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Adler-Bardeen theorem in path integral formulation

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Abstract. The higher order contributions to Jacobian in Fujikawa's path integral framework is considered and the form of anomaly equation in higher orders is established. An argument for the Adler-Bardeen theorem in this formulation is given.

Keywords. Adler-Bardeen theorem; path integral formulation.

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1. Introduction

In recent years Fujikawa has derived the various known anomalies in path integral formalism in a series of remarkable papers (Fujikawa 1979–82; 1985). The path integral measures in such a formulation are not invariant under appropriate field transformations leading to nontrivial Jacobian factors. In these formulations, these nontrivial Jacobian factors are the sources of anomalies. Results of Fujikawa have been extended in various ways (Balachandran *et al* 1982; Andrinov *et al* 1982; Verstagen 1984; Reuter 1985).

All these results are valid only upto one loop approximation. This has been explained in detail in §2, and also elsewhere (Joglekar and Misra 1986). The object of this work is to take a major step in extending these results to higher orders as far as chiral anomaly is concerned. We consider, in this work the regularized Jacobian factor

$$B_M(x) \equiv \sum_n \phi_n^\dagger(x)\gamma_5 \exp(-\not{D}^2/M^2)\phi_n(x) \tag{1}$$

where $\phi_n(x)$ are the eigenfunctions of the operator $\not{D}$. We show that $B_M(x)$ has the form

$$B_M(x) = -\frac{1}{16\pi^2}\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) + \frac{O(6)}{M^2} + \frac{O(8)}{M^4} + \ldots + \frac{O(2n)}{M^{2n-4}} + \ldots \tag{2}$$

where $O(6)$, $O(8)$, ... are gauge-invariant operators of dimension 6, 8, ... etc. We show in §2 that all these series must be taken into account if one is to go beyond one loop order. This is the object of this work.

In § 3 we shall establish the form of the operators $O(2n)$. This form will enable us in § 4 to obtain the form of the contributions coming from the higher order term in the above series to the anomaly equation. In § 4 we establish that the anomaly equation in the path integral formulations takes the form in higher orders

$$\partial_\mu J_5^\mu(x) = 2im_0\bar{\psi}(x)\gamma_5\psi(x) - \frac{1}{8\pi^2}\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) + f(g, \ln M^2)\partial_\mu J_5^\mu(x) \tag{3}$$

where the last term in anomaly equation comes from the series in (2). This is the major result of this paper.

To prove Adler-Bardeen theorem (Adler and Bardeen 1969) one would have to show that $f(g^2, \ln M^2)$ in (3) vanishes. This, we have been unable to do entirely within the path integral formulation. However in § 5 we shall give an argument not entirely within the path integral formulation to prove the Adler-Bardeen theorem.

2. Preliminary

In this section we shall fix our notations, give a brief account of Fujikawa's path integral derivation of anomaly and point out the reasons as to why these results are of one loop order.

We shall mainly use the notations of (Fujikawa 1979, 1980a). γ-matrices are those used in Bjorken and Drell (1965). γ^0 is hermitian and γ^k $(k=1, 2, 3)$ are antihermitian. $\gamma^4 \equiv i\gamma^0$ is hermitian. The hermitian γ_5 is defined by $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3\gamma^4$.

We shall consider the Lagrange density of SU(N) Yang-Mills field coupled to fermions in the fundamental representation of SU(N):

$$\mathscr{L} = \bar{\psi} i\gamma^\alpha D_\alpha \psi - m_0 \bar{\psi}\psi + \frac{1}{2g^2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}). \tag{4}$$

The path-integral in Euclidean space is defined by first continuing L to Euclidean spacetime.

After the Wick rotation $x^0 \to -ix^4$ and $A_0 \to iA_4$, A_4 is assumed to be antihermitian. Then as γ^α $(\alpha = 1, 2, 3, 4)$ are antihermitian

$$\not{D} = \sum_{\alpha=1}^{4} \gamma^\alpha D_\alpha = \sum_{\alpha=1}^{4} \gamma^\alpha(\partial_\alpha + A_\alpha) \tag{5}$$

is a hermitian operator in the Euclidean space time x^α $(\alpha = 1, 2, 3, 4)$. After the Wick rotation, the metric becomes $g_{\mu\nu} = g^{\mu\nu} = (-1, -1, -1, -1)$. In (4)

$$iA_\mu \equiv g_0 A_\mu^a T^a$$

$$[T^a, T^b] = if^{abc}T^c$$

$$\mathrm{Tr}(T^aT^b) = \tfrac{1}{2}\delta^{ab}$$

$$F^{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \equiv g_0 T^a F_{\mu\nu}^a. \tag{6}$$

Fujikawa (1979, 1980a) defines the path integral by first expanding the fermionic Grassmann variables ψ and $\bar{\psi}$ in terms of eigenfunctions of the operator $\not{D}$. We shall assume that the system is enclosed in a large space-time box. Then the eigenvalues of the operator $\not{D}$ are discrete and real. The eigenfunctions satisfy

$$\not{D}\phi_n(x) = \lambda_n \phi_n(x) \tag{7}$$

where $\phi_n(x)$ are 4-spinors and satisfy the orthogonality and completeness properties:

$$\int \mathrm{d}^4x\, \phi_n^\dagger(x)\phi_m(x) = \delta_{nm} \tag{8}$$

and

$$\sum_n \phi_n(x)\phi_n^\dagger(y) = \delta^4(x-y)\mathbf{1}. \tag{9}$$

One expands ψ and $\bar{\psi}$ as

$$\psi(x) = \sum_n a_n \phi_n(x)$$

$$\bar{\psi}(x) = \sum_n b_n \phi_n^\dagger(x) \tag{10}$$

where a_n and b_n are independent Grassmann variables. Then the path-integral measure is *defined* as

$$\mathscr{D}\psi\,\mathscr{D}\bar{\psi} \equiv \prod_n \mathrm{d}a_n \prod_n \mathrm{d}b_n. \tag{11}$$

It is this definition that correctly reproduces the chiral anomaly to one loop order as the change in the measure under the chiral transformations. The anomaly equation in the path-integral formulation is derived as the W-T identity obtained by applying the local chiral transformations to the functional integral

$$W[J_\mu, \xi, \bar{\xi}] = \int \mathscr{D}A_\mu \mathscr{D}\psi \mathscr{D}\bar{\psi} \mathscr{D}c \mathscr{D}\bar{c} \exp i\{S_{\mathrm{eff}}[A, c, \bar{c}, \psi, \bar{\psi}] + \int \mathrm{d}^4x [J^\mu A_\mu + \bar{\xi}\psi + \bar{\psi}\xi]\} \tag{12}$$

$$W[0, 0, 0] \equiv 1$$

where $\mathscr{D}\psi\,\mathscr{D}\bar{\psi}$ have been defined in (11) earlier, and $S_{\mathrm{eff}}[A, c, \bar{c}, \psi, \bar{\psi}]$ is the effective action including the Faddeev-Popov ghost terms in terms of the ghost fields c and $\bar{c}$.

The local chiral transformations are:

$$\psi(x) \rightarrow \psi'(x) \equiv \exp[i\alpha(x)\gamma_5]\psi(x)$$

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x) \equiv \bar{\psi}(x)\exp[i\alpha(x)\gamma_5] \tag{13}$$

under the infinitesimal chiral transformations the Lagrange density of (4) is transformed into

$$\mathscr{L}(x) \rightarrow \mathscr{L}(x) - \partial_\mu \alpha(x) J_5^\mu(x) - 2m_0 i\alpha(x)\bar{\psi}(x)\gamma_5\psi(x)$$

$$J_5^\mu \equiv \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) \tag{14}$$

while under the chiral transformations of (13) the coefficients a_n and b_n transform as,

$$a'_m = \sum_n C_{mn} a_n$$

and

$$\bar{b}'_m = \sum C_{mn} \bar{b}_n \tag{15}$$

where

$$C_{mn} = \int \mathrm{d}^4x\, \phi_m^\dagger(x)\exp[i\alpha(x)\gamma_5]\phi_n(x). \tag{16}$$

Taking into account the Grassmann nature of a_n and $\bar{b}_n$ we have

$$\prod_m \mathrm{d}a'_m = (\det C)^{-1} \prod_n \mathrm{d}a_n$$

$$\prod_m \mathrm{d}\bar{b}'_m = (\det C)^{-1} \prod_n \mathrm{d}b_n \tag{17}$$

The Jacobian factor $(\det C)^{-1}$ evaluated for infinitesimal $\alpha(x)$ is

$$(\det C)^{-1} = \exp - i \int \mathrm{d}^4 x \alpha(x) B(x)$$

where

$$B(x) = \sum_n \phi_n^\dagger(x) \gamma_5 \phi_n(x). \tag{18}$$

Thus

$$\mathscr{D}\psi' \mathscr{D}\bar{\psi}' = \exp\{-2i \int \mathrm{d}^4 x \alpha(x) B(x)\} \mathscr{D}\psi \mathscr{D}\bar{\psi}$$

$B(x)$ of (18) is an ill-defined quantity and one evaluates it by introducing a cut off M:

$$B_M(x) \equiv \sum_n \phi_n^\dagger(x) \gamma_5 \phi_n(x) \exp(-\lambda_n^2/M^2). \tag{19}$$

As $\phi_n(x)$ are functionals of A_μ, $B_M(x)$ is a functional of A_μ. Fujikawa (1979, 1980a) evaluates $B_M(x)$ by going over to the plane-wave basis or equivalently as follows:

$$\begin{aligned}
B_M(x) &= \sum_n \phi_n^\dagger(x) \gamma_5 \exp(-\lambda_n^2/M^2) \phi_n(x) \\
&= \sum_n \phi_n^\dagger(x) \gamma_5 \exp(-\not{D}^2/M^2) \phi_n(x) \\
&= \lim_{y \to x} \sum \phi_n^\dagger(y) \gamma_5 \exp(-\not{D}^2/M^2) \phi_n(x) \\
&= \lim_{y \to x} \mathrm{Tr} \sum_n \gamma_5 \exp(-\not{D}^2/M^2) \phi_n(x) \phi_n^\dagger(y) \\
&= \lim_{y \to x} \mathrm{Tr} \{\gamma_5 \exp(-\not{D}^2/M^2) \delta^4(x-y) \mathbf{1}\} \\
&= \lim_{y \to x} \mathrm{Tr}\, \gamma_5 \exp(-\not{D}^2/M^2) \frac{1}{(2\pi)^4} \int \mathrm{d}^4 x \exp[ik(x-y]
\end{aligned} \tag{20}$$

where we have used (9). After some simplification one can express $B_M(x)$ as,

$$B_M(x) = \mathrm{Tr}\, \gamma_5 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \exp\left(-\frac{k_\mu k_\mu}{M^2}\right) \exp\left[\frac{D^2 + 2ik \cdot D + (i/2)\sigma \cdot F}{M^2}\right].$$

Further changing the variable to $k'_\mu = k_\mu/M$ and dropping primes:

$$B_M(x) = M^4 \mathrm{Tr}\left\{\gamma_5 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \exp(-k_\mu k_\mu) \exp\left[\frac{D^2}{M^2} + \frac{2ik \cdot D}{M} + \frac{i\sigma_{\mu\nu} F^{\mu\nu}}{2M^2}\right]\right\}. \tag{21}$$

Fujikawa (1979, 1980) evaluates $B(x)$ as

$$B(x) \equiv \lim_{M\to\infty} B_M(x) \tag{22}$$

and the result is

$$B(x) = -\frac{1}{16\pi^2}\mathrm{Tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu}) \tag{23}$$

where

$$\tilde{F}^{\mu\nu} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \quad (\varepsilon^{1234}=1).$$

Fujikawa (1979, 1980a) has further shown that the limit in (22) is independent of the regularizing function $f(\lambda_n^2/M^2)$ in (19) provided $f(0)=1$ and

$$f(\infty)=f'(\infty)=f''(\infty)= \ldots =0. \tag{24}$$

When the result for $B(x)$ in (23) is combined with the W-T identity so obtained from the generating functional of (12) is the well-known anomaly equation:

$$\partial_\mu J_5^\mu = 2m_0 i\bar{\psi}\gamma_5\psi - \frac{i}{8\pi^2}\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}). \quad \text{(modulo equation of motion terms)} \tag{25}$$

Here we wish to make a few comments. The procedure used by Fujikawa (1979, 1980a) for regularizing $B(x)$ by (19) is an adhoc one. The anomaly equation should be obtained directly in the regularized form, in which the current and various Green's functions are directly regularized. This can be done by regularizing the Lagrange density L itself as shown by Verstegen (1984). We shall assume that we are working in the context of such a regularized Lagrange density even though we shall not, for the present, deal with it directly. When this is done the result for the anomaly term in the regularized anomaly equation is a term of the form $\langle B_M(x)\rangle_{J,\xi,\bar{\xi}}$ where

$$\langle B_M(x)\rangle_{J,\xi,\bar{\xi}} = \int \mathscr{D}A_\mu \mathscr{D}C\mathscr{D}\bar{C}\mathscr{D}\psi\mathscr{D}\bar{\psi}B_M[A]\exp i\{S_{\mathrm{eff}} + \text{source terms}\}$$

where the action and hence the Green's functions are regularized in terms of an effective cut off M. Thus not only is $B_M(x)$ regularized by a cut off on the large eigenvalues of $\not{D}$, but also are the Green's functions similarly regularized. These Green's functions will in higher orders, be divergent, containing divergences in powers of M^2 modulo powers of $\log M^2$. Now in evaluating the anomaly we are interested in

$$\lim_{M^2\to\infty} \langle B_M(x)\rangle$$

and not $\langle \lim_{M^2\to\infty} B_M(x)\rangle$. As shown below, these will generally differ in higher orders.

To see this we note that $B_M(x)$ of (21) viz

$$B_M(x) = M^4\mathrm{Tr}\left\{\gamma_5\int\frac{\mathrm{d}^4k}{(2\pi)^4}\exp(-k_\mu k_\mu)\exp\left[\frac{D^2}{M^2} + \frac{2ik\cdot D}{M} + \frac{i\sigma_{\mu\nu}F^{\mu\nu}}{2M^2}\right]\right\}$$

is actually a series in $1/M^2$ of the form

$$B_M(x) = -\frac{1}{16\pi^2}\mathrm{Tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu}) + \frac{O(6)}{M^2} + \frac{O(8)}{M^4} + \ldots + \frac{O(2n)}{M^{2n-4}} + \ldots \tag{26}$$

where $O(6)$, $O(8)$, ..., $O(2n)$ ... are operators (of gauge fields) of dimensions 6, 8, ..., $2n$, ... which are as shown later, local and gauge-invariant. Thus the expression in (25) is

$$\langle B_M(x)\rangle = -\frac{1}{16\pi^2}\langle \mathrm{Tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}]\rangle_{J,\xi,\bar{\xi}} + \frac{\langle O(6)\rangle_{J,\xi,\bar{\xi}}}{M^2} + \ldots . + \frac{\langle O(2n)\rangle_{J,\xi,\bar{\xi}}}{M^{2n-4}} + \ldots . \tag{27}$$

Now a typical term $\langle O(2n)\rangle_{J,\xi,\bar{\xi}}$ generates Green's functions of $O(2n)$ which will contain divergences in powers of (M^2). This will be analyzed in detail in § 4. For the present we shall only take an example. Suppose $\langle O(6)\rangle$ contains terms that go like M^2 or $M^2(\ln M^2)^p$. (This actually happens). Then

$$\lim_{M^2\to\infty}\frac{\langle O(6)\rangle}{M^2}\neq 0$$

and may in fact diverge. Thus in higher orders it is not true that

$$\lim_{M^2\to\infty}\langle B_M(x)\rangle = \left\langle \lim_{M^2\to\infty} B_M(x)\right\rangle. \tag{28}$$

In the lowest nontrivial order, however, the result of (28) is true. This is because the Green's functions of $O(2n)$ are to be taken in the tree order which are always finite. Hence

$$\lim_{M^2\to\infty}\frac{\langle O(2n)\rangle}{M^{2n-4}} = 0$$

in the lowest nontrivial order (one loop order).

The above clearly illustrates the limitation of Fujikawa's discussion (Fujikawa 1979, 1980a). To prove Adler-Bardeen theorem, it is clear that the series of (27) will have to be dealt with. This is the purpose of this paper.

The one loop character of Fujikawa's results is also clearly evident in the context of trace anomaly where the approach has been able only to yield the lowest order ($O(g^2)$) result for trace anomaly. We shall deal with this elsewhere (Joglekar and Anuradha, in preparation).

In § 3 we shall take up the structure of the operators $O(2n)$ in the series of (27). In § 4 we shall deal with the Green's functions of these operators and extract the form of the contribution to the anomaly coming from the Jacobian in higher loop orders.

3. The general form of $O(2n)$

In this section we shall deduce the general form of the operators $O(2n)$ and show that they can be expressed as

$$O(2n) = \partial^\mu\partial^\nu O^{(1)}_{\mu\nu}(2n-2) + \partial^\mu O^{(2)}_\mu(2n-1) \tag{29}$$

where $O_\mu^{(2)}(2n-1)$ is given by

$$O_\mu^{(2)}(2n-1)=\left(-\frac{1}{4}\right)^{n-2}\frac{1}{(n-2)!}\sum_m \phi_m^\dagger(x)[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m(x). \tag{30}$$

The number of commutators in the above expressions is $(2n-5)$. We shall further show that $O_{\mu\nu}^{(1)}(2n-2)$ and $O_\mu^{(2)}(2n-1)$ are gauge-invariant local operators of gauge fields of dimensions $(2n-2)$ and $(2n-1)$ respectively.

We shall present the proof of the above statements for the case $n=3$ in this section. The proof for the case $n>3$ is along similar lines and is presented in the Appendix *B*.

We begin with the expression for $B_M(x)$:

$$B_M(x)=\sum_n \phi_n^\dagger(x)\gamma_5\exp(-\lambda_n^2/M^2)\phi_n(x) \tag{31}$$

$$=-\frac{1}{16\pi^2}\mathrm{Tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}]+\frac{O(6)}{M^2}+\frac{O(8)}{M^4}+\ldots+\frac{O(2n)}{M^{2n-4}}+\ldots \tag{32}$$

We thus see that with $n\geqslant 3$,

$$O(2n)=\frac{1}{(n-2)!}\left.\frac{\partial^{n-2}}{\partial(M^{-2})^{n-2}}B_M(x)\right|_{M^2\to\infty} \tag{33}$$

In particular,

$$O(6)=\left.\frac{\partial}{\partial M^{-2}}B_M(x)\right|_{M^2\to\infty} \tag{34}$$

and from (31),

$$\frac{\partial}{\partial M^{-2}}B_M(x)=\sum_m(-\lambda_m^2)\phi_m^\dagger(x)\gamma_5\exp(-\lambda_m^2/M^2)\phi_m(x). \tag{35}$$

We simplify the expression of (35) by using the following identity (for proof see Appendix A) twice, viz

$$\phi_m^\dagger(x)X\gamma_5\phi_m(x)\equiv\frac{1}{-2\lambda_m}\{\phi_m^\dagger(x)[X,\not{D}]\gamma_5\phi_m(x)+\partial^\mu[\phi_m^\dagger(x)\gamma_\mu X\gamma_5\phi_m(x)\} \tag{36}$$

provided $\lambda_m\neq 0$.

We obtain, for $\lambda_m\neq 0$,

$$\phi_m^\dagger(x)\gamma_5\phi_m(x)=-\frac{1}{2\lambda_m}\partial^\mu[\phi_m^\dagger(x)\gamma_\mu\gamma_5\phi_m(x)]$$

$$=\left(-\frac{1}{2\lambda_m}\right)^2\partial^\mu\{\phi_m^\dagger(x)[\gamma_\mu,\not{D}]\gamma_5\phi_m(x)+\partial^\nu[\phi_m^\dagger(x)\gamma_\nu\gamma_\mu\gamma_5\phi_m(x)]\}$$

$$=\frac{1}{4\lambda_m^2}\{\partial^2(\phi_m^\dagger(x)\gamma_5\phi_m(x))+\partial^\mu[\phi_m^\dagger(x)[\gamma_\mu,\not{D}]\gamma_5\phi_m(x)]\}. \tag{37}$$

Thus using the result of (37) into (35), we obtain

$$\frac{\partial}{\partial M^{-2}}B_M(x)=-\tfrac{1}{4}\partial^2\sum_{\lambda_m\neq 0}\phi_m^\dagger\gamma_5\exp(-\lambda_m^2/M^2)\phi_m$$

$$-\tfrac{1}{4}\partial^\mu\sum_{\lambda_m\neq 0}\phi_m^\dagger[\gamma_\mu,\not{D}]\gamma_5\exp(-\lambda_m^2/M^2)\phi_m. \tag{38}$$

The restriction $\lambda_m\neq 0$ on the summations in (38) can be removed because the additional terms are identically zero

$$-\partial^2\sum_{\lambda_m=0}\phi_m^\dagger\gamma_5\phi_m-\partial^\mu\sum_{\lambda_m=0}\phi_m^\dagger[\gamma_\mu,\not{D}]\gamma_5\phi_m\equiv 0 \tag{39}$$

as seen by using $\not{D}\phi_m=0$; $\phi_m^\dagger\not{D}=\phi_m^\dagger(\overleftarrow{\not{\partial}}+\not{A})$ (for $\lambda_m=0$) in the second term on the left hand side of (39). Replacing $\exp(-\lambda_m^2/M^2)\phi_m$ by $\exp(-\not{D}^2/M^2)\phi_m$ we obtain,

$$O(6)=\partial^\mu\partial^\nu O_{\mu\nu}^{(1)}(4)+\partial^\mu O_\mu^{(2)}(5)\equiv\partial^\mu\partial^\nu[g_{\mu\nu}O^{(1)}(4)]+\partial^\mu O_\mu^{(2)}(5) \tag{40}$$

where

$$O^{(1)}(4)=\lim_{M^2\to\infty}-\tfrac{1}{4}\sum_m\phi_m^\dagger\gamma_5\exp(-\not{D}^2/M^2)\phi_m \tag{41}$$

$$O_\mu^{(2)}(5)=\lim_{M^2\to\infty}-\tfrac{1}{4}\sum_m\phi_m^\dagger[\gamma_\mu,\not{D}]\gamma_5\exp(-\not{D}^2/M^2)\phi_m. \tag{42}$$

$O^{(1)}(4)$ and $O_\mu^{(2)}(5)$ can be calculated as was done by Fujikawa (1979–1981) and are clearly local operators of gauge fields as seen by direct calculation (see Appendix B; equation (B.14)). They are also gauge-invariant operators because as shown in Appendix B any operator of the form

$$\lim_{M^2\to\infty}\sum_m\phi_m^\dagger(x)f(\not{D})\exp(-\not{D}^2/M^2)\phi_m(n)$$

where $f(\not{D})$ is a matrix operator function of $\not{D}$ is gauge-invariant. $O^{(1)}(4)$ and $O_\mu^{(2)}(5)$ have dimension four and five respectively. We give the results for $O^{(1)}(4)$ and $O_\mu^{(2)}(5)$ for completeness

$$O^{(1)}(4)=\frac{1}{64\pi^2}\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$$

$$O_\mu^{(2)}(5)=\frac{1}{96\pi^2}[-\tfrac{1}{2}\partial_\mu(F\tilde{F})+\tfrac{1}{2}D^{ab\eta}F_{\eta\delta}^b\tilde{F}_\mu^{a\delta}] \tag{43}$$

$$\equiv\frac{1}{96\pi^2}[-\tfrac{1}{2}\partial_\mu(F\tilde{F})]+\tilde{O}_\mu^{(2)}(5).$$

The results stated at the beginning of this section (see (29) and (30)) have been proved in

Appendix B along similar lines. The form of operator $O^{(1)}_{\mu\nu}(2n-2)$ is not needed as it will be shown in §4 that it does not contribute to the anomaly equation.

4. General form of higher order corrections

In this section we shall derive the general form of higher order contributions coming from the Jacobian using the general form of $O(2n)$ established in the previous section and certain results on the renormalization of gauge invariant operators (Joglekar and Lee 1976; Kluberg-Stern and Zuber 1975; Dixon and Taylor 1974; Deans and Dixon 1975).

We shall assume that Green's functions of operators have been suitably regularized using a cut off M on the large eigenvalues of the operator $\not{D}$. (Our results are of general nature of the form of the higher order contributions and hence probably independent of the details of regularization.) The Green's functions will contain divergences of the kind $M^{2p}(\ln m^2)^9$ where p is restricted from above by power counting. Consider now Green's functions of $O(2n)$ whose form is given in (29) and (30). In (27) for $\langle B_M(x)\rangle$, $O(2n)$ appears in the form $\langle O(2n)\rangle/M^{2n-4}$. Hence in the limit $M^2\to\infty$ only that part of $\langle O(2n)\rangle$ which diverges as M^{2n-4} or worse could possibly contribute to $\lim_{M^2\to\infty}\langle B_M(x)\rangle$. It is thus clear that the contribution to $\langle B_M(x)\rangle$ from higher order terms comes entirely from divergences in Green's functions $\langle O(2n)\rangle$ and these too of sufficiently high order. Now $O(2n)$ is a linear combination of terms that involve two local gauge-invariant operators $O^{(1)}_{\mu\nu}$ and $O^{(2)}_{\mu}$ [see (30)]. We thus need to know the divergence structure of the Green's functions of local gauge invariant operators. This has already been established in the context of dimensionally regularized Green's functions (Joglekar and Lee 1976; Kluberg-Stern and Zuber 1975; Dixon and Taylor 1974; Deans and Dixon 1975). We state the results below:

(A) The operators that mix under renormalization with gauge invariant operators are either of the following three varieties:

(a) Gauge invariant operators;

(b) Operators of the form

$$\frac{\delta\tilde{S}}{\delta A^a_\mu}\frac{\delta F}{\delta(\partial_\mu c^a)}+\frac{\delta\tilde{S}}{\delta\psi_a}H_a+\frac{\delta\tilde{S}}{\delta\bar{\psi}_b}\bar{H}_b+\mathscr{G}_0 F$$

where $\tilde{S}$ = gauge invariant action + ghost action and F is a functional of A^a_μ, c^a, $\bar{c}^a$, ψ, $\bar{\psi}$ with correct ghost structure (Joglekar and Lee 1976). $\mathscr{G}_0$ is the BRS variation operator for A, c, ψ, $\bar{\psi}$.

(c) $$\frac{\delta\tilde{S}}{\delta C^a}X_a[A, c, \bar{c}, \psi, \bar{\psi}]$$

where X_a is a local functional of its arguments.

(B) The above set of operators is closed under renormalization to all orders. (Of course all these operators must have the same Lorentz structure, parity and ghost number as the original operator.)

We shall not need these results in their entirety. We shall only need the following corollaries to these results.

Corollary 1: In a Yang-Mills theory with fermions only, there is *no* operator of dimension two that can mix with a gauge-invariant operator to any order. (Proof: There is no such operator amongst the types (a), (b), (c) listed above.)

Corollary 2: The only pseudovector group-scalar operator of dimension three (in a Y-M theory with fermions only) that can mix with a gauge invariant operator is $\bar{\psi}\gamma_\mu\gamma_5\psi$. This is again so because the operators in (b) and (c) have dimensions four or more.

It should be mentioned that the results A and B above have been proved in the context of a different regularization. The question as to if they are valid when regularization is in terms of the cutoff on eigenvalues of $\not{D}$ is under study by us. But in any case the two corollaries stated above will be still valid. This is so because they depend purely on the form of the W-T identity of gauge invariant operators which is the same for either regularization.

We now come back to the problem at hand. The divergence structure of $O(2n)$ is determined by two gauge-invariant operators $O^{(1)}_{\mu\nu}(2n-2)$ and $O^{(2)}_{\mu}(2n-1)$. First consider $O^{(1)}_{\mu\nu}(2n-2)$. We need to worry only about the divergences in $\langle O^{(1)}_{\mu\nu}(2n-2)\rangle$ that go like M^{2n-4} or worse. By counting dimensions such divergences are necessarily of form

$$M^{2n-4}(\ln M^2)^q \times \text{local operator of dimension two.}$$

But $O^{(1)}_{\mu\nu}$ is a gauge-invariant operator. From corollary 1, there is no gauge invariant operator of dimension two it can mix with. Hence, such terms must be absent. Hence

$$\lim_{M^2\to\infty} \frac{\partial^\mu\partial^\nu\langle O^{(1)}_{\mu\nu}(2n-2)\rangle}{M^{2n-4}}=0. \tag{44}$$

By a similar logic the only divergences in $\langle O^{(1)}_{\mu}(2n-1)\rangle$ we need to worry about are of the form

$$(M^2)^{n-2}\,(\ln M^2)^q \times \text{local pseudovector operator of dimension three.}$$

By corollary 2 above, the only such operator is $\bar{\psi}\gamma_\mu\gamma_5\psi$. Hence as $M^2\to\infty$

$$\frac{\langle O^{(1)}_{\mu}(2n-1)\rangle}{M^{2n-4}}=\frac{1}{2}f_n(g^2,\ln M^2)\bar{\psi}\gamma_\mu\gamma_5\psi+O\left(\frac{1}{M^2}\right) \tag{45}$$

where $f_n(g^2, \ln M^2)$ is a power series in g^2 and $\ln M^2$.

Combining results of (44) and (45) we obtain

$$\lim_{M^2\to\infty}\langle B_M(x)\rangle=-\frac{1}{16\pi^2}\langle \mathrm{Tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}]\rangle+\frac{1}{2}f(g^2,\ln M^2)\partial^\mu\langle\bar{\psi}\gamma_\mu\gamma_5\psi\rangle \tag{46}$$

where

$$f(g^2,\ln M^2)=\sum_{n=3}^{\infty}f_n(g^2,\ln M^2).$$

The result of (46) is the main result of our paper on the structure of the higher order corrections to the Jacobian.

Here we would like to make a number of comments. Firstly, a direct calculation shows that $f_n(g^2,\ln M^2)$ is a power series in g^2 beginning as g^4. Secondly, all the operators $O(2n)$, $n \geqslant 3$ will generally contribute to a given order in $(g^2)^m$ to $f(g^2, \ln M^2)$. This makes the evaluation of $f(g^2, \ln M^2)$ technically difficult. Finally, we note that we have not been fully able to exploit the form of $O(2n)$ and $O_\mu^{(2)}(2n-1)$ in order to obtain information on $f(g^2, \ln M^2)$. We hope to do this in future.

5. Adler-Bardeen theorem

We have shown in the previous section the following form for anomaly equations in the path integral formulation valid to all orders obtained from (46).

$$\partial_\mu J_5^\mu = \frac{2im_0}{1+f(g^2, \ln M^2)} \bar{\psi}\gamma_5\psi - \frac{1}{8\pi^2} \frac{1}{1+f(g^2, \ln M^2)} \mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}). \tag{47}$$

Proving Adler-Bardeen theorem within path integral formulation requires proving $f(g^2, \ln M^2) = 0$. Owing to the technical complexities, we have not been able to show this directly. We shall, however, present an indirect argument for the same.

Fujikawa (1979, 1980a) has shown that the usual Pauli Villars treatment is an ideal perturbative realization of the path integral formulation. In the formulation the anomaly comes not from the Jacobian (these cancel between the regulator field and the fermion field measures), but from the regulator field terms from the anomaly equation. In the standard perturbative treatment, it is well known that the coefficient of the naive divergence term $2im_0\bar{\psi}\gamma_5\psi$ is not modified to all orders. This follows very simply from an elementary diagramatic considerations (Adler 1970). If we borrow just this much from the perturbative approach, (47) implies that

$$f(g^2, \ln M^2) = 0.$$

This in turn proves the Adler-Bardeen theorem to all orders:

$$\partial_\mu J_5^\mu = 2im_0\bar{\psi}\gamma_5\psi - \frac{1}{8\pi^2} \mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}). \tag{48}$$

Appendix A

In this appendix we shall prove the identity of (36) viz for $\lambda_m \neq 0$, and any operator X,

$$\phi_m^\dagger(x) X\gamma_5\phi_m(x) = -\frac{1}{2\lambda_m}\{\phi_m^\dagger(x)[X, \not{D}]\gamma_5\phi_m(x) + \partial^\mu[\phi_m^\dagger(x)\gamma_\mu X\gamma_5\phi_m(x)]\} \tag{36}$$

$$\begin{aligned}\phi_m^\dagger(x) X\gamma_5\phi_m(x) &= \frac{1}{\lambda_m}\phi_m^\dagger(x) X\gamma_5\not{D}\phi_m(x) \\ &= -\frac{1}{\lambda_m}\phi_m^\dagger(x) X\not{D}\gamma_5\phi_m(x) \\ &= -\frac{1}{\lambda_m}\{\phi_m^\dagger[X, \not{D}]\gamma_5\phi_m + \phi_m^\dagger\not{D}X\gamma_5\phi_m\}\end{aligned} \tag{A.1}$$

Now using

$$\phi_m^\dagger(x)[-\overleftarrow{\partial\!\!\!/}+A\!\!\!/]=\lambda_m\phi_m^\dagger(x) \tag{A.2}$$

One obtains, for any Y,

$$\phi_m^\dagger D\!\!\!\!/\,Y=\lambda_m\phi_m^\dagger Y+\partial^\mu(\phi_m^\dagger\gamma_\mu Y) \tag{A.3}$$

Using (A.3) in (A.1) and simplifying, one obtains equation (36) given above.

Appendix B

In this appendix we shall derive the general form of the operator $O(2n)$ stated at the beginning of § 3.

From equation (33), we have

$$\begin{aligned} O(2n) &= \lim_{M^2\to\infty}\frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial(M^{-2})^{n-2}}B_M(x) \qquad n\geqslant 3 \\ &= \lim_{M^2\to\infty}\frac{1}{(n-2)!}\sum_m(-\lambda_m^2)^{n-2}\phi_m^\dagger\exp(-\lambda_m^2/M^2)\gamma_5\phi_m. \end{aligned} \tag{B.1}$$

We shall first prove by induction that $(-\lambda_m^2)^{n-2}\phi_m^\dagger\gamma_5\phi_m$ has the form ($n\geqslant 3$; $\lambda_m\neq 0$)

$$\begin{aligned} (-\lambda_m^2)^{n-2}\phi_m^\dagger(x)\gamma_5\phi_m(x) &= \partial^\mu\partial^\nu\{\phi_m^\dagger K_{\mu\nu}^{(n)}\gamma_5\phi_m\} \\ &\quad +(-\tfrac{1}{4})^{n-2}\partial^\mu\{\phi_m^\dagger[[\ldots\ldots[\gamma_\mu,D\!\!\!\!/\,],D\!\!\!\!/\,]\ldots D\!\!\!\!/\,]\gamma_5\phi_m\} \end{aligned} \tag{B.2}$$

where the last term has $(2n-5)$ commutators. For $n=3$, the result is true by virtue of equation (37).

Let the result be true for $n=r$. In other words, let

$$\begin{aligned} (-\lambda_m)^{r-2}\phi_m^\dagger\gamma_5\phi_m &= \partial^\mu\partial^\nu[\phi_m^\dagger K_{\mu\nu}^{(r)}\gamma_5\phi_m] \\ &\quad +(-\tfrac{1}{4})^{r-2}\partial^\mu\{\phi_m^\dagger\underbrace{[[\ldots[\gamma_\mu,D\!\!\!\!/\,],D\!\!\!\!/\,]\ldots D\!\!\!\!/\,]}_{\leftarrow 2r-5 \text{ commutators}\rightarrow}\gamma_5\phi_m\} \end{aligned} \tag{B.3}$$

for some $K_{\mu\nu}^{(r)}$. We shall prove the result for $n=r+1$. Consider

$$\begin{aligned} (-\lambda_m^2)^{r-1}\phi_m^\dagger\gamma_5\phi_m &= (-\lambda_m^2)\{(-\lambda_m^2)^{r-2}\phi_m^\dagger\gamma_5\phi_m\} \\ &= -\lambda_m^2\{\partial^\mu\partial^\nu[\phi_m^\dagger K_{\mu\nu}^{(r)}\gamma_5\phi_m] \\ &\quad +(-\tfrac{1}{4})^{r-2}\partial^\mu(\phi_m^\dagger[[\ldots[\gamma_\mu,D\!\!\!\!/\,],D\!\!\!\!/\,]\ldots D\!\!\!\!/\,]\gamma_5\phi_m)\} \end{aligned} \tag{B.4}$$

Applying equation (36) to the second term on the right hand side,

$$\partial^\mu(\phi_m^\dagger\underbrace{[[\ldots[\gamma_\mu,D\!\!\!\!/\,],D\!\!\!\!/\,]\ldots D\!\!\!\!/\,]}_{\leftarrow 2r-5 \text{ commutators}\rightarrow}\gamma_5\phi_m)$$

$$=\partial^{\mu}\left\{-\frac{1}{2\lambda_m}\phi_m^{\dagger}[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m\right.$$

$\leftarrow 2r-4$ commutators $\rightarrow$

$$-\frac{1}{2\lambda_m}\partial^{\nu}[\phi_m^{\dagger}\gamma_\nu[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m].$$

$\leftarrow 2r-5$ commutators $\rightarrow$ (B.5)

Applying equation (36) once more to the first term on the right hand side one obtains:

$$-\lambda_m^2\partial^{\mu}(\phi_m^{\dagger}[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m)$$
$$=-\tfrac{1}{4}\partial^{\mu}\{\phi_m^{\dagger}[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m\}$$

$\leftarrow 2r-3$ commutators $\rightarrow$

$$-\tfrac{1}{4}\partial^{\mu}\partial^{\nu}\{\phi_m^{\dagger}\gamma_\nu[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m\}$$

$\leftarrow 2r-4$ commutators $\rightarrow$ (B.6)

While the second term in (B.5) yields

$$(-\lambda_m^2)\left(-\frac{1}{2\lambda_m}\right)\partial^{\mu}\partial^{\nu}\{\phi_m^{\dagger}\gamma_\nu[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m\}$$
$$=\tfrac{1}{2}\partial^{\mu}\partial^{\nu}\{\phi_m^{\dagger}\gamma_\nu[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\not{D}\phi_m\}$$

$\leftarrow 2r-5$ commutators $\rightarrow$ (B.7)

Combining (B.5), (B.6), (B.7) in (B.4) one obtains

$$(-\lambda_m^2)^{r-1}\phi_m^{\dagger}\gamma_5\phi_m$$
$$=\partial^{\mu}\partial^{\nu}\{\phi_m^{\dagger}K_{\mu\nu}^{(r)}\gamma_5\not{D}^2\phi_m+(-\tfrac{1}{4})^{r-1}\phi_m^{\dagger}\gamma_\nu[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m$$

$\leftarrow 2r-4$ commutators $\rightarrow$

$$+\tfrac{1}{2}(-\tfrac{1}{4})^{r-3}\phi_m^{\dagger}\gamma_\nu[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\not{D}\phi_m\}$$

$\leftarrow 2r-5$ commutators $\leftarrow$

$$+(-\tfrac{1}{4})^{r-2}\partial^{\mu}\{\phi_m^{\dagger}[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m\}$$

$\leftarrow 2r-3$ commutators $\rightarrow$

$$\equiv\partial^{\mu}\partial^{\nu}\{\phi_m^{+}K_{\mu\nu}^{(r+1)}\gamma_5\phi_m$$
$$+(-\tfrac{1}{4})^{r-1}\partial^{\mu}\{\phi_m^{\dagger}[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m\} \quad \text{(B.8)}$$

where

$$K_{\mu\nu}^{(r+1)}=K_{\mu\nu}^{(r)}\not{D}^2+(-\tfrac{1}{4})^{r-1}\gamma_\nu[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]$$

$\leftarrow 2r-4$ commutators $\rightarrow$

$$-\tfrac{1}{2}(-\tfrac{1}{4})^{r-2}\gamma_\nu[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\not{D}$$

with

$$K_{\mu\nu}^{(3)}=-\tfrac{1}{4}g_{\mu\nu}\,\mathbf{1}. \quad \text{(B.9)}$$

In (B.8) we have proved the result of (B.2) for $n=r+1$. Hence the proof of (B.2) by induction is complete.

Using the result of (B.2) in (B.1), we obtain,

$$O(2n)=\lim_{M^2\to\infty}\frac{1}{(n-2)!}\left\{\partial^\mu\partial^\nu\sum_{\lambda_m\neq 0}\phi_m^\dagger K_{\mu\nu}^{(n)}\gamma_5\exp(-\not{D}^2/M^2)\phi_m\right.$$
$$+(-\tfrac{1}{4})^{n-2}\partial^\mu\sum_{\lambda_m\neq 0}\phi_m^\dagger[[\ldots\,[\gamma_\mu,\not{D}],\not{D}]\ldots\,\not{D}]\gamma_5$$
$$\left.\times\exp(-\not{D}^2/M^2)\phi_m\right\}. \tag{B.10}$$

The restriction $\lambda_m\neq 0$ can be removed as in the case of $O(6)$. (See equations (38) and (39)). To see this, we consider the additional terms needed to remove the restriction in the curly bracket of (B.10). They are

$$\partial^\mu\partial^\nu\sum_{\lambda_m=0}\phi_m^\dagger K_{\mu\nu}^{(n)}\gamma_5\phi_m+(-\tfrac{1}{4})^{n-2}\partial^\mu\sum_{\lambda_m=0}\phi_m^\dagger[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]\gamma_5\phi_m.$$

These can be shown to vanish as in the case of $O(6)$ using (B.9). We thus have

$$O(2n)=\partial^\mu\partial^\nu O_{\mu\nu}^{(1)}(2n-2)+\partial^\mu O_\mu^{(1)}(2n-1) \tag{B.11}$$

where

$$O_{\mu\nu}^{(1)}=\lim_{M^2\to\infty}\frac{1}{(n-2)!}\sum_m\phi_m^\dagger K_{\mu\nu}^{(n)}\gamma_5\exp(-\not{D}^2/M^2)\phi_m \tag{B.12}$$

and

$$O_\mu^{(2)}=\lim_{M^2\to\infty}\frac{1}{(n-2)!}(-\tfrac{1}{4})^{n-2}\sum_m\phi_m^\dagger[[\ldots[\gamma_\mu,\not{D}],\not{D}]\ldots\not{D}]$$
$$\times\exp(-\not{D}^2/M^2)\gamma_5\phi_m.$$
$$\leftarrow 2n-5\text{ commutators}\rightarrow \tag{B.13}$$

To prove the result stated at the beginning of § 3, we have to show that $O^{(1)}$ and $O^{(2)}$ are local gauge invariant operators of gauge fields. To this end we first note that on account of (B.9), $K_{\mu\nu}^{(n)}$ is a polynomial in $\not{D}$ say $K_{\mu\nu}^n(\not{D})$. Thus $O_{\mu\nu}^{(1)}(2n-2)$ can be evaluated as done for the anomaly in § 2 viz;

$$O_{\mu\nu}^{(1)}(2n-2)=\lim_{M^2\to\infty}\frac{1}{(n-2)!}\sum_m\phi_m^\dagger(x)K_{\mu\nu}^{(n)}(\not{D})\gamma_5\exp(-\not{D}^2/M^2)\phi_m(x)$$
$$=\lim_{M^2\to\infty}\frac{1}{(n-2)!}\int\frac{d^4k}{(2\pi)^4}K_{\mu\nu}^{(n)}(\not{D}+i\not{k})\exp[-(\not{D}+i\not{k})^2/M^2]. \tag{B.14}$$

In the limit $M^2\to\infty$ only a finite number of terms contribute to $O_{\mu\nu}^{(1)}(2n-2)$ and the resultant expression is local in A_μ. A similar argument applies to $O_\mu^{(2)}(2n-1)$.

To prove the gauge invariance of $O_{\mu\nu}^{(1)}$ and $O_{\mu\nu}^{(2)}$, we note that they are both of the form

$$\sum_m\phi_m^\dagger(x)f(\phi)\phi_m(x)\exp(-\lambda_m^2/M^2)$$

where $f(\not{D})$ is a matrix polynomial in $\not{D}$.

This is invariant under a gauge transformation as under a gauge transformation, the basis functions ϕ_m change to

$$\phi_m(x)\to\phi'_m(x)=\exp[i\alpha(x)]\,\phi_m(x);\quad \phi^\dagger_m(x)\to\phi^{\dagger\prime}_m(x)=\phi^\dagger_m(x)\exp[-(i\alpha(x)]$$

where as

$$f(\not{D})\phi_m(x)\to f(\not{D}')\phi'_m(x)=\exp[(i\alpha(x)]\,f(\not{D})\phi_m(x). \tag{B.15}$$

Hence etc.

Appendix C

In this appendix we show that $f_n(g^2, \ln M^2)$ occurring in equation (45) are nontrivial (i.e. individually non-zero) by working out f_3 to the lowest nontrivial order in g^2 viz $O(g^4)$.

From equation (43), $\tilde{O}^2_\mu(5)$ has the form

$$\tilde{O}^{(2)}_\mu(5)=Cg_0^2D^{ab\eta}F^b_{\eta\delta}\tilde{F}^{a\mu\delta}_\mu=C'g_0^2\frac{\delta S_0}{\delta A^a_\delta}\tilde{F}^{a\delta} \tag{C.1}$$

where C and C' are constants and $S_0=\frac{1}{2g^2}\int \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\mathrm{d}^4x$.

We can express $O^{(2)}_\mu(5)$ as

$$\tilde{O}^{(2)}_\mu(5)=C'g_0^2\left[\frac{\delta S_{\text{eff}}}{\delta A^a_\delta}\tilde{F}^{a\delta}_\mu-ig_0\bar{\psi}\gamma^\delta T^a\psi\tilde{F}^{a\delta}_\mu-\frac{\delta S_{\text{ghost}}}{\delta A^a_\delta}\tilde{F}^{a\delta}_\mu\right]. \tag{C.2}$$

We shall show that $\partial^\mu\langle O^{(2)}_\mu(5)\rangle$ leads to a nontrivial divergence when one considers the two-fermion matrix elements of $\partial^\mu O^{(2)}_\mu(5)$ in one loop approximation. This would mean an nontrivial $f_3(g^2, \ln M^2)$ to $O(g^4)$.

Consider the two-fermion matrix element of $\tilde{O}^{(2)}_\mu(5)$ of (C.2) in the one loop approximation. The last term involving S_{ghost} does not contribute to one loop diagrams as fermions do not couple to ghosts directly. On account of equations of motion valid to one loop approximation we have

$$\left\langle\frac{\delta S_{\text{eff}}}{\delta A^a_\delta}\tilde{F}^{a\mu\delta}\right\rangle=-J^a_\delta\langle\tilde{F}^{a\mu\delta}\rangle=\frac{\delta\Gamma}{\delta\langle A^a_\delta\rangle}\langle\tilde{F}^{a\mu\delta}\rangle$$

$$=\frac{\delta\Gamma}{\delta\langle A^a_\delta\rangle^R}\langle Z^{-1/2}\tilde{F}^{a\mu\delta}\rangle. \tag{C.3}$$

Thus the effective dimensions of this term in $O^{(2)}_\mu(5)$ is reduced from five to two as $\delta\Gamma/\delta\langle A^a_\delta\rangle^R$ is finite. This term has at best $\log M^2$ divergences coming from $Z^{-1/2}$. Thus it does not contribute to $\lim\limits_{M^2\to\infty}\frac{\langle O(6)\rangle}{M^2}$.

It is straightforward to verify that the middle term in the right hand side of (C.2) does indeed lead to quadratically divergent two-fermion proper vertex. This leads to a divergence of $O(g_0^4M^2)$. Thus $f_3(g^2, \ln M^2)\neq 0$ to $O(g^4)$.

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