psi_f$  in ECM has been considered (Pande and Chaudhary 1984b) which yields

$$\lambda_E = \lambda_{\text{ECM}} \left[ 1 + 3.844 \left\{ \lambda_s - \lambda_{\text{ECM}} \right\} \xi_s^{2/3} / (\lambda_s + 2\lambda_{\text{ECM}}) \right] \quad (15)$$

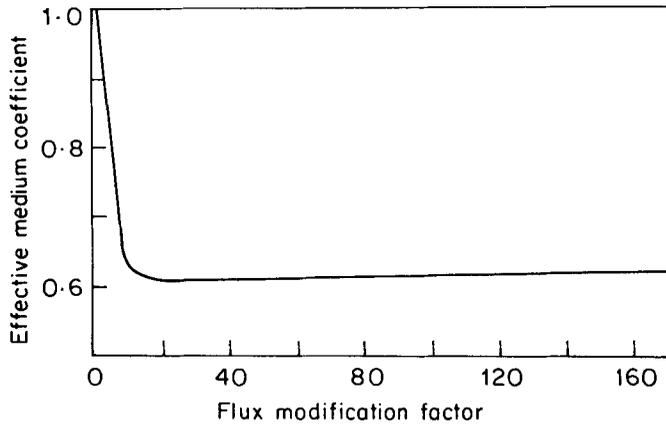


Figure 1. Effective medium coefficient  $b$  versus flux modification factor  $a$  indicating the variation in  $b$  over full span of  $a$ .

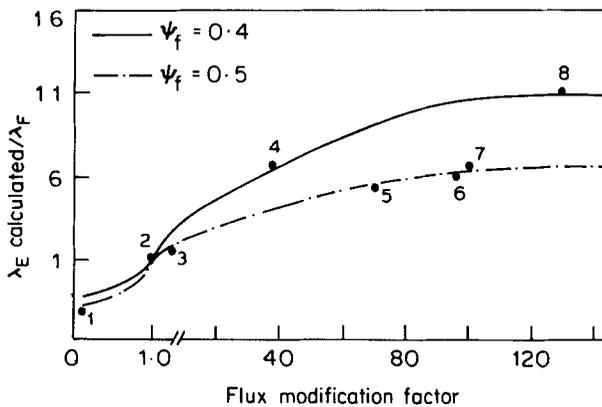


Figure 2. A graph showing ETC variation over fluid conductivity at various values of flux modification factor  $a$ . The numbers along with the solid points on the curve show the experimental values of ETC as follows: 1. Uranium oxide and sodium. 2. Glass beads and water. 3. Water-saturated soil. 4. Glass beads in air. 5. Zirconia powder dry. 6. Steel beads in water. 7. Brick sand dry. 8. Dry soil.

where  $\xi_s = (0.5 - \psi_f)$ .

Here the value of  $\psi_f$  has been chosen (0.4 and 0.5) empirically but it covers all natural systems. In fact one is free to choose any value for  $\psi_f$  but in that case it would be difficult to compare the results from naturally dispersed systems.

Figure 2 shows that  $\lambda_E$  changes very sharply in the region  $0.6 < (\lambda_s/\lambda_f) < 10$  with a point of inflection at  $(\lambda_s/\lambda_f) = 1$ . Between  $(\lambda_s/\lambda_f) = 10$  and 50, the increase in  $\lambda_E/\lambda_f$  is proportional while thereafter it becomes asymptotic. In figure 2 one may find the experimental values of ETC of glass beads in air and water, steel beads in air, water and oil, zirconia powder, dry soil, water-saturated soil, uranium oxide in sodium and molybdenum, copper solder system etc. The experimental values of ETC of some of these systems are shown as numbers and as solid points on the curves. Using figures 1

Table 1. ETC ( $\text{Wm}^{-1}\text{K}^{-1}$ ) of air and water-saturated systems

Systems	$a$	$b$	$\psi_g$	$\lambda_{\text{cal}}^a$	$\lambda_{\text{cal}}^b$	$\lambda_{\text{cal}}^c$	$\lambda_{\text{Expt}}$
Glass beads and air (Verschoor and Schuit 1950)	37.586	0.644	0.40	0.137	0.255	0.185	0.188
Loose glass and air (Chaurasia 1976)	42.928	0.645	0.40	0.127	0.250	0.181	0.178
Zirconia powder and air (Godbee 1966)	67.273	0.649	0.42	0.141	0.341	0.248	0.229
Zirconia powder and air (Ratcliffe 1969)	95.143	0.652	0.53	0.073	0.178	0.114	0.119
Stone concrete and air (Pande <i>et al</i> 1984)	96.154	0.652	0.56	0.101	0.194	0.122	0.111
Cement loose and air (Pande <i>et al</i> 1984)	100.0	0.652	0.56	0.087	0.197	0.124	0.118
Brick sand and air (Pande <i>et al</i> 1984)	109.616	0.653	0.49	0.658	0.285	0.204	0.195
Dune sand and air (Pande <i>et al</i> 1984)	128.615	0.657	0.485	1.41	0.317	0.232	0.220
Dune sand and air (Chaudhary <i>et al</i> 1969)	128.615	0.657	0.486	1.50	0.358	0.267	0.274
			0.4502	1.52	0.376	0.279	0.289
			0.4394	1.56	0.396	0.291	0.302
			0.4297	1.59	0.415	0.301	0.312
			0.4052	1.67	0.467	0.326	0.336
Penner Leda clay (Penner <i>et al</i> 1975) (water saturated)	6.00	0.804	0.498	1.203	1.203	0.984	0.930
			0.505	1.192	1.192	0.949	0.920

<sup>a</sup> Maxwell (1904); <sup>b</sup> Lichtnecker (1924); <sup>c</sup> Present relation (15).

and 2 it is easy to calculate the effective thermal conductivity of all these systems whose porosity is 0.4 and 0.5. For ETC at other porosity values one should plot a curve similar to figure 2 at that value of  $\psi_f$ . Plotting of figure 1 is not essential. One may find the ETC using (15) for that value of  $\psi_f$  in conjunction with figure 2.

A comparison of the calculated values of  $\lambda_E$  with the reported experimental results is given in table 1 for some highly porous two-phase materials. The calculations have been performed using Maxwell and Lichtnecker relations and our equation (15). The maximum deviation between calculated and experimental results using (15) is 10%, while using the Maxwell relation the continuous medium in systems 2 to 9 and system 1 (table 1) has been taken as solid and fluid respectively.

The present analysis (equation (14)) as carried out above indicates that the shape of the container which influences the flux modification ratio  $a$  locally allows an increase and decrease in the value of effective thermal conductivity of a stationary medium as well. In fact by plotting (14) with respect to  $a$  one may numerically evaluate the variation of ETC with respect to  $a$ .

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### List of symbols

$a$	flux modification factor
$b$	effective medium coefficient
$F$	structural function
$n$	dispersion exponent
$\lambda$	thermal conductivity
$c$	continuous phase
$d$	dispersed phase
$s$	solid phase
$A$	air phase
$f$	fluid phase
ECM	effective continuous medium
$E$	effective value
$\psi$	volume fraction
$\delta\psi$	small dispersion amount
$\xi$	volume fraction which is higher than 0.5

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