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Wave reflection from inhomogeneous media at normal incidence

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Abstract. Wave reflection from a medium with continuously varying refractive index is examined. A differential equation is developed, the solution of which yields the back-reflected part of a wave of unity amplitude incident on a non-homogeneous medium at normal incidence.

Keywords. Reflectance; transmittance; wave reflection; inhomogeneous media.

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1. Introduction

In this paper reference will often be made to a paper by D'Souza (1986, hereafter referred to as paper I), where we had defined reflectance ρ as the coefficient of the back-reflected wave when a wave of unity magnitude is incident on a scatterer. Transmittance t is similarly defined. Thus a scatterer is completely defined by a set of four quantities (ρ_L, ρ_R, t_L, t_R). The subscripts R and L denote rightward and leftward incidence. A symmetric scatterer has an axis of symmetry and if this is taken as the origin then $\rho_L = \rho_R = \rho$ and $t_L = t_R = t$. In paper I it was further shown that if two symmetric scatterers are separated by a distance l, (ρ_1, t_1) and (ρ_2, t_2) being the characteristics of the two scatterers, then the two scatterers can be replaced by a single scatterer whose reflectance is given by

$$\rho_{12} = \rho_1 + \frac{t_1^2 \rho_2 \exp(2ikl)}{1 - \rho_1 \rho_2 \exp(2ikl)}, \tag{1}$$

where k is the wave vector. If, on the other hand, the two scatterers are not symmetric and are characterized by the sets (ρ_{L1}, ρ_{R1}, t_{L1}, t_{R1}) and (ρ_{L2}, ρ_{R2}, t_{L2}, t_{R2}) then the equivalent reflectance is given by

$$R_L = \rho_{L1} + \frac{\rho_{L2} t_{L1} t_{R1} \exp(2ikl)}{1 - \rho_{L2} \rho_{R1} \exp(2ikl)}, \tag{2a}$$

and the transmittance by

$$T_L = \frac{t_{L1} t_{L2} \exp(ikl)}{1 - \rho_{L2} \rho_{R1} \exp(2ikl)}. \tag{2b}$$

Relations (1) and (2) are true for all wave phenomena including electromagnetic waves, electron waves and seismic waves and this is evident from the derivations in paper I though there it is derived exclusively for electron waves.

2. Continuously varying medium

The continuously varying medium is represented in figure 1. The direction x is as indicated. Here we denote by $y(x)$ the combined reflectance of the total medium from the origin to the point x. At x we take a slice of the succeeding medium of width Δx and combine it with $y(x)$ according to equation (1). It will be noted that if Δx is sufficiently small then the slice can be considered a square symmetric barrier. If in equation (1) the distance l is brought to zero, it reduces to

$$\rho_{12}=\rho_1+t_1^2\rho_2/(1-\rho_1\rho_2). \tag{3}$$

We let $\Delta\rho$ be the reflectance of the small slice alone, and t be its transmittance. As Δx tends to zero, $\Delta\rho$ will go to zero and t will go to 1. Let $t = 1-\Delta t$, where Δt goes to zero as Δx tends to zero. Then using (3)

$$y(x+\Delta x)=\Delta\rho+\frac{(1-\Delta t)^2 y(x)}{1-\Delta\rho y(x)},$$

$$y(x+\Delta x)-y(x)=\frac{(\Delta\rho-y)(1-\Delta\rho y(x))+(1-\Delta t)^2 y(x)}{1-\Delta\rho y(x)}.$$

Ignoring the squared terms of $\Delta\rho$ and Δt and ignoring $\Delta\rho y(x)$ in comparison with '1' in the right side denominator

$$y(x+\Delta x)-y(x)=\Delta\rho-y+y^2\Delta\rho+(1-2\Delta t)y(x).$$

Dividing throughout by Δx and taking the limit $\Delta x\to 0$

$$\mathrm{d}y/\mathrm{d}x=(1+y^2)(\mathrm{d}\rho/\mathrm{d}x)-2(\mathrm{d}t/\mathrm{d}x)y. \tag{4}$$

We claim the above equation to be generally true for electromagnetic waves, electron waves and seismic waves and we demonstrate and elaborate the case for EM waves below.

2.1 *Case I: Electromagnetic wave*

Here $\mathrm{d}\rho/\mathrm{d}x$ and $\mathrm{d}t/\mathrm{d}x$ for square barriers are first derived. The barrier is shown in figure 2. The wave impedances for regions I, II and III are respectively η_1, η_2, and η_3, the wave impedance being defined as the ratio E/H for a travelling wave. Jordan and Balmain (1983) give the following relations

$$\rho_{L1}=(\eta_2-\eta_1)/(\eta_2+\eta_1),\quad t_{L1}=2\eta_2/(\eta_1+\eta_2),\quad \rho_{R1}=(\eta_1-\eta_2)/(\eta_1+\eta_2),$$

$$t_{R1}=2\eta_1/(\eta_1+\eta_2),\quad \rho_{L2}=(\eta_1-\eta_2)/(\eta_1+\eta_2),\quad t_{L2}=2\eta_1/(\eta_1+\eta_2),$$

$$\rho_{R2}=(\eta_2-\eta_1)/(\eta_1+\eta_2),\quad t_{R2}=2\eta_2/(\eta_1+\eta_2).$$

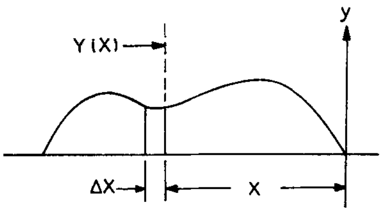

Figure 1. The inhomogeneous barrier.

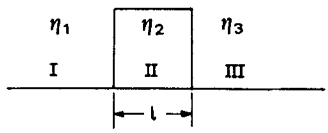

Figure 2. The square dielectric barrier.

Letting

$$\rho=(\eta_2-\eta_1)/(\eta_1+\eta_2),$$
$$\rho_{L1}=\rho=-\rho_{R1}=-\rho_{L2}=\rho_{R2},$$
$$t_{L1}=1+\rho=t_{R2},\ t_{R1}=t_{L2}=1-\rho.$$

Thus from (2a)

$$\rho_\beta=\rho-\frac{(1-\rho^2)\rho\exp(2ikl)}{1-\rho^2\exp(2ikl)},$$
$$=\frac{\rho[1-\exp(2ikl)]}{1-\rho^2\exp(2ikl)},$$
$$=\frac{(\eta_2^2-\eta_1^2)[1-\exp(2ikl)]}{(\eta_1+\eta_2)^2-(\eta_1-\eta_2)^2\exp(2ikl)},$$

where l is the width of the barrier.

In the limit as $l=\Delta x\to 0$

$$\Delta\rho=\frac{(\eta_2^2-\eta_1^2)(-2ik\Delta x)}{(\eta_1+\eta_2)^2-(\eta_1-\eta_2)^2}$$
$$\Delta\rho/\Delta x=\left(\frac{\eta_2^2-\eta_1^2}{4\eta_1\eta_2}\right)(-2ik). \tag{5}$$

Similarly (2b) becomes

$$t_\beta=\frac{(1-\rho^2)\exp(ikl)}{1-\rho^2\exp(2ikl)}$$
$$1-t_\beta=\Delta t=\frac{[1-\exp(ikl)]-\rho^2[\exp(2ikl)-\exp(ikl)]}{1-\rho^2\exp(2ikl)}.$$

In the limit $l=\Delta x\to 0$

$$\Delta t/\Delta x=(-ik-\rho^2 ik)/(1-\rho^2)$$
$$=[-(1+\rho^2)/(1-\rho^2)]ik,$$
$$=[-2(\eta_1^2+\eta_2^2)/4\eta_1\eta_2]ik \tag{6}$$

Let us consider the specific case of an electromagnetic wave propagating in a non-magnetic dielectric. Here

$$\eta=(\mu/\varepsilon)^{1/2}; \quad \text{thus} \quad (\eta_1/\eta_2)=(\varepsilon_2/\varepsilon_1)^{1/2}=(\varepsilon_r)^{1/2}=n$$

n is the refractive index of the medium and ε is the permittivity of the medium. Also $c_1/c_2=n=k/k_0$ where c_1 and c_2 are the velocities of propagation in free space and in the medium respectively, and k_0 is the free space wave number. Thus

$$\Delta\rho/\Delta x=\frac{1-(\eta_1/\eta_2)^2}{4(\eta_1/\eta_2)}\times -2ik,$$

$$=\frac{1-n^2}{2}\times -ik_0$$

and

$$\Delta t/\Delta x=\tfrac{1}{2}(1+n^2)\times -ik_0.$$

Thus using (4)

$$\mathrm{d}y/\mathrm{d}x=\frac{1-n^2}{2}(-ik_0)(1+y^2)-2y\left(\frac{1+n^2}{2}\right)(-ik_0),$$

$$\mathrm{d}y/\mathrm{d}x=\frac{-ik_0}{2}[(1-y)^2-n^2(x)(1+y)^2]. \tag{7}$$

For the special case of ionospheric propagation, Jordan and Balmain (1983) give $n^2=1-[81N(x)]/f^2$ where $N(x)$ is the electron density and f the frequency; thus (7) becomes

$$\mathrm{d}y/\mathrm{d}x=-\frac{ik_0}{2}\left[(1-y)^2-\left(1-\frac{81N(x)}{f^2}\right)(1+y)^2\right]. \tag{8}$$

Here y is the net reflectance of the backscattered wave. Through equation (8) backscattering from the ionosphere can be calculated.

2.2 *Case II*: *Seismic waves*

For such waves the derivation is analogous, equations (5) and (6) being the same except that the wave impedances are termed σ instead of η.

2.3 *Case III*: *Electron waves*

Schiff (1968) gives the equivalent calculations for a square potential barrier (figure 3).

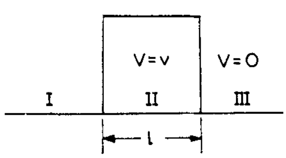

Figure 3. The square potential barrier.

$$\rho_\beta = \frac{(k^2+\beta^2)(1-\exp(-2\beta l)}{(k+i\beta)^2-(k-i\beta)^2\exp(-2\beta l)} \tag{9}$$

where $\beta = \left(\frac{2m(V-E)}{\hbar^2}\right)^{1/2}$, $k = (2mE/\hbar^2)^{1/2}$,

l is the barrier width and m the electronic mass.
Thus

$$d\rho/dx = (i/2k)(k^2+\beta^2)$$

and
$$t_B = \frac{i4k\beta\exp(-\beta l)}{(k+i\beta)^2-(k-i\beta)^2\exp(-2\beta l)}. \tag{10}$$

Thus $dt/dx = (i/2k)(\beta^2-k^2)$.
Finally

$$dy/dx = \frac{i(k^2+\beta^2)}{2k}(1+y^2) - 2y\frac{i}{2k}(\beta^2-k^2)$$

K is independent of x but β will be a function of x depending on the nature of the barrier $V(x)$. Thus

$$dy/dx = \frac{i}{2k}\left[(1+y^2)\frac{2mV(x)}{\hbar^2} - \frac{4m}{\hbar^2}y(V(x)-2E)\right]. \tag{11}$$

3. An illustrative example

As an illustrative example we assume the $F1$ layer of the ionosphere to contain a uniform density of electrons of 4×10^{11} electrons/m^3 and extending from a height of 130 km to 230 km i.e. a width of 100 km. Now

$$n^2 = 1-[81N(x)]/f^2 = 1-[81N(x)4\pi^2]/k_0^2c^2$$

here c is the velocity of light. Accordingly (8) becomes

$$dy/dx = -\frac{ik_0}{2}\left[(1-y)^2 - \left(1-\frac{81N_0 4\pi^2}{c^2k_0^2}\right)(1+y)^2\right].$$

For a uniform density N_0, the solution is

$$y(k_0) = \frac{(k_c^2/k_0^2)\sin nk_0 l}{(2-(k_c/k_0)^2)\sin nk_0 l + 2i(1-k_c/k_0)^2)^{1/2}\cos k_0 nl} \tag{12}$$

Here $k_c = \frac{81N_0 4\pi^2}{c^2}$ and $n^2 = 1-(k_c/k_0)^2$

$= 0{\cdot}192$/metre

A plot of the envelope of $|y(k)|$ vs k is shown in figure 4. It is interesting to note that there is significant reflection from this layer even above the critical frequency. Another note of significance is the existence of closely spaced nulls (not shown) but evident from

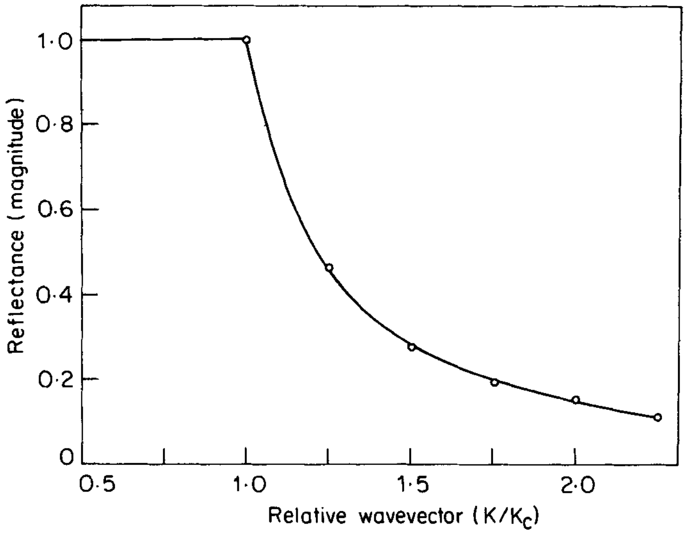

Figure 4. Plot of reflectance vs wavevector.

equation (12). The relation is not to be confused with total reflection from the ionosphere.

4. Conclusions

In this paper wave back-scattering at normal incidence from a non-homogeneous medium has been examined. The results find immediate applicability in electron waves, seismic waves and electromagnetic waves. The results should be useful in examining back-scattering from the ionosphere in constructing non-homogeneous dielectric optical filters and in electron back-scattering without invoking the WKBJ approximation. It is also possible to examine reflection of seismic waves from non-homogeneous media.

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